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LENS SPACES AND DEHN SURGERY

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ABSTRACT. The question of when a lens space arises by Dehn surgery is discussed with a characterization given for satellite knots. The lens space $L(2, 1)$, i.e. real projective 3-space, is shown to be unobtainable by surgery on a symmetric knot.

The problem of when a lens space can be obtained by performing Dehn surgery on a knot in the 3-sphere has been of interest to topologists for some time. It is known that certain lens spaces can arise by Dehn surgery on torus knots (Moser [Mo]), certain pretzel knots (Fintushel-Stern [FS]), and certain nontrivial satellite knots (in fact, certain cables of torus knots, Bailey-Rolfsen [BR]).

In this note we show how some recent developments in 3-manifold theory shed more light on this problem. In Theorem 1 we use recent results of Culler-Gordon-Luecke-Shalen [CGLS], Gabai [Ga], Gordon [Go], and Scharlemann [S] to characterize how a lens space can be obtained by surgery on a (nontrivial) satellite knot. Theorem 1 was originally proven by Wu [Wu]. We present here a somewhat more concise proof, discovered independently, which makes deeper use of the critical theorems of Gabai and Gordon. Similar results have also been obtained by Wang [Wa_1] and Hempel [H]. We then specialize to the question of when real projective 3-space, i.e. the lens space $L(\pm 2, 1)$, can be obtained by Dehn surgery. Using results of Thompson [T] and Wang [Wa_2], we show in Theorem 2 that no surgery on a nontrivial symmetric knot yields this manifold.

Theorem 1. *If nontrivial Dehn surgery on a satellite knot yields a manifold with cyclic fundamental group, then the knot is a cable of a torus knot and the knot and surgery coefficient are as in* [Go, Theorem 7.5 (iii), $k = 2$]. *I.e. the knot is the $(2pq \pm 1, 2)$-cable on a (p, q)-torus knot, the surgery coefficient is $4pq \pm 1$, and the resulting manifold is $L(4pq \pm 1, 4q^2)$.*

Corollary. *If a lens space L is obtained by Dehn surgery on a satellite knot then* $|\pi_1(L)| \geq 23$.

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Proof of Theorem 1. Our notation will mostly follow that of [Go]. Let K be a nontrivial knot in the 3-sphere S^3, let J be a knot nontrivially contained in $S^1 \times D^2$ with winding number $w \geq 0$, and let $J(K)$ be the satellite of K determined by $J \subseteq S^1 \times D^2$. Further let $(J; r)$ be the manifold obtained from $S^1 \times D^2$ by r-surgery on J, $r \neq 1/0$; and let $(J(K); r)$ be the manifold obtained from S^3 by r-surgery on $J(K)$. Since $J(K)$ is not a torus knot, Corollary 1 of [CGLS] implies that $(J(K); r)$ does not have cyclic fundamental group unless r is an integer; we therefore consider only surgery with an integral coefficient m (so that our notation agrees with that of [Go]).

We begin by reviewing what is known when $(J; m)$ has compressible boundary.

(1) By the proof of [S, Corollary 5.2], $w \neq 0$.

(2) The manifold $(J; m)$ is homeomorphic to $V \# M$ where $V \cong S^1 \times D^2$ and M is some closed 3-manifold. Denoting the g.c.d. of w and m by (w, m), we have [by Go, Lemma 3.3] that $H_1(M) \cong \mathbf{Z}_{(w,m)}$, and a meridian of V has slope m/w^2 relative to the original $S^1 \times D^2$. Moreover, by the proof of [Go, Lemma 3.6], w and $m/(w, m)$ are relatively prime. It follows that m/w^2 has denominator at least w, and is therefore not an integer unless $w = 1$.

(3) By [Ga, Theorem 1.1], either $H_1(M) \neq 0$ (i.e. $(w, m) \neq 1$) or $(J; m)$ is homeomorphic to $S^1 \times D^2$ and J is a 0- or 1-bridge braid. If $w = 1$, then we must have the second possibility; but this implies that J is a core of $S^1 \times D^2$. Hence $w \geq 2$.

Now suppose that $(J(K); m)$ has cyclic fundamental group for some integer m. Then $(J; m)$ has compressible boundary and all of the above applies. It follows that $(J(K); m) \cong (K; m/w^2) \# M$. Hence $(K; m/w^2)$ has cyclic fundamental group, and since m/w^2 is not an integer, Corollary 1 of [CGLS] implies that K is a torus knot. Hence the fundamental group of $(K; m/w^2)$ is nontrivial, and so the fundamental group of M must be trivial. By (3) above, this implies that $(J; m)$ is homeomorphic to $S^1 \times D^2$ and J is a 0- or 1-bridge braid. A 0-bridge braid is a cable, so it only remains to eliminate the possibility that J is a 1-bridge braid.

