PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 96, Number 1, January 1986

NONEXISTENCE OF SOME NONPARAMETRIC SURFACES OF PRESCRIBED MEAN CURVATURE

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ABSTRACT. If $\Omega \subset \mathbf{R}^2$ has a reentrant corner, the Dirichlet problem for the equation of prescribed mean curvature H with zero boundary value has no solution when H has constant nonzero sign.

Suppose Ω is a bounded open subset of $\mathbf{R}^n$, $\phi \in C^0(\partial\Omega)$, and $H \in C^0(\Omega)$. Does there exist a solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem

$$\text{(1)} \qquad \operatorname{div}(Tu) = nH \quad \text{on } \Omega, \qquad u = \phi \quad \text{on } \partial\Omega,$$

where $Tu = \nabla u/W$ and $W = \sqrt{1 + |\nabla u|^2}$? Using barrier arguments, Graham Williams [5] has proven this

THEOREM. *Given $n \geqslant 2$, $K \in (0, 1/\sqrt{n-1})$, $R > 0$, and Ω with locally Lipschitz boundary satisfying a uniform exterior sphere condition of radius R, for $\|H\|_\infty$ small enough there exists $\varepsilon > 0$ depending on n, K, R, and $\|H\|_\infty$ such that whenever $\phi \in C^{0,1}(\partial\Omega)$ has Lipschitz constant $\leqslant K$ and $\sup\phi - \inf\phi < \varepsilon$, there exists $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ which satisfies* (1).

It has been known for some time that if the data is small and $\partial\Omega$ is smooth, (1) has a (classical) solution [4].

We will say that $\Omega \subset \mathbf{R}^2$ has a reentrant corner at $P \in \partial\Omega$ if for some a, α, β with $\beta - \alpha > \pi$ and $a > 0$, $\{(r,\theta) | 0 < r < a, \alpha \leqslant \theta \leqslant \beta\} \subset \Omega$, where (r,θ) represents polar coordinates about P. The purpose of this note is to prove the following

THEOREM. *Let $n = 2$ and Ω be a bounded open subset of $\mathbf{R}^2$ which has reentrant corner at $P \in \partial\Omega$. Suppose $H \in C^0(\Omega)$ with $H < 0$ and $\phi \in C^0(\partial\Omega)$ with $\phi(P) = 0$ and $\phi \geqslant 0$. Then there is no solution $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ of* (1).

Using the results of [1], we can determine the behavior of the variational solution of (1) near P when $\partial\Omega \setminus \{P\}$ has positive curvature. The case $n = 2$ is interesting because of its relationship to the "membrane analogy" of engineering, which we discuss later.

PROOF. Suppose a solution u of (1) exists. We may assume P is the origin and $-3\pi/4 < \alpha = -\beta$. Let V be the interior of the nonconvex quadrilateral symmetric with respect to the x-axis determined by the points $(r,\theta) = (0,0)$, (a,β), $(a,-\beta)$,

Received by the editors April 15, 1985.
1980 *Mathematics Subject Classification*. Primary 35J25, 35J60.

$(a, 0)$. Set $m = \inf\{u(r,\theta) | r = a/4, -\beta \leqslant \theta \leqslant \beta\}$. By the strict maximum principle, $u > 0$ on Ω and so $m > 0$.

Define $k \in C^0(\partial V)$ by setting $k = 0$ on $\partial V \cap S$, where $S = \{(r,\theta) | r < a/2\}$, and $k = m$ on the two sides of ∂V not touching P and by requiring k to be linear on each of the remaining portions of $\partial\Omega$.

Now let f be the variational solution of the Dirichlet problem for the minimal surface equation in V with boundary values k. Then $f \in C^2(V) \cap C^0(\bar{V} \setminus \{P\})$ and $f = k$ on $\partial V \setminus \{P\}$. Notice $0 \leqslant f \leqslant m$ and $u \geqslant m$ on $r = a/4$, $-\beta \leqslant \theta \leqslant \beta$, so $f \leqslant u$ on $r = a/4$, $-\beta \leqslant \theta \leqslant \beta$. Since $u \geqslant 0$ and $k = 0$ on $\partial V \cap S$, we see that $0 \leqslant f \leqslant u$ in $V \cap \{(r,\theta) | r < a/4\}$. This implies $f \in C^0(\bar{V})$, in contradiction to [**2**, pp. 146–147]. Q.E.D.

Suppose Ω is connected and simply connected in $\mathbf{R}^2$. Consider a uniform bar with cross section Ω. If we apply couples to the ends and twist the bar, then the stress function g satisfies

$$\Delta g = 2H \quad \text{on } \Omega, \qquad g = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $H = -GA$, G is the modulus of rigidity of the bar, and A is the angle of twist per unit length of the bar. The membrane analogy is the assumption that $g \approx u$ and $\nabla g \approx \nabla u$ on Ω, where g solves (2) and u solves (1) with $n = 2$ and $\phi \equiv 0$. If Ω has a reentrant corner, no solution of (1) exists and the analogy is invalid. For a discussion of the membrane analogy and some experiments it inspired, see [**3**].

ACKNOWLEDGEMENT. I wish to thank Professor Gulliver for his comments and encouragement.

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