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A DENSE SET OF OPERATORS QUASISIMILAR TO NORMAL + COMPACT

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ABSTRACT. The algebra of all bounded linear operators acting on a complex separable infinite dimensional Hilbert space is the disjoint union of two dense subsets: Every operator in one of them is quasisimilar to an operator of the form "normal + compact," and every operator in the complement is not quasisimilar to an operator of that form.

1. Introduction. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded, linear operators acting on $\mathcal{H}$. Two operators T, T' in $\mathcal{L}(\mathcal{H})$ are *similar* if there exists W invertible in $\mathcal{L}(\mathcal{H})$ such that $T' = WTW^{-1}$ (in symbols, $T' \sim T$). T and T' are *quasisimilar* ($T' \sim_{\mathrm{qs}} T$) if there exists X, Y in $\mathcal{L}(\mathcal{H})$ with $\ker X = \ker X^* = \ker Y = \ker Y^* = \{0\}$ such that $XT' = TX$ and $T'Y = YT$. (Here X^* denotes the adjoint of the operator X. Clearly, the conditions $\ker X = \ker X^* = \{0\}$ mean that X has trivial kernel and dense range.) The notion of quasisimilarity plays a very important role in connection with invariant subspace problems (see, e.g., [**14**]).

Let $\mathcal{K}$ and $\mathcal{N}$ denote the ideal of all compact operators and the set of all normal operators, respectively. A well-known corollary of the celebrated Brown-Douglas-Fillmore (BDF) Theorem says that $\mathcal{N} + \mathcal{K} = \{T \in \mathcal{L}(\mathcal{H})\colon T = N + K,\ N \in \mathcal{N},\ K \in \mathcal{K}\}$ is a closed subset of $\mathcal{L}(\mathcal{H})$ [**4**]. A recent article of C. Apostol, H. Bercovici, C. Foiaş, and C. Pearcy [**1**] provides some interesting information about the *quasisimilarity orbit* of $\mathcal{N} + \mathcal{K}$:

$$(\mathcal{N} + \mathcal{K})_{\mathrm{qs}} = \{T \in \mathcal{L}(\mathcal{H})\colon T \sim_{\mathrm{qs}} T' \text{ for some } T' \in \mathcal{N} + \mathcal{K}\}.$$

What kinds of operators can we expect to find in this family? The answer is, in a certain sense, all kinds of operators.

To make this more precise: observe that if $T \sim_{\mathrm{qs}} T'$ and $T' \sim T''$, then $T \sim_{\mathrm{qs}} T''$; therefore, $(\mathcal{N} + \mathcal{K})_{\mathrm{qs}}$ includes the *similarity orbit*

$$\mathcal{S}(T) = \{WTW^{-1}\colon W \in \mathcal{L}(\mathcal{H}) \text{ is invertible}\}$$

of every T in $(\mathcal{N} + \mathcal{K})_{\mathrm{qs}}$.

Since $(\mathcal{N} + \mathcal{K})_{\mathrm{qs}}$ is invariant under similarity, we can expect to find a simple characterization of its norm closure, $[(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}]^-$, by using the machinery developed in the monograph [**2**, **8**]. It is easily seen that (S): *Membership in $\mathcal{N} + \mathcal{K}$ does not imply any kind of restriction in the shape, number of components, or connectivity of the spectrum, $\sigma(T)$, or the essential spectrum, $\sigma_e(T)$, of an operator T in this*

