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THE CONVERGENCE OF MOMENTS IN THE CENTRAL LIMIT THEOREM FOR ρ-MIXING SEQUENCES OF RANDOM VARIABLES

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ABSTRACT. In this paper we establish maximal inequalities for ρ-mixing sequences and, as a consequence, we obtain the convergence of the expected value of functions of partial sums to the corresponding ones of the normal distribution.

1. Introduction and notations. Let $\{X_k\}_{k\in Z}$ be a strictly stationary sequence of random variables on a probability space (Ω, K, P). Denote by F_n^m the σ-algebra generated by the variables $(X_n, \ldots, X_m)$, $\|\cdot\|_2$ the norm in L_2, $S_0 = 0$, $S_n := \sum_{i=1}^n X_i$, $\sigma_n^2 = \operatorname{var} S_n$, $N(0,1)$ the standard normal random variable, and $[x]$ the greatest integer $\leq x$.

Define

$$\rho(n) = \sup |EXY - EXEY|/\|X\|_2\|Y\|_2, \qquad X \in L_2(F_{-\infty}^0),\ Y \in L_2(F_n^\infty),$$

and

$$\lambda(n) = \sup |P(AB) - P(A)P(B)|/(P(A)P(B))^{1/2}, \qquad A \in F_{-\infty}^0,\ B \in F_n^\infty,\ P(A)P(B) \neq 0.$$

It is easy to see that $\rho(n)$ is a nondecreasing sequence and

$$\lambda(n) \leq \rho(n). \tag{1.0}$$

On the other hand by Theorem 1.1 of Bradley-Bryc [**4**],

$$\rho(n) \leq 3000\lambda(n)(1 - \log \lambda(n)).$$

The sequence is said to be ρ-mixing (Kolmogorov, Rozanov (1960)) if $\rho(n) \to 0$ (equivalently $\lambda(n) \to 0$) as $n \to \infty$.

The following theorem is well known (Ibragimov [**9**]).

THEOREM 1.0. *Suppose that the strictly stationary ρ-mixing sequence $\{X_k\}$ satisfies $E|X_0|^{2+\delta} < \infty$ for some $\delta > 0$ and, in addition,*

$$EX_0 = 0 \quad \text{and} \quad \sigma_n \to \infty \quad \text{as } n \to \infty. \tag{1.1}$$

Then $\{S_n/\sigma_n\}$ converges in distribution as $n \to \infty$ to $N(0,1)$. Moreover if for each $t \in [0,1]$ we put $W_n(t) = S_{[nt]}/\sigma_n$, then, under the same conditions, $\{W_n\}$ is weakly

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convergent to the standard Brownian motion process on $[0,1]$, *denoted by* W *in the sequel.*

The convergence of moments in the central limit theorem was established for independent sequences by Bernstein [**2**] and extended to martingales by Hall [**7**] and to strong and uniform mixing sequences by Yokoyoma [**16**]. In Peligrad [**15**, Corollary 2.2] the convergence of the absolute moments of order $2+\delta$ with $0<\delta<1$, for ρ-mixing sequences of random variables was established. The proofs of all these results are based on a very tedious computation of bounds for $E|S_n|^p$, where $p>2$, in terms of σ_n. The same kind of technique might be generalized in order to extend Peligrad's result for $\delta>1$. But this kind of technique is not helpful if we are interested in the convergence of more general quantities such as $Eq(|S_n|/\sigma_n)$ or $Eq(|\max_{1\le i\le n} S_i|/\sigma_n)$ with q a function belonging to a certain class of functions. In order to approach such a problem we shall first establish maximal inequalities for ρ-mixing sequences. Basing our proof on these inequalities we shall obtain the following result.

Denote by Q the class of functions $q\colon [0,\infty)\to[0,\infty)$ satisfying the conditions:
(A_1) q is continuous and $q(0)=0$.
(A_2) $q(x)/x^{2+\delta}$ is nondecreasing for some $\delta>0$ (for all x sufficiently large).
(A_3) $q(2x)\le cq(x)$ with some constant $c>0$ (for all x sufficiently large).