Suppose that J is a 1-bridge braid. In §3 of [Ga] there are associated to J two integers, the braid width $b (1 \leq b \leq w - 2)$ and $t (1 \leq t \leq w - 1)$. By [Ga, Lemma 3.5] and the remarks just before [Ga, Definition 3.3], the surgery coefficient m is equal to $\pm(wt + d)$ depending on the orientation convention, where d is either b or $b + 1$. On the other hand, as K is a torus knot and as $\pi_1(K; m/w^2)$ is cyclic, [Go, Corollary 7.4] implies that m is congruent to $\pm 1 \bmod w^2$. (Note that w and m are relatively prime since $H_1(M) = 0$.) Hence d is congruent to $\pm 1 \bmod w$. Now b is equal to either d or $d - 1$ and $1 \leq b \leq w - 2$, so either $b = 1$ or $b = w - 2$. Replacing J by its mirror image replaces b by $w - b - 1$ (see the remark after [Ga, Lemma 3.4]), so it is enough to consider the case $b = 1$. But this means that J is a (2,1)-cable of a cable (see the remark after [Ga, Examples 3.8]), so that $J(K)$ is a cable of a cable of a torus knot. This contradicts [Go, Theorem 7.5]. □

Theorem 2. *Real projective* 3*-space cannot be obtained by Dehn surgery on a nontrivial symmetric knot* K *in the* 3*-sphere.*

Proof of Theorem 2. Recall that a knot K in S^3 is symmetric if it is invariant under the action on S^3 of some nontrivial finite group G. Without loss of generality we may take G to be a cyclic group $\mathbf{Z}_n$. From Moser [Mo] we conclude that K is not a torus knot. We then reduce to the strongly invertible case by appealing to Wang [Wa_2]. He proves that if K admits an action of a cyclic group $\mathbf{Z}_n$ and is not a torus knot, then if $n > 2$ or if $n = 2$ and the action is fixed point free, no nontrivial surgery on K yields a lens space. He also shows that if $n = 2$ and K is disjoint from the fixed point set, then no nontrivial surgery can yield real projective 3-space. The only symmetry left to consider is a strong inversion.

So suppose that surgery on some nontrivial strongly invertible knot K gives RP^3. As before, we begin by recalling some general facts about surgery on strongly invertible knots. The strong inversion on K extends to an involution on each of the manifolds $(K; r)$ obtained from S^3 by performing r-fold surgery on K. For each $(K; r)$ the quotient under this involution is the 3-sphere and hence each $(K; r)$ double branch covers S^3. Moreover, the branch set of this covering can be obtained by removing a trivial tangle from the unknot (i.e. the image of the axis of the strong inversion) and replacing it in a manner determined by the surgery coefficient r. See, for example, Montesinos [M] or, for a more explicit construction, [B].

As K is not a torus knot, we can apply the cyclic surgery theorem of [CGLS] to conclude that the surgery coefficient r is at a distance 1 from the meridian of K. It follows that the removal and replacement of the trivial tangle corresponding to r is in fact the attachment of a band to the unknot. The core of this band is the image of our original knot K under the quotient map $(K; 1/0) \to S^3$; again see [M] or [B].

By Hodgson and Rubenstein [HR], lens spaces uniquely double branch cover the 3-sphere with branch set the appropriate two-bridge knot or link. For real projective space this two-bridge link is the Hopf link. We conclude that obtaining RP^3 by surgery on K corresponds in the quotient S^3 to obtaining the Hopf link by attaching a band to the unknot.

Now we apply a theorem of Thompson [T, Corollary 3] which states that there is a unique band which creates the unknot from the Hopf link, or, dually, that there is a unique band which creates the Hopf link from the unknot. It follows that we have one of the two pictures in Figure 1, and that our original knot K is unknotted.

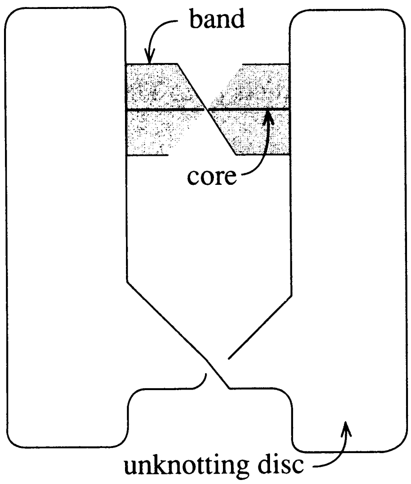

FIGURE 1.

We close with some questions and conjectures.

(1) Is it possible to obtain a lens space L with $|\pi_1(L)| < 5$ by Dehn surgery on a nontrivial knot? Conjecture: No.

(2) It is possible to obtain a lens space L with $|\pi_1(L)| < 18$ by Dehn surgery on a nontorus knot? Conjecture: No.

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