PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 4, April 1990

A NONZERO COMPLEX SEQUENCE WITH VANISHING POWER-SUMS

ANDREW LENARD

(Communicated by R. Daniel Mauldin)

ABSTRACT. In response to a question of W. M. Priestley, we construct a nonvanishing infinite sequence of complex numbers (a_j) such that all the power-sums $\sum_j a_j^r$ $(r = 1, 2, 3, \ldots)$ vanish.

If a finite number of complex numbers $a_1, \ldots, a_n$ have the property that

$$\sum_1^n a_j^r = 0 \qquad (r = 1, 2, \ldots, n), \tag{1}$$

then it follows that all of them must be zero. For if not, then there is a $k \leq n$ and k distinct nonvanishing ones among the a_j, say $b_1, \ldots, b_k$, such that the first k of the equations (1) may be written

$$\sum_1^k b_j^r n_j = 0 \qquad (r = 1, 2, \ldots, k), \tag{2}$$

where the n_j are strictly positive integers. This requires the vanishing of the determinant of the $k \times k$ matrix b_j^r and this, in turn, the vanishing of at least one of the b_j $(1 \leq j \leq k)$, contrary to the assumption.

The question was raised by W. M. Priestley [1] whether this state of affairs has an infinite analogue so that the infinitely many equations

$$\sum_1^\infty a_j^r = 0 \qquad (r = 1, 2, 3, \ldots) \tag{3}$$

would imply that all of the a_j vanish. He conjectured that this is not the case. We confirm this by generating an explicit example of an infinite sequence, starting with the number 1, and such that it satisfies all of the equations (3). The convergence concept used in (3) is that of ordinary (rather than absolute) convergence (i.e., convergence of partial sums).

First some notation. If s', s'', etc., are finite sequences of complex numbers (finite in number) we denote by (s', s'', etc.) the finite sequence obtained

Received by the editors June 27, 1989 and, in revised form, August 28, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 40A05.

by writing the given sequences one after another. It proves convenient to use a further abbreviated notation for the case when some of the given sequences are identical, especially when they occur many times. For instance, we write $(s', s'')_{1,27}$ for the sequnce obtained by writing down one copy of the sequence s' followed by 27 copies of the sequence s''. If s is a sequence and λ a complex number, λs denotes the sequence whose terms are those of s multiplied by λ. We use the abbreviation α_n to denote $\exp(i\pi/n)/n$ for $n = 1, 2, \ldots$.

Define sequences s_0, s_1, $s_2, \ldots$ with ever increasing number of terms inductively by the following formulae

$$\begin{aligned} s_0 &= (1), \\ s_1 &= (s_0, \alpha_1 s_0)_{1,1^1}, \\ s_2 &= (s_1, \alpha_2 s_1)_{1,2^2}, \\ s_3 &= (s_2, \alpha_3 s_2)_{1,3^3}, \end{aligned} \tag{4}$$

and so forth. In a more explicit notation,

$$s_1 = (1, -1),$$

and

$$s_2 = (1, -1, i/2, -i/2, i/2, -i/2, i/2, -i/2, i/2, -i/2).$$

s_3 has 280 terms: $(1+27) \times 10 = 280$, and s_4 has 71,960 terms: $(1+256) \times 280 = 71{,}960$. Each of these sequences is an extension of the previous one, obtained from the latter by multiplying it with a (small) complex factor and adjoining (many) copies of this scaled version to the original one. The sum of the terms of s_1 vanishes. Since s_2 consists of scaled versions of s_1 this is still true of s_2, but the complex multiplier was so chosen that the sum of the squares of the terms of s_2 vanishes as well. Continuing, the sum of the terms and the sum of the squares of the terms of s_3 vanishes, for the same reason as before, but again the complex multiplier chosen makes the sum of the cubes vanish as well; and this pattern continues.

Let a_n denote the nth term in the sequence s_j, defined and independent of j for all sufficiently large j. Then the infinite sequence $s = (a_1, a_2, a_3, \ldots)$ satisfies all the equations (4). Indeed, if r is any positive integer, one can view s as a succession of infinitely many copies of s_r scaled with complex factors of ever-decreasing modulus. Since the rth power-sum of s_r vanishes, the partial sums of the rth power-sum of s are in effect partial sums of the rth power-sum of s_r, finite in number, multiplied with factors of ever-decreasing modulus. This shows that the infinite power-sum is convergent and converges to zero, as required.

Acknowledgment

The writer is indebted to two of his colleagues, Grahame Bennett and Scott McCollough, for interesting discussions, and to Professor Freeman J. Dyson of the Institute for Advanced Study for telling him of the problem in a casual conversation.

References

1. W. M. Priestley, private communication.

Department of Mathematics, Indiana University, Bloomington, Indiana, 47405