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PHRAGMÉN-LINDELÖF THEOREM FOR THE MINIMAL SURFACE EQUATION

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ABSTRACT. It is proved that if u satisfies the minimal surface equation in an unbounded domain Ω which is properly contained in a half plane, then the growth property of u depends on Ω and the boundary value of u only.

1. Introduction. The purpose of this paper is to establish a Phragmén-Lindelöf Theorem for the minimal surface equation in $\mathbf{R}^2$. We prove that if u satisfies the minimal surface equation in an unbounded domain Ω, which is properly contained in a half plane, then the growth property of u depends on Ω and $u\,|_{\partial\Omega}$ only, without requiring any other restriction for u. In this respect, the Phragmén-Lindelöf Theorem for the minimal surface equation is better than that of the linear equations. We remark that if u satisfies the Laplace equation with vanishing boundary value in a sector domain Ω_α with angle $0 < \alpha < \pi$, then we cannot conclude that $u \equiv 0$. If we want to establish a maximum principle on Ω_α, we must impose some restriction on the growth of u at infinity [**12**, Chapter 2, §9]. In fact, for the Laplace equation in an unbounded domain Ω, the growth property of u cannot be determined completely by $u\,|_{\partial\Omega}$ alone. There are various types of restriction on the growth of u at infinity in Phragmén-Lindelöf Theorems for general equations. For these results, the reader is referred to [**1**, **2**, **3**, **5**, **6**, **7**, **10**, **12**, **13**].

On the other hand, if u satisfies the minimal surface equation with vanishing boundary value in Ω_α, then $u \equiv 0$ [**8**, p. 256]. So it is natural to conjecture that if u satisfies the minimal surface equation in Ω_α, then the growth property of u depends on $u\,|_{\partial\Omega}$ only.

We prove that if $\Omega \subset \{(x,y)\,|\,y > 0,\ -f(y) < x < f(y)\}$ where $f \in C^0[0,\infty)$, $f \geq 0, f(t)$ increases as t increases, then the previous conjecture is true (Main Theorem). Our estimates depend on the shape of Ω, and the behavior of $u\,|_{\partial\Omega}$ does not enter the picture explicitly.

We emphasize that in such a domain Ω, the solutions for the minimal surface equation with vanishing boundary value may not be unique, but the Main Theorem is still true. (e.g. in $\Omega = \{(x,y)\,|\,-\sqrt{(\cosh y)^2 - 1} < x < \sqrt{(\cosh y)^2 - 1}, y > 0\}$, we have two solutions with vanishing boundary value $u \equiv 0$ and $u = \sqrt{(\cosh y)^2 - x^2 - 1}$). Since in a half plane, the bound of the solutions with vanishing boundary value does not even exist, the domain must be properly contained in a half plane.

Some examples and remarks can be found in §3.

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2. Main theorem. Throughout the whole article, Ω will be a connected domain (bounded or unbounded) in $\mathbf{R}^2$, and for any function $u \in C^1(\Omega)$, Tu will denote the vector $Du/\sqrt{1+|Du|^2}$ where Du is the gradient vector of u.

For latter arguments of comparison, a particular function will be utilized. Its definition and basic properties are stated in the following lemma whose proof follows by direct computation:

LEMMA 2.1. *Let* $v = (1/c)\cdot(x^2 - x_0^2)/(y-y_0) + a(y-y_0) + b$ *in* $\Omega = (-x_0, x_0) \times (-\infty, y_0)$ *where* c, a, x_0 *are positive constants and* b, y_0 *are constants. Then we have*

(i) $\lim_{y\to y_0} v = +\infty$,

(ii) $\operatorname{div} Tv \le 0$ *in* Ω *for* $0 < c \le 4a$,

(iii) $v(x,y) \ge v(x_0,y) = v(-x_0,y)$ *for every* $y < y_0$ *and* $x \in (-x_0, x_0)$.

