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FUNCTION SPACES AND LOCAL CHARACTERS OF TOPOLOGICAL SPACES

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ABSTRACT. We write $V \simeq W$ to mean that the two linear topological spaces V and W are linearly homeomorphic. In this paper we prove: (1) There are compact spaces X, Y for which $C_p(X) \simeq C_p(Y)$ and $\chi(X) \neq \chi(Y)$ are satisfied. (2) For each infinite cardinal κ, there are spaces X, Y for which $C_p(X) \simeq C_p(Y)$, $\chi(X) = \omega$ and $\psi(Y) = \kappa$. (3) For each infinite cardinal κ, there are spaces X, Y for which $C_p(X) \simeq C_p(Y)$, $\pi_\chi(X) = \omega$ and $\pi_\chi(Y) = \kappa$.

All topological spaces considered here are Tychonoff spaces. For an arbitrary topological space X, the space $C_p(X)$ ($C_s(X)$ in [**2**], $C_\pi(X)$ in [**4**]) is the set of all real-valued continuous functions on X with the topology of pointwise convergence. The space $C_p(X)$ is a linear topological space under the algebraic operations being defined pointwise. We write $V \simeq W$ if two linear topological spaces V and W are linearly homeomorphic. Topological spaces X, Y are said to be l-equivalent [**1**] if $C_p(X) \simeq C_p(Y)$. The weak dual of $C_p(X)$ is denoted by $L_p(X)$. It is well known that $C_p(X) \simeq C_p(Y)$ if and only if $L_p(X) \simeq L_p(Y)$ [**5, 6**]. Further, each topological space X is embedded in $L_p(X)$ as a Hamel basis. Hence, if compact spaces X and Y are l-equivalent, then $\chi(Y) \leq |L_p(X)| \leq |R^\omega \times X^\omega| \leq 2^\omega \cdot 2^{\chi(X)}$.

In this note we will establish the following results.

EXAMPLE 1. There are l-equivalent compact spaces X and Y for which $\chi(X) = \omega$ and $\chi(Y) = 2^\omega$ are satisfied.

EXAMPLE 2. For each infinite cardinal κ, there are l-equivalent countably compact normal spaces X and Y for which $\chi(X) = \psi(X) = \omega$ and $\chi(Y) = \psi(Y) = \kappa$ are satisfied.

EXAMPLE 3. For each infinite cardinal κ, there are l-equivalent topological spaces X and Y for which $\pi_\chi(X) = \omega$ and $\pi_\chi(Y) = \kappa$ are satisfied.

Let Y be a subspace of a topological space X. We define $C_p(X; Y) = \{f \in C_p(X) \colon f(Y) = \{0\}\}$. Pavlovskii [**5**] showed the following lemma.

LEMMA 1. *If Y is a retract of a space X, then $C_p(X) \simeq C_p(Y) \times C_p(X; Y)$.*

For a closed subset Y of a normal space X, let X/Y be the quotient space of X obtained by collapsing Y to one point $*$. In this case, the following lemma is a tedious exercise.

LEMMA 2. $C_p(X; Y) \simeq C_p(X/Y; *)$.

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1. Example 1. Let us recall the Alexandroff double circle $C_1 \cup C_2$ [**3**]. The underlying set of this compact space is the union of two concentric circles $C_i = \{(x,y) \in R^2 : x^2 + y^2 = i\}$, where $i = 1, 2$. The subspace C_1 is the unit circle S^1 with its natural topology and the subspace C_2 is a discrete space of cardinality 2^ω. Obviously C_1 is a retract of $C_1 \cup C_2$.

CLAIM 1. Let $A(2^\omega)$ be the one-point compactification of a discrete space of cardinality 2^ω. Then $C_p(S^1 \oplus A(2^\omega)) \simeq C_p(C_1 \cup C_2)$.

PROOF. Let a be a point of C_2. Then

$$\begin{aligned} C_p(C_1 \cup C_2) &\simeq C_p(C_1 \cup (C_2 - \{a\})) \times C_p(\{a\}) \\ &\simeq C_p(C_1) \times C_p(C_1 \cup (C_2 - \{a\}); C_1) \times R \\ &\simeq C_p(C_1) \times C_p((C_1 \cup (C_2 - \{a\}))/C_1; *) \times R \\ &\simeq C_p(C_1) \times C_p((C_1 \cup (C_2 - \{a\}))/C_1) \\ &\simeq C_p(S^1) \times C_p(A(2^\omega)) \simeq C_p(S^1 \oplus A(2^\omega)). \end{aligned}$$

Since $\chi(C_1 \cup C_2) = \omega$ and $\chi(S^1 \oplus A(2^\omega)) = 2^\omega$, it follows that the desired example is obtained.