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family. ($\sigma_e(T)$ is the spectrum of the canonical projection of T in the quotient Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}$.) Recall that $T \in \mathcal{L}(\mathcal{H})$ is semi-Fredholm if $\operatorname{ran} T$ is closed and either $\operatorname{nul} T := \dim\ker T$, or $\operatorname{nul} T^*$ is finite. In this case, the index of T is defined by $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$ [**7**, **13**]. It is easy to check that quasisimilar operators have the same nullity and the same conullity. Thus, if $T \sim_{\mathrm{qs}} T'$ and both T and T' are semi-Fredholm, then $\operatorname{ind} T = \operatorname{ind} T''$. However, a well-known example of T. B. Hoover [**12**] shows that if $T \sim_{\mathrm{qs}} T'$ and T is semi-Fredholm, then it is not necessarily true that T' is semi-Fredholm; indeed, we can actually have $\sigma(T) \neq \sigma(T')$ and $\sigma_e(T) \neq \sigma_e(T')$. Therefore, (F): *There is no index obstruction, a priori, that will prevent an operator T from being a member of $(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$.*

Properties (S) and (F), and the result of [**2**, Chapter 9] strongly suggest that $[(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}]^-$ must be equal to $\mathcal{L}(\mathcal{H})$. This is, indeed, the case.

THEOREM 1. *Both $(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$ and its complement $\mathcal{L}(\mathcal{H})\backslash(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$ are uniformly dense in $\mathcal{L}(\mathcal{H})$.*

That $\mathcal{L}(\mathcal{H})\backslash(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$ is dense in $\mathcal{L}(\mathcal{H})$ is actually a corollary of [**9**, Theorem 3]: $\mathcal{L}(\mathcal{H})$ includes a dense subset (D) of operators $T \sim A \oplus B$, where (i) $\sigma(A)\cap\sigma(B) = \varnothing$, (ii) $\lambda_A - A$ and $(\lambda_B - B)^*$ are semi-Fredholm operators of index $-\infty$ (for suitably chosen points λ_A, λ_B in $\sigma(T)$), and (iii) every T' quasisimilar to T is actually similar to T. Thus, $(D)_{\mathrm{qs}} = (D)$. Since $\operatorname{ind}(\lambda - A) = 0$ for all λ in $\rho_{\mathrm{s\text{-}F}}(A)$, the semi-Fredholm domain, of every A in $\mathcal{N}+\mathcal{K}$, we conclude that

$$(D)\cap(\mathcal{N}+\mathcal{K})_{\mathrm{qs}} = \varnothing.$$

2. $(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$ **is dense in** $\mathcal{L}(\mathcal{H})$. It will be shown that $(\mathcal{N}+\mathcal{K})_{\mathrm{qs}}$ includes a sufficiently large (dense) family of operators. Let $T \in \mathcal{L}(\mathcal{H})$ and let $\varepsilon > 0$ be given. The spectrum of T can be written as the following disjoint union:

$$\sigma(T) = \sigma_0(T) \cup \Sigma_0(T) \cup \Gamma \cup \left[\bigcup\{\Omega'_k\colon\ -\infty \le k \le \infty\}\right],$$

where

(i) $\sigma_0(T)$ is the set of isolated points λ of $\sigma(T)$ such that the corresponding Riesz subspace, $\mathcal{H}(\lambda; T)$, is finite dimensional (i.e., the set of all normal eigenvalues of T);

(ii) $\sigma_e(T)\backslash\rho_{\mathrm{s\text{-}F}}(T) = \Sigma_0(T)\cup\Gamma$, where Γ is perfect and $\Sigma_0(T)$ is at most denumerable;

(iii) $\Omega'_0 = \operatorname{interior}\{\lambda \in \rho_{\mathrm{s\text{-}F}}(T)\cap\sigma(T)\colon \operatorname{ind}(\lambda - T) = 0\}$; and

(iv) $\Omega'_k = \{\lambda \in \rho_{\mathrm{s\text{-}F}}(T)\colon \operatorname{ind}(\lambda - T) = k\}$ $(k \neq 0)$.

Any of these subsets can be empty. In order to simplify the proof, it will be assumed that $\Sigma_0(T)$, Γ, $\bigcup\{\Omega'_k\colon k > 0\}$, and $\bigcup\{\Omega'_k\colon k < 0\}$ are nonempty sets. The necessary modifications for the cases when some of these sets are empty will be left to the reader.