Now, we have

THEOREM 2.2. *Let* $\Omega \subset (-x_0,x_0)\times(0,y_0)$ *and let* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. *Suppose that*

(i) $\operatorname{div} Tu \ge 0$ *in* Ω,

(ii) $u|_{\partial\Omega\cap[-x_0,x_0]\times\{y\}} \le ay+b$, *where* a, x_0, y_0 *are positive constants,* b *is a constant,* $0 \le y < y_0$.

Then if $y_0 - x_0/(2a) > 0$, *we have* $u(x,y) \le ay_0 + b$ *for every* $(x,y) \in (-x_0,x_0)\times(0, y_0 - x_0/(2a)) \cap \Omega$.

PROOF. Let $v = (1/(4a))\cdot(x^2-x_0^2)/(y-y_0) + ay + b$. Since $u|_{\partial\Omega} \le v|_{\partial\Omega}$ and div Tu $\ge 0 \ge$ div Tv in Ω, we have $u \le v$ in Ω. Noting that $v(x,y) \le ay_0 + b$ for every $(x,y) \in (-x_0,x_0)\times(0,y_0-x_0/(2a)) \cap \Omega$, one immediately completes the proof.

REMARK. Since

$$v|_{(-x_0,x_0)\times\{y_0\}} = +\infty,$$

there are no restrictions for $u|_{\partial\Omega\cap[-x_0,x_0]\times\{y_0\}}$. The Main Theorem follows from this idea:

MAIN THEOREM. *Let* $\Omega \subset \Omega_1 = \{(x,y) \mid y > 0,\ -f(y) < x < f(y)\}$ *where* $f, g \in C^0[0,\infty)$, $f, g \ge 0$, $g(0) = 0$, $f(t)$, $g(t)/t$ *increase as* t *increases, and let* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. *Suppose that*

(i) $\operatorname{div} Tu \ge 0$ *in* Ω,

(ii) $u|_{\partial\Omega\cap[-f(y),f(y)]\times\{y\}} \le g(y)$ *where* $y \in [0,\infty)$,

(iii) $0 < \beta(y) \equiv f(y)/(2g(y)) < 1$ *for some* $y_1 > 0$ *and every* $y > y_1$,

(iv) $\beta(y)$ *decreases in* $[y_1, \infty)$.

Then $u(x,y) \le g(y/(1-\beta(y)))$ *for every* $(x,y) \in \Omega$ *where* $y > y_1$.

PROOF. Fixing $y_2 > y_1 > 0$ and noting that for every $0 \le y < y_2$, we have $u|_{\partial\Omega\cap[-f(y),f(y)]\times\{y\}} \le g(y) \le (g(y_2)/y_2)y$. By Theorem 2.2, we have $u(x,y) \le g(y_2)$ where $0 \le y \le y_2 - \beta(y_2)y_2$. Now for every $y_3 > y_1$, we have $y_3/(1-\beta(y_3)) > y_3 > y_1$ and $y_3 = (1-\beta(y_3))y_3/(1-\beta(y_3)) \le y_3/(1-\beta(y_3)) - \beta(y_3/(1-\beta(y_3)))y_3/(1-\beta(y_3))$, and we obtain $u(x,y_3) < g(y_3/(1-\beta(y_3)))$.

REMARK. In the above theorem, f controls the increasing rapidity of width of the defined domain Ω and g controls the increasing rapidity of boundary value. It is natural that we require $g(y)/y$ to increase as y increase.

3. Examples and remarks. Now, we give some examples for the applications of the Main Theorem.

EXAMPLE 3.1. Let $\Omega = \{-y < x < y \,|\, y > 0\}$ and let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Suppose that $\operatorname{div} Tu \geq 0$ in Ω and $u(\pm y, y) \leq y^m$ for every $y \geq 0$, where $m > 1$ is a positive constant. Then for $y \geq 1$, we have $\beta(y) = (2y^{m-1})^{-1} \leq 1$, and $u(x,y) \leq (y/(1-(2y^{m-1})^{-1})^m = y^m + (m/2)y + O(y^{2-m})$ as $y \to \infty$.