2. Example 2. For each infinite cardinal κ, let $W(\kappa+1)$ be the space of all ordinals less than $\kappa + 1$ with the usual interval topology. Let $V(\kappa+1)$ be the subspace of $W(\kappa+1)$ consisting of all ordinals whose cofinality is less than ω_1.

LEMMA 3. *Let $F(\kappa+1)$ be the subspace of all limit points in $V(\kappa+1)$. Then $F(\kappa+1)$ is a retract of $V(\kappa+1)$.*

PROOF. For each α in $V(\kappa+1)$, let

$$r(\alpha) = \min\{\beta : \alpha \leq \beta,\ \beta \in F(\kappa+1)\}.$$

Then $r : V(\kappa+1) \to F(\kappa+1)$ is a retraction.

CLAIM 2. Let $A(\kappa)$ be the one-point compactification of a discrete space of cardinality κ. Then $C_p(A(\kappa) \oplus F(\kappa+1)) \simeq C_p(V(\kappa+1))$.

PROOF.

$$\begin{aligned} C_p(V(\kappa+1)) &\simeq R \times C_p(V(\kappa+1)) \\ &\simeq R \times C_p(V(\kappa+1); F(\kappa+1)) \times C_p(F(\kappa+1)) \\ &\simeq R \times C_p(V(\kappa+1)/F(\kappa+1); *) \times C_p(F(\kappa+1)) \\ &\simeq C_p(V(\kappa+1)/F(\kappa+1)) \times C_p(F(\kappa+1)) \\ &\simeq C_p((V(\kappa+1)/F(\kappa+1)) \oplus F(\kappa+1)). \end{aligned}$$

Since $V(\kappa+1)$ is countably compact, the quotient space $V(\kappa+1)/F(\kappa+1)$ must be homeomorphic with the space $A(\kappa)$. Hence

$$C_p(V(\kappa+1)) \simeq C_p(A(\kappa) \oplus F(\kappa+1)).$$

It is obvious that

$$\chi(V(\kappa+1)) = \psi(V(\kappa+1)) = \omega,$$
$$\chi(A(\kappa) \oplus F(\kappa+1)) = \psi(A(\kappa) \oplus F(\kappa+1)) = \kappa.$$

3. Example 3. First consider regular cardinals. For each infinite regular cardinal λ, let $U(\lambda+1)$ be the subspace of $W(\lambda+1)$ consisting of all ordinals α such that $\mathrm{cf}(\alpha) \leq \omega$ or $\alpha = \lambda$.

CLAIM 3. $C_p(U(\lambda+1)) \simeq C_p(U(\lambda+1)/\{\omega,\lambda\})$.

PROOF.

$$\begin{aligned} C_p(U(\lambda+1)) &\simeq C_p(U(\lambda+1);\{\omega,\lambda\}) \times C_p(\{\omega,\lambda\}) \\ &\simeq C_p(U(\lambda+1)/\{\omega,\lambda\};*) \times R^2 \\ &\simeq C_p(U(\lambda+1)/\{\omega,\lambda\}) \times R \\ &\simeq C_p(U(\lambda+1)/\{\omega,\lambda\}). \end{aligned}$$

Note that $\pi_\chi(U(\lambda+1)) = \lambda$, $\pi_\chi(U(\lambda+1)/\{\omega,\lambda\}) = \omega$. Now consider a singular cardinal κ and let $\{\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ be a transfinite sequence of regular cardinals such that

$$\sup\{\lambda_\alpha : \alpha < \mathrm{cf}(\kappa)\} = \kappa.$$

Let $T(\kappa)$ be the topological sum of $\{U(\lambda_\alpha+1) : \alpha < \mathrm{cf}(\kappa)\}$, and let $S(\kappa)$ be the topological sum of $\{U(\lambda_\alpha+1)/\{\omega,\lambda_\alpha\} : \alpha < \mathrm{cf}(\kappa)\}$. Then, by Theorem 2.6 in [**4**],

$$C_p(T(\kappa)) \simeq C_p(S(\kappa)).$$

Further, it is obvious that $\pi_\chi(T(\kappa)) = \kappa$, $\pi_\chi(S(\kappa)) = \omega$.

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