THEOREM 1.1. *Suppose* $\{X_k\}$ *is a strictly stationary sequence satisfying* (1.1) *and* $Eq(|X_0|)<\infty$ *for some* $q\in Q$. *Then*

$$Eq(|S_n|/\sigma_n) \text{ converges to } Eq(|N(0,1)|) \quad \text{as } n\to\infty, \tag{1.2}$$

$$Eq\left(\left|\max_{1\le i\le n} S_i\right|/\sigma_n\right) \text{ converges to } Eq(M) \quad \text{as } n\to\infty, \tag{1.3}$$

where $P(M\le\alpha)=\sqrt{2/\pi}\int_0^\alpha \exp(-u^2/2)\,du$, $\alpha\ge 0$, *and*

$$Eq\left(\max_{1\le i\le n}|S_i|/\sigma_n\right) \text{ converges to } Eq(M^*) \quad \text{as } n\to\infty, \tag{1.4}$$

where

$$P(M^*<b)=\sum_{k=-\infty}^{\infty}(-1)^k P((2k-1)b<N(0,1)<(2k+1)b), \qquad b\ge 0.$$

In the context of this theorem the conditions imposed to q are not particularly restrictive. Condition A_1 is imposed only in order to simplify some computations, in general only the behavior of q at infinity is important for Theorem (1.1).

Condition A_2 is "natural" in our context. Bradley [**3**, Corollary 3] proved that the existence of moments of order $2+\delta$, with some $\delta>0$, is a "minimal" moment condition that implies Theorem 1.0 without any other additional assumptions on the "mixing rate" (i.e., the speed at which $\{\rho(n)\}$ approaches 0).

Condition A_3, which is known as the Δ_2 condition, states that $q(x)$ is of finite upper type. If, for instance, $q(x)/x^l$ is decreasing for some $l>2$, condition A_3 is satisfied with $c=2^l$. (For more details about "upper type" see for instance Gustavsson and Peetre [**6**] and for further properties connected with A_3 see Bingham and Goldie [**1**].)

A large variety of functions belong to Q. Functions such as

$$x^{2+\delta}[\log(2+x)]^{\alpha} \quad \text{with } x \geq 0,\ \delta > 0, \text{ and } \alpha \text{ real},$$

or

$$x^{2+\delta}[\exp(\log(2+x))^{\alpha}]^{\beta} \quad \text{with } x \geq 0,\ \delta > 0, \text{ and } \alpha, \beta \text{ reals}, \alpha < 1,$$

satisfy A1–A3.

The main tool in proving Theorem 1.1 is Proposition 2.3. This also provides a new proof of Theorem 1.0.

2. Auxiliary results and proof of Theorem 1.1. We shall start by establishing the following lemma which is an extension of Ottaviani and Hoffman-Jørgensen's inequalities from the independent case to ρ-mixing sequences. The extension of these inequalities to the ϕ-mixing case can be found in Peligrad [**14**] (see also Iosifescu-Theodorescu [**11**, Lemma 1.1.6]).

LEMMA 2.1. *Suppose $\{X_k\}$ is a strictly stationary sequence and suppose that for some integers r, n, with $n/r \geq 2$ and $a > 0$, we have*

$$2 \max_{2r \leq i \leq n} P(|S_i| > a) + [2n/r]^{1/2}\rho(r) \leq b < 1. \tag{2.1}$$

Then, for every $x \geq 5a$ we have

$$\begin{aligned} &P\left(\max_{1\leq i\leq n}|S_i| > x\right) \\ &\quad \leq 2(1-b)^{-1}\left[\max_{2r\leq i\leq n} P(|S_i| > x/5) + [n/r]P\left(\max_{1\leq i\leq 2r}|S_i| > x/5\right)\right] \end{aligned} \tag{2.2}$$

and

$$\max_{2r\leq i\leq n} P(|S_i| > x) \leq bP\left(\max_{1\leq i\leq n}|S_i| > x/5\right) + 2[n/r]P\left(\max_{1\leq i\leq 2r}|S_i| > x/5\right). \tag{2.3}$$