Let $d(\lambda) = \operatorname{dist}[\lambda, \sigma_e(T)]$. For each λ in $\sigma_0(T)$, let $J(\lambda)$ be the Jordan form of the nilpotent $T - \lambda|\mathcal{H}(\lambda; T)$, and let $Q = \bigoplus_{n=1}^{\infty} n^{-1} q_n^{(\infty)}$, where q_n is the $n\times n$ nilpotent Jordan cell (acting on $\mathbf{C}^n$). Clearly, Q is a universal quasinilpotent operator (in the sense of [**2**, **8**], i.e., $\sigma(Q) = \{0\}$ and $Q^k \notin \mathcal{K}$ for $k = 1, 2, \ldots$); moreover, Q is unitarily equivalent to the orthogonal direct sum of denumerably many copies of $q_1 = 0$, denumerably many copies of $\frac{1}{2}q_2 = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}$, etc.

For each bounded open set $\Omega = \operatorname{interior}(\Omega^-)$, the operator $N(\Omega)$ = multiplication by λ on $L^2(\Omega, dm_2)$ (where m_2 denotes the planar Lebesgue measure) admits a decomposition of the form

$$N(\Omega) = \begin{pmatrix} N_+(\Omega) & C(\Omega) \\ 0 & N_-(\Omega) \end{pmatrix} \begin{matrix} B^2(\Omega) \\ L^2(\Omega) \ominus B^2(\Omega), \end{matrix}$$

where $B^2(\Omega) :=$ the closure of the rational functions with poles outside Ω^-, and is invariant under $N(\Omega)$; furthermore, the subnormal operator $N_+(\Omega) = N(\Omega)|B^2(\Omega)$ is essentially normal (i.e., $[N_+(\Omega)^*, N_+(\Omega)] = N_+(\Omega)^* N_+(\Omega) - N_+(\Omega) N_+(\Omega)^*$ is compact) and satisfies the conditions: $\sigma(N_+(\Omega)) = \Omega^-$ is a spectral set (in the sense of von Neumann) for $N_+(\Omega)$, $\sigma_e(N_+(\Omega)) = \partial\Omega$ (the boundary of Ω), $\operatorname{nul}(\lambda - N_+(\Omega)) = 0$, and $\operatorname{ind}(\lambda - N_+(\Omega)) = -\operatorname{nul}(\lambda - N_+(\Omega))^* = -1$ for all λ in Ω (see, e.g., [**5**; **8**, Chapter 3] for details).

Let N be a diagonal normal operator of uniform infinite multiplicity such that $\sigma(N) = \sigma_e(N) = \Gamma$. Define

$$\begin{aligned} A = {} & \left[\bigoplus\{\lambda + d(\lambda)J(\lambda)\colon\ \lambda \in \sigma_0(T)\}\right] \\ & \oplus \left[\bigoplus\{\lambda + Q\colon\ \lambda \in \Sigma_0(T)\}\right] \oplus N \\ & \oplus \left\{ \bigoplus_{1 \le k \le \infty} N_+(\Omega_{-k})^{(k)} \right\} \oplus \left\{ \bigoplus_{1 \le k \le \infty} N_+(\Omega_k^*)^{*(k)} \right\}, \end{aligned}$$

where $\Omega_k = \operatorname{interior}(\Omega_k')^-$ $(k \neq 0)$, $\Omega^* = \{\bar{\lambda}\colon\ \lambda \in \Omega\}$, and $R^{(k)}$ denotes the direct sum of k copies of the operator R acting in the usual fashion on the orthogonal direct sum of k copies of its underlying space.

It is a straightforward exercise to check that A spectrally dominates T (in the sense of [**2**, p. 5]) and therefore, by [**2**, Theorem 9.1], there exists T' in $\mathcal{L}(\mathcal{H})$ *similar to* A such that $\|T - T'\| < \varepsilon$.

