PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 97, Number 1, May 1986

# ON A QUESTION OF FEIT

PAMELA A. FERGUSON[1] AND ALEXANDRE TURULL

ABSTRACT. The following theorem is proved: Assume $\chi$ is an irreducible complex character of the finite group $G$ and $G$ is $\pi$-solvable where $\pi$ is the set of prime divisors of $\chi(1)$. Then $G$ contains an element of order $f(\chi)$.

**Introduction.** All groups in this paper are finite. The question referred to in the title is the following: Let $\chi$ be an irreducible complex character of a finite group $G$ and let $f(\chi)$ be the smallest positive integer such that $\{\chi(g)|g \in G\} \subseteq Q(\alpha)$ where $\alpha$ is a primitive $f(\chi)$th root of unity. Does $G$ contain an element of order $f(\chi)$?

By using factorizations of quasi-primitive irreducible characters into products of characters we obtain the following result:

THEOREM. *Assume $\chi$ is an irreducible complex character of a finite group $G$ and $G$ is $\pi$-solvable where $\pi$ is the set of primes dividing $\chi(1)$. Then $G$ contains an element of order $f(\chi)$.*

This theorem yields the following Corollary which was proved independently by Amit and Chillag [**1**].

COROLLARY. *If $\chi$ is an irreducible character of a solvable group $G$, then $G$ contains an element of order $f(\chi)$.*

We wish to thank Professor I. M. Isaacs and the referee for carefully reading the original manuscript and making helpful suggestions. In particular, the notation $f(\chi)$ was suggested by Professor Isaacs in honor of W. Feit.

**1.** We introduce some definitions and notation. Let $\sigma$ be a nonempty set of primes and $h$ be an element in a group $G$, $h$ is a $\sigma$-element if $\langle h\rangle$ is a $\sigma$-group. Any $g \in G$ may be written uniquely as $g = g_\sigma g_{\sigma'}$, where $g_\sigma$ is a $\sigma$-element, $g_{\sigma'}$ is a $\sigma'$-element and both $g_\sigma$ and $g_{\sigma'}$ are powers of $g$. The elements $g_\sigma$ and $g_{\sigma'}$ are called the $\sigma$-part and $\sigma'$-part of $g$.

If $\Omega$ is a Galois extension of the field $\Omega_1, G(\Omega/\Omega_1)$ denotes the Galois group of $\Omega$ over $\Omega_1$.

If $\chi \in \mathrm{Irr}(G)$ and $\sigma$ is a set of primes, then $\chi$ is $\sigma$-special provided that $\chi(1)$ is a $\sigma$-number and that for all subnormal subgroups $S$ of $G$ and all irreducible constituents $\theta$ of $\chi_S$, the determinantal order $O(\theta)$ is a $\sigma$-number.

PROOF OF THEOREM. The result is clear for linear characters so by induction on $\chi(1)$ we assume that the theorem is true for irreducible characters $\Phi$ and groups $H$ satisfying the hypothesis if $\Phi(1) < \chi(1)$. If $\chi = \Phi^G$ where $\Phi \in \mathrm{Irr}(H)$ and

Received by the editors January 30, 1985 and, in revised form, May 1, 1985.
1980 *Mathematics Subject Classification*. Primary 20C15.
*Key words and phrases*. Prime character, $\pi$-special character, $f(\chi)$.
[1] Partially supported by a National Science Foundation Grant.

$|H| < |G|$, then $f(\chi)|f(\Phi)$. Since $\Phi(1)$ is a proper divisor of $\chi(1)$, $H$ contains an element of order $f(\Phi)$ and the theorem follows. Thus, we may assume that $\chi$ is primitive.

Let $f(\chi) = \prod_{i=1}^{n} p_i^{a_i}$ where $a_i \geq 1$ and the $p_i$ are distinct primes. Let $p_i^{r_i}$ denote the order of a Sylow $p_i$-subgroup of $G$ for $i = 1, \ldots, n$. Then $\Omega_i$ denotes the field of $(|G|/p_i^{r_i - a_i + 1})$th roots of unity over $Q$. Let $\Omega$ be a field of $|G|$th roots of unity over $Q$. By the definition of $f(\chi)$, we may choose a $\theta_i \in G(\Omega/\Omega_i)$ such that $\theta_i$ does not leave $\chi$ invariant for $i = 1, \ldots, n$. We first show that no product of an odd number of distinct elements in $\{\theta_1, \ldots, \theta_n\}$ leaves $\chi$ invariant. Assume otherwise; then since $G(\Omega/Q)$ is abelian, we may choose notation so that $\beta = \prod_{i=1}^{r} \theta_i$ leaves $\chi$ invariant, for some $r \geq 1$.