PROOF. First some notations:

$$M_n = \max_{1\leq i\leq n}|S_i|, \qquad R_n(x) = P\left(\max_{1\leq i\leq n}|S_i| > x\right),$$

$$Q_n(x) = P(|S_n| > x), \qquad N_{r,n}(x) = \max_{r\leq i\leq n} P(|S_i| > x),$$

$$E_i(x) = (M_{i-1} < x \leq |S_i|).$$

Although the proof of (2.2) is similar to that of Remark 1 in Oodaira and Yoshihara [**12**], or of Lemma (3.8) in Peligrad [**13**], we give it here because of some differences that occur. Let $l = [n/r]$. It is easy to see that

$$\begin{aligned} P(M_n > x) \leq{}& P(|S_n| > x/5) + \sum_{i=0}^{l-2} P\left(\bigcup_{j=1}^{r} E_{ir+j}(x), |S_n - S_{(i+2)r}| > 2x/5\right) \\ &+ \sum_{i=0}^{l-2} P\left(\bigcup_{j=1}^{r}(E_{ir+j}(x), |S_{(i+2)r} - S_{ir+j}| > 2x/5)\right) \\ &+ \sum_{i=(l-1)r+1}^{n} P(E_i(x), |S_n - S_i| > 4x/5). \end{aligned}$$

By (1.0),

$$\begin{aligned}R_n(x) \le Q_n(x/5) &+ \sum_{i=0}^{l-2} P\left(\bigcup_{j=1}^{r} E_{ir+j}(x)\right) P(|S_n - S_{(i+2)r}| > 2x/5)\\ &+ \rho(r)\sum_{i=0}^{l-2} P^{1/2}\left(\bigcup_{j=1}^{r} E_{ir+j}(x)\right) P^{1/2}(|S_n - S_{(i+2)r}| > 2x/5)\\ &+ \sum_{i=0}^{l-2} P\left(\max_{1\le j\le r} |S_{(i+2)r} - S_{ir+j}| > 2x/5\right)\\ &+ P\left(\max_{(l-1)r+1\le i\le n} |S_n - S_i| > 4x/5\right).\end{aligned}$$

By Cauchy-Schwarz' inequality, stationarity, and a simple computation, we get

$$\begin{aligned}R_n(x) \le Q_n(x/5) &+ 2N_{2r,n}(x/5)R_n(x) + 2^{1/2}\rho(r)(l-1)^{1/2}R_n^{1/2}(x)N_{2r,n}^{1/2}(x/5)\\ &+ 2(l-1)R_{2r}(x/5) + 2R_{2r}(2x/5).\end{aligned}$$

By the obvious inequality $ab \le a^2 + b^2/4$ for every real a and b and by (2.1) we get

$$(1-b)R_n(x) \le \tfrac{5}{4}N_{2r,n}(x/5) + 2lR_{2r}(x/5).$$

This proves (2.2). In order to prove (2.3) let m be an integer such that $2r \le m \le n$. Let $p = [m/r]$. We have successively

$$\begin{aligned}Q_m(x) &= P(|S_m| > x, M_m > x/5)\\ &\le \sum_{i=0}^{p-2} P\left(\bigcup_{j=1}^{r} E_{ir+j}(x/5), |S_m - S_{(i+2)r}| > 2x/5\right)\\ &\quad + \sum_{i=0}^{p-2} P\left(\bigcup_{j=1}^{r} (E_{ir+j}(x/5), |S_{(i+2)r} - S_{ir+j-1}| > 2x/5)\right)\\ &\quad + \sum_{i=(p-1)r+1}^{m} P(E_i(x/5), |S_m - S_{i-1}| > 4x/5).\end{aligned}$$

Whence, by (1.0), Cauchy-Schwarz' inequality, and stationarity, we obtain

$$\begin{aligned}Q_m(x) &\le 2R_m(x/5)N_{2r,m}(x/5) + [2m/r]^{1/2}\rho(r)R_m^{1/2}(x/5)N_{2r,m}^{1/2}(x/5)\\ &\quad + (p-1)P\left(\max_{0\le j\le r} |S_{2r} - S_j| > 2x/5\right) + 2R_{2r}(x/5)\\ &\le bR_n(x/5) + 2lR_{2r}(x/5).\end{aligned}$$

Now (2.3) follows by taking the maximum for $m, 2r \le m \le n$.