Thus, in order to complete the proof of Theorem 1, it suffices to show that $A \in (\mathcal{N} + \mathcal{K})_{\mathrm{qs}}$.

Claim. Let M be a normal operator such that

$$\sigma(M) = \sigma_e(M) = \sigma(T) \backslash [\sigma_0(T) \cup \Omega_0'];$$

then there exists K compact such that $A \sim_{\mathrm{qs}} M + K$.

The following auxiliary result has a very simple proof (see, e.g., [**9** or **12**]).

LEMMA 2. *If* $\{A_n\}_{n=1}^\infty$ *and* $\{B_n\}_{n=1}^\infty$ *are bounded families of operators such that* $A_n \sim_{\mathrm{qs}} B_n$ (*or* $A_n \sim B_n$) *for each* n, *then* $\bigoplus_{n=1}^\infty A_n \sim_{\mathrm{qs}} \bigoplus_{n=1}^\infty B_n$.

Since a Jordan nilpotent J is similar to ηJ for all $\eta > 0$, it readily follows from Lemma 2 that

$$\begin{aligned} A_0 := {} & \left[\bigoplus\{\lambda + d(\lambda)J(\lambda)\colon\ \lambda \in \sigma_0(T)\}\right] \\ & \oplus \left[\bigoplus\{\lambda + Q\colon\ \lambda \in \Sigma_0(T)\}\right] \sim_{\mathrm{qs}} L - C, \end{aligned}$$

where L is a normal operator satisfying

$$\sigma(L) = \sigma_e(L) = [\Sigma_0(T) \cup \sigma_0(T)]^- \backslash \sigma_0(T)$$

and C is a compact operator satisfying $\|C\| < \max\{\operatorname{dist}[\lambda, \sigma_e(T)]\colon\ \lambda \in \sigma_0(T)\} + \varepsilon$.

Let $\Omega, N(\Omega)$, and $N_+(\Omega)$ be as above, with $\Omega = \Omega_k$ for some $k \neq 0$. If μ is a positive Borel measure with $\text{support}(\mu) \subset \Omega$ and $N(\Omega;\mu)$, $B^2(\Omega;\mu)$, $N_+(\Omega;\mu) = N(\Omega;\mu)|B^2(\Omega;\mu)$ defined exactly as above, with $dm_2|\Omega$ replaced by $dm_2|\Omega + d\mu$, then it is not difficult to check that

$$B^2(\Omega) = B^2(\Omega;\mu)$$

and that the identity map $\iota(\mu)\colon B^2(\Omega) \to B^2(\Omega;\mu)$ implements a similarity between $N_+(\Omega)$ and $N_+(\Omega;\mu)$. In [**6**, Lemma 3], K. R. Davidson showed that, given $\delta > 0$, μ can be chosen so that $\|[N_+(\Omega;\mu)^*, N_+(\Omega;\mu)]\| < \delta$.

Observe that

$$A_+ := \bigoplus_{1\le k\le\infty} N_+(\Omega_{-k})^{(k)} = \bigoplus_n N_+(\Phi_{-n}),$$

where $\Phi_{-n} = \Omega_{-k}$ for exactly k indices n $(1 \le k \le \infty)$. For each n, we can find a measure μ_n with $\text{support}(\mu_n) \subset \Phi_{-n}$, such that $\|[N_+(\Phi_{-n};\mu_n)^*, N_+(\Phi_{-n};\mu_n)]\| < \delta/n$. It follows from Lemma 2 that A_+ is quasisimilar to the essentially normal subnormal operator

$$A'_+ = \bigoplus_n N_+(\Phi_{-n};\mu_n)$$

($[A'^*_+, A'_+] \in \mathcal{K}$ and $\|[A'^*_+, A'_+]\| < \delta$).