REMARK 3.2. As [8, p. 256], with the help of the general maximum principle and a suitable solution of Dirichlet's problem [8, II, 7.2] the following maximum principle can be proved: Let $\Omega = \{-y < x < y \,|\, y > 0\}$ (or a sector domain Ω_α with angle $0 < \alpha < \pi$), and let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a solution of the minimal surface equation in Ω. Then

(i) if $u(\pm y, y) \leq ay + c$, we have $u(x,y) \leq ay + c$,

(ii) if $u(\pm y, y) \geq ay + c$, we have $u(x,y) \geq ay + c$,

where a and c are constants. Further,

(iii) if $u(\pm y, y) \leq g(y)$ where $g(y) \in C^0[0,\infty) \cap C^2(0,\infty)$ and $g''(y) \leq 0$ for every $y > 0$, we have $u(x,y) \leq g(y)$,

(iv) if $u(\pm y, y) \geq h(y)$ where $h(y) \in C^0[0,\infty) \cap C^2(0,\infty)$ and $h''(y) \geq 0$ for every $y > 0$, we have $u(x,y) \geq h(y)$.

(In (iii), we have $u(\pm y, y) \leq g(y_0) + g'(y_0)(y - y_0)$ for every fixing $y_0 > 0$, by (ii) $u(x,y) \leq g(y_0) + g'(y_0)(y - y_0)$ and $u(x, y_0) \leq g(y_0)$.)

In Example 3.1, if $u(\pm y, y) \leq y^m$ where $0 < m \leq 1$ is a constant, by (iii) we have $u(x,y) \leq y^m$, but the Main Theorem will not give the optimal result.

In Example 3.1, if $u(\pm y, y) \leq y^m$ where $1 < m$ is a constant, by (iv) $u(x,y)$ may be greater than y^m, it seems that $u(x,y) = y^m + (m/2)y + O(y^{2-m})$ is a good estimate.

Now, we give two examples to explain how to estimate the solutions with vanishing boundary value.

EXAMPLE 3.3. Let $\Omega = \{-\sinh y < x < \sinh y \,|\, y > 0\}$ and let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Suppose that $\operatorname{div} Tu \geq 0$ in Ω and $u\,|_{\partial\Omega} \leq 0$. Consider $g(y) = c \sinh y$, where $c > \frac{1}{2}$ is a constant to be specified. Then $\beta(y) = 1/2c < 1$ and $u(x,y) \leq c \sinh(y/(1-(2c)^{-1})) \leq (\frac{c}{2}) e^{y/(1-(2c)^{-1})}$. Let $c = y/2 > 1$, then

$$\begin{aligned} u(x,y) &\leq \frac{y}{4} \cdot e^{y/(1-y^{-1})} = \frac{y}{4} e^{y(1+\frac{1}{y}+\frac{1}{y^2}+\cdots)} \\ &\leq (e + O(1/y)) \frac{y}{4} e^y \quad \text{as } y \to \infty. \end{aligned}$$

EXAMPLE 3.4. In the case of catenoid, $u = \sqrt{(\cosh y)^2 - x^2}$ and the defined domain $\Omega = \{-\cosh y < x < \cosh y \,|\, y > 0\}$. Since $\Omega \subset \{-\sinh(y+1) < x < \sinh(y+1) \,|\, y > -1\}$ and $u\,|_{\partial\Omega} \leq c \sinh(y+1)$, where $c > 1$. By Example 3.3, we have $u(x,y) = O((y+1) \cdot e^{y+1}) = O(ye^y)$.

REMARK 3.5. In Example 3.4, the actual growth behavior is $\cosh y = O(e^y)$, and our estimate $O(ye^y)$ are not optimal. In fact, the slower the growth of u on $\partial\Omega$, the poorer the estimates of u in Ω.

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