PROPOSITION 2.2. *Suppose $\{X_k\}$ is a strictly stationary sequence. Suppose that for some integers n and r with $[n/r] \ge 2$ and $a > 0$ (2.1) holds and, in addition,*

$$d^{-1} := 1 - 2b(1-b)^{-1}c^5 > 0. \tag{2.4}$$

(Here b is defined by (2.1), *and c by* A_3.*) Then*

$$Eq\left(\max_{1\le i\le n}|S_i|\right)\le dc^5\left[q(a)+6(1-b)^{-1}[n/r]Eq\left(\max_{1\le i\le 2r}|S_i|\right)\right]. \tag{2.5}$$

PROOF. By combining (2.2) and (2.3) we get, for $x > 25a$

$$R_n(x)\le 2(1-b)^{-1}[bR_n(x/25)+3[n/r]R_{2r}(x/25)]. \tag{2.6}$$

According to Proposition 2.7 of Hoffmann-Jørgensen [8], because q is continuous and $q(0)=0$ we have

$$Eq(M_n)=\int_0^\infty R_n(t)\,dq(t). \tag{2.7}$$

So, by (2.6) and (2.7) we have

$$Eq(M_n)\le q(25a)+2(1-b)^{-1}[bEq(25M_n)+3[n/r]Eq(25M_{2r})].$$

(2.5) follows now by condition A_3.

PROPOSITION 2.3. *Under the conditions of Theorem* 1.1, *there is a constant* $K>0$ *such that*

$$Eq\left(\max_{1\le i\le n}|S_i|/\sigma_n\right)\le K \quad \textit{for every } n\ge 1.$$

PROOF. The proof follows by induction on n. First some remarks:

(i) Let us mention first that we can find a sequence $r_n = o(n)$, such that $(n/r_n)\rho(r_n)\to 0$ as $n\to\infty$. To see this we take first $k_n=o(n)$. Then we can find a sequence $j_n\to\infty$, $j_n=o(n)$ such that $j_n\rho(k_n)\to 0$ as $n\to\infty$. We now take $v_n=[n/j_n]$ and $r_n=\max(v_n,k_n)$.

(ii) On the other hand, it is known (Ibragimov [9, Theorem 2.1]) that $\sigma_n^2=nh(n)$, where $h(x)$ is a slowly varying function on R^+. Using Karamata representation for slowly varying functions (see Ibragimov-Linnik [10, Appendix 1, p. 394]) it is easy to see that if $\delta>0$, $0<\varepsilon<\delta/2$, and $r=r_n=o(n)$ as $n\to\infty$ we have

$$(n/2r)^{1+\varepsilon}=o(\sigma_n/\sigma_{2r})^{2+\delta}\quad\text{as } n\to\infty. \tag{2.8}$$

Now let η be a real number $0<\eta<1/(1+2c^5)$. Denote $b^*:=2/a^2+\eta$ and $d^{*-1}:=1-2b^*(1-b^*)^{-1}c^5$. Choose a sufficiently large such that

$$b^*<1\quad\text{and}\quad d^{*-1}>0. \tag{2.9}$$

By remarks (i) and (ii), it is possible to choose an integer N such that for every $n>N$ we can find $r=r_n$ such that

$$\begin{aligned}
&\text{(a)}\quad [2n/r]^{1/2}\rho(r)<\eta,\\
&\text{(b)}\quad \max_{2r\le i\le n}P(|S_i|>a\sigma_n)\le 1/a^2,\\
&\text{(c)}\quad n/2r\le(2r/n)^{\varepsilon}(\sigma_n/\sigma_{2r})^{2+\delta},\\
&\text{(d)}\quad 12d^*(1-b^*)^{-1}c^5(2r/n)^{\varepsilon}<2^{-1}.
\end{aligned} \tag{2.10}$$

Now choose a constant K such that

$$\begin{aligned}
&\text{(a)}\quad Eq(M_k/\sigma_k)\le K\quad\text{for every } k\le N,\\
&\text{(b)}\quad d^*c^5\cdot q(a)\cdot K^{-1}+2^{-1}<1.
\end{aligned} \tag{2.11}$$

We shall prove that (2.11) can be extended for every $n > N$ with the same constant K.