Similarly, $A_- = \bigoplus_{1\le k\le\infty} N_+(\Omega^*_k)^{*(k)}$ is quasisimilar to an essentially normal cosubnormal operator A'_- ($\|[A'^*_-, A'_-]\| < \delta$).

On the other hand, it follows from [**10**, Corollary 2.4 and its proof] (see also [**11**]) that there exists a compact operator G, with $\|G\| < \delta$, such that $M - G = \bigoplus_{j=1}^\infty(\mu_j + Q_j)$, where $\mu_j \subset \sigma(N)$ and Q_j is a nilpotent acting on a subspace $\mathcal{H}_j$ of finite dimension. (To see this, observe that each component of $\sigma(M)$ intersects $\sigma(N)$.) Then

$$\mu_j + Q_j = \begin{pmatrix} \mu_j & & & & \\ & \mu_j & & * & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \mu_j \end{pmatrix} \begin{matrix} e_1(j) \\ e_2(j) \\ \cdot \\ \cdot \\ \cdot \\ e_{d_j}(j) \end{matrix}$$

with respect to a suitable orthonormal basis $\{e_i(j)\}_{i=1}^{d_j}$ of $\mathcal{H}_j$. Since $\sigma(N)$ is perfect, for each j we can pick d_j distinct eigenvalues $\mu_j(1), \mu_j(2), \ldots, \mu_j(d_j)$ of N such that $\max_i |\mu_j - \mu_j(i)| < \varepsilon/j$, $j = 1, 2, \ldots$. Clearly, $\bigcup_{j=1}^\infty \{e_i(j)\}_{i=1}^{d_j}$ is an orthonormal basis of $\mathcal{H}$ and the diagonal normal operator D, defined by $De_i(j) = [\mu_j - \mu_j(i)]e_i(j)$, is compact and satisfies $\|D\| < \varepsilon$; furthermore, $M - (G + D) = \bigoplus_{j=1}^\infty F_j$, where

$$F_j = \begin{pmatrix} \mu_j(1) & & & & \\ & \mu_j(2) & & * & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & \cdot & \\ & & & & \mu_j(d_j) \end{pmatrix} \begin{matrix} e_1(j) \\ e_2(j) \\ \cdot \\ \cdot \\ \cdot \\ e_{d_j}(j) \end{matrix}$$

is similar to the diagonal normal operator $N_j = \operatorname{diag}\{\mu_j(1), \mu_j(2), \ldots, \mu_j(d_j)\}$. By Lemma 2,

$$M - (G+D) = \bigoplus_{j=1}^{\infty} F_j \sim_{\mathrm{qs}} N' = \bigoplus_{j=1}^{\infty} N_j.$$

Since each eigenvalue of N' is an eigenvalue of N, and N has uniform infinite multiplicity, it readily follows that N' is a direct summand of N, that is, $N = N' \oplus N''$.

Now we put all the ingredients together. By Lemma 2, we have

$$\begin{aligned} A &= A_0 \oplus N \oplus A_+ \oplus A_- = A_0 \oplus N' \oplus N'' \oplus A_+ \oplus A_- \sim_{\mathrm{qs}} B \\ &:= (L - C) \oplus [M - (G+D)] \oplus N'' \oplus A'_+ \oplus A'_- \\ &= [(L \oplus M \oplus N'') \oplus (A'_+ \oplus A'_-)] - [C \oplus (G+D) \oplus 0 \oplus 0 \oplus 0]. \end{aligned}$$

Here $L \oplus M \oplus N''$ is a normal operator with spectrum and essential spectrum equal to $\sigma(M)$, and

$$\sigma(M) \supset \sigma(A'_+ \oplus A'_-) = \left(\bigcup_{k \neq 0} \Omega_k \right)^- .$$

Since $A'_+ \oplus A'_-$ is essentially normal, and $C \oplus (G+D) \oplus 0 \oplus 0 \oplus 0$ is compact, it follows that B is also essentially normal with $\sigma_e(B) = \sigma_e(M) \supset \sigma(A'_+ \oplus A'_-)$ and $\sigma(B) = \sigma_e(B) \cup \sigma_0(T)$. By using the BDF Theorem, we conclude that $B \simeq M + K$ for some compact operator K, where $\simeq$ denotes unitary equivalence [**4**].