Suppose $p_i|\chi(1)$ for some $i = 1, \ldots, r$. Since $G(\Omega/Q)$ is abelian, we may assume $p_1|\chi(1)$. By [**4**, Corollary 2.7], $\chi = \chi_1\Phi$ where $\chi_1$ and $\Phi$ are primitive irreducible $p_1$-special and $p_1'$-special characters. $G$ is $p_1$-solvable, hence $p_1'$-separable, and so $f(\Phi)$ is a $p_1'$-number by [**3**, Proposition 6.3(a)].

If $\theta \in G(\Omega/Q)$ and $\sigma$ is any set of primes, then it is direct to see that whenever $\gamma$ is an irreducible $\sigma$-special character so is $\theta \circ \gamma$. Hence $\beta \circ \chi_1$ and $\beta \circ \Phi$ are $p_1$-special and $p_1'$-special. Now $\chi_1\Phi = \chi = \beta \circ \chi = (\beta \circ \chi_1)(\beta \circ \Phi)$ and [**4**, Theorem 2.2] yield $\chi_1 = \beta \circ \chi_1$. By [**3**, Proposition 6.3(a)], $f(\chi_1)$ is a $p_1$-number so $\theta_i \circ \chi_1 = \chi_1$ for $i = 2, \ldots, r$. Since $G(\Omega/Q)$ is abelian, it follows that $\chi_1 = \beta \circ \chi_1 = \theta_1 \circ \chi_1$. Now $f(\Phi)$ a $p_1'$-number implies that $\theta_1 \circ \Phi = \Phi$. However, $\theta_1 \circ \chi = (\theta_1 \circ \chi_1)(\theta_1 \circ \Phi) = \chi_1\Phi = \chi$ contradicts the choice of $\theta_1$. Therefore, $(\prod_{i=1}^{r} p_i, \chi(1)) = 1$.

Now set $p = p_1$. By [**4**, Corollary 2.7], $\chi = \Phi\lambda$, where $\Phi$ and $\lambda$ are irreducible, primitive $\pi$-special and $\pi'$-special characters. Since $\chi(1)$ is a $\pi$-number, $\lambda$ is linear. As in the previous paragraph, $\beta \circ \Phi$ and $\beta \circ \lambda$ are $\pi$-special and $\pi'$-special. Thus, $\Phi\lambda = \chi = \beta \circ \chi = (\beta \circ \Phi)(\beta \circ \lambda)$ and [**4**, Theorem 2.2] imply that $\lambda = \beta \circ \lambda$. Since $\beta$ fixes $\lambda$, $\beta$ fixes all powers of $\lambda$. In particular, $\beta$ fixes $\lambda_p$ and $\lambda_{p'}$, where $\lambda_p$ is the $p$-part of $\lambda$ and $\lambda_{p'}$ is the $p'$-part. It is clear that $f(\lambda_p)$ is a $p$-number and $f(\lambda_{p'})$ is a $p'$-number. Hence, $\theta_i \circ \lambda_p = \lambda_p$ for $i = 2, \ldots, r$ and $\theta_1 \circ \lambda_{p'} = \lambda_{p'}$. Since $G(\Omega/Q)$ is abelian, $\lambda_p = \beta \circ \lambda_p = \theta_1 \circ \lambda_p$. By [**3**, Proposition 6.3(a)], $f(\Phi)$ is a $\pi$-number. Thus, $\theta_1 \circ \Phi = \Phi$ and $\theta_1 \circ \chi = (\theta_1 \circ \Phi)(\theta_1 \circ \lambda_p)(\theta_1 \circ \lambda_{p'}) = \Phi\lambda_p\lambda_{p'} = \chi$. Again this is a contradiction. Therefore, no product of an odd number of distinct $\theta_i$ leaves $\chi$ invariant. Hence, by [**2**, Theorem 2], there is an element $g \in G$ such that $\chi(g) \notin \Omega_i$ for any $i = 1, \ldots, n$. Thus, $f(\chi)|\,|\langle g\rangle|$.

## REFERENCES

1. G. Amit and D. Chillag, *On a question of Feit*, Pacific J. Math. (to appear).
2. R. Brauer, *A note on theorems of Burnside and Blichtfeldt*, Proc. Amer. Math. Soc. **15** (1964), 31–34.
3. D. Gajendragadkar, *A characteristic class of characters of finite $\pi$-separable groups*, J. Algebra **59** (1979), 237–259.
4. I. Isaacs, *Primitive characters, normal subgroups, and M-groups*, Math. Z. **177** (1981), 267–284.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124