Let us assume as an induction hypothesis that (2.11) holds for all k, $N \le k \le n-1$, and let us prove (2.11) for $k = n$. We apply Proposition 2.2 to the sequence $\{X_i/\sigma_n\}_i$. Using (2.10)(a), (b) and (2.9), because $n > N$, we have

$$Eq(M_n/\sigma_n) \le d^* c^5 q(a) + 6d^*(1-b^*)^{-1}c^5[n/r]Eq(M_{2r}/\sigma_n)$$

whence by (2.10)(c), (d) and the fact that $q(x)/x^{2+\delta}$ is an increasing function we get

$$Eq(M_n/\sigma_n) \le d^* c^5 q(a) + 2^{-1}Eq(M_{2r}/\sigma_{2r}).$$

Now by the induction hypothesis and (2.11)(b) we have the desired result.

PROOF OF THEOREM 1.0. Since, under conditions of this theorem, $W_n(t)$ has asymptotically independent increments, by Theorem 19.2 of Billingsley [**5**], the weak convergence of W_n to W is a consequence of the uniform integrability of $(W_n^2(t))$ for every $t \in [0,1]$ plus the tightness of $(W_n(t))$. Also, by [**5**, Theorem 8.4], we can see that the uniform integrability of $(\max_{1\le i\le n} S_i^2/\sigma_n^2)$ assures the tightness and at the same time Theorem 1.0. Now, the uniform integrability of $(\max_{1\le i\le n} S_i^2/\sigma_n^2)$ is a consequence of Proposition 2.3.

PROOF OF THEOREM 1.1. It is well known that the family $\{q(|Y_k|)\}$ is uniformly integrable if and only if there is a function $f(x)$ such that $f(x)/q(x)$ is increasing to $+\infty$ as $x \to \infty$ and $\sup_k Ef(|Y_k|) < \infty$. Let us put $f(x) = q(x) \cdot h(x)$, where $h(x)$ is increasing to $+\infty$ as $x \to \infty$.

If $q(x) \in Q$, without loss of generality we can consider $f(x) \in Q$ too. (This can be achieved by decreasing $h(x)$ if necessary such that, for instance, $h(x)/\log x$ is a decreasing function.)

Now because $Eq(|X_0|) < \infty$, the family $\{q(|X_k|)\}$ is uniformly integrable, so there is a function $f(x) \in Q$ such that $f(x)/q(x) \to \infty$ and $Ef(|X_0|) < \infty$. According to Proposition 2.3 we can find a constant K_1 such that for every $n \ge 1$,

$$Ef\left(\max_{1\le i\le n} |S_i|/\sigma_n\right) \le K_1.$$

This implies that the family $\{q(\max_{1\le i\le n} |S_i|/\sigma_n)\}$ is uniformly integrable. Because the invariance principle holds under the conditions of Theorem 1.1 (i.e., $W_n(t)$ is weakly convergent to W) then by [**5**, Corollary 1 of Theorem 5.1, p. 31] it follows that $\sup_{t\in[0,1]} W_n(t)$ converges weakly to $\sup_{t\in[0,1]} W(t)$, and $\sup_{t\in[0,1]} |W_n(t)|$ converges weakly to $\sup_{t\in[0,1]} |W(t)|$.

It is well known [**5**, pp. 72, 79] that

$$P\left(\sup_{t\in[0,1]} W(t) \le a\right) = \sqrt{\frac{2}{\pi}}\int_0^a \exp(-u^2/2)\,du, \qquad a \ge 0,$$

and

$$P\left(\sup_{t\in[0,1]} |W(t)| < b\right) = \sum_{k=-\infty}^{\infty} (-1)^k P((2k-1)b < N(0,1) < (2k+1)b), \qquad b \ge 0.$$

Now by [**5**, Theorem 5.4, p. 32] we get the conclusion of this theorem.

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