The proof of Theorem 1 is now complete. □

3. Concluding remarks. (i) K. R. Davidson's result that μ can be chosen so that $\|[N_+(\Omega;\mu)^*, N_+(\Omega;\mu)]\| < \delta$ is only necessary for the case when $\Omega_{+\infty} \cup \Omega_{-\infty} \neq \emptyset$. For the case when $\operatorname{ind}(\lambda - T)$ is finite for all $\lambda \in \rho_{\text{s-F}}(T)$, we can use an argument of [**4**] (based on the Berger-Shaw trace inequality) in order to show that

$$\|[N_+(\Omega_k)^*, N_+(\Omega_k)]\| \leq \frac{1}{\pi}\operatorname{area}(\Omega_k) \to 0 \qquad (|k| \to \infty).$$

However, with the help of Davidson's result we can obtain the following, more precise, result:

If M is replaced by a normal operator M' such that $\sigma_e(M') = \sigma(M)$, $\sigma(M') = \sigma_e(M') \cup \sigma_0(T)$, and $\operatorname{nul}(\lambda - M') = \dim \mathcal{H}(\lambda; T)$ for each λ in $\sigma_0(T)$, then the compact operator C can be chosen so that $\|C\| < \varepsilon$. Recall that

$$\|[(A'_+ \oplus A'_-)^*,\ (A'_+ \oplus A'_-)]\| < \delta$$

and A'_+ (A'_-) is subnormal (cosubnormal), and therefore $\sigma(A'_+)$ $(\sigma(A'_-))$ is a spectral set for A'_+ (A'_-). According to a very recent quantitative version of the BDF Theorem, due to I. D. Berg and K. R. Davidson [**3**], if δ is small enough, then there exists E compact with $\|E\| < \varepsilon$, such that

$$(L \oplus M' \oplus N'') \oplus (A'_+ \oplus A'_-) - E \simeq M'.$$

It readily follows that $A \sim_{\mathrm{qs}} M' + K'$, where $K' \in \mathcal{K}$ and $\|K'\| < 3\varepsilon$.

(ii) It follows from [**2**, Theorem 9.1] that given a compact subset Δ of Γ such that every component of Γ intersects Δ, the direct summand N (of A) can be replaced by a certain operator N_Δ such that $\sigma(N_\Delta) = \sigma_e(N_\Delta) = \Delta$. Furthermore, the normal operator M can be chosen so that $\sigma(M) = \sigma_e(M)$ is any compact set including $\Delta' = \Sigma_0 \cup \Delta \cup [\bigcup\{\Omega_k\colon\ k \neq 0\}]$ such that each component of $\sigma(M)$ intersects Δ'. In particular, we can take $N_\Delta = N$ and $\sigma(M) = \sigma_e(M) = \sigma(T)\backslash\sigma_0(T)$, or $\sigma(M) = \sigma_e(M) =$ a closed disk including $\sigma(T)$. (In the latter case, the compact perturbation can be chosen of arbitrarily small norm.)

(iii) *Conjecture.* $\mathcal{L}(\mathcal{H}) = (\mathcal{N} + \mathcal{K})_{\mathrm{qs}} + \mathcal{K}$. More precisely, given T in $\mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K_\varepsilon \in \mathcal{K}$, with $\|K_\varepsilon\| < \varepsilon$, such that $T - K_\varepsilon \sim_{\mathrm{qs}} A$ for some $A \in \mathcal{N} + \mathcal{K}$.

Minor modifications of the proof of Theorem 1 show that the above Conjecture is true, at least, for direct sums of essentially normal operators.

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