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WHEN IS A KIRILLOV ORBIT A LINEAR VARIETY?

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ABSTRACT. It is well known that a Kirillov orbit is a linear variety if and only if the corresponding irreducible representation is square integrable modulo its kernel ([1], Theorem 1.1). Now we give a new representation-theoretic criterion for a Kirillov orbit being a linear variety in terms of weak containment and tensor products of group representations.

Let G be a nilpotent simply connected Lie group and $\mathfrak{G}$ its Lie algebra. For $\pi \in \hat{G}$, the set of equivalence classes of irreducible unitary representations of G, let Ω_π be the corresponding Kirillov orbit in the dual space $\mathfrak{G}^*$ of $\mathfrak{G}$. The orbit corresponding to the one-dimensional identity representation 1 of G is $\{0\}$, and the orbit corresponding to the conjugate representation $\bar{\pi}$ of π is $-\Omega_\pi$. For $\pi, \rho \in \hat{G}$ an element $\sigma \in \hat{G}$ is weakly contained in the tensor product $\pi \otimes \rho$ of π and ρ if and only if $\Omega_\sigma \subseteq \overline{\Omega_\pi + \Omega_\rho}$ [2, Lemma 2.1]. Therefore 1 is weakly contained in $\pi \otimes \bar{\pi}$ for every π (compare [4, Lemma 1]). Now we investigate under which condition 1 is weakly contained in $\pi \otimes \bar{\rho}$ for $\rho \neq \pi$ (compare [3, 2]).

THEOREM. *The following properties of* $\pi \in \hat{G}$ *are equivalent*:
(i) Ω_π *is a linear variety*;
(ii) *if* 1 *is weakly contained in* $\pi \otimes \bar{\rho}$ *for* $\rho \in \hat{G}$, *then* $\rho = \pi$.

PROOF. Let Ω_π be a linear variety and $0 \in \overline{\Omega_\pi - \Omega_\rho}$. Then $\Omega_\pi = p + M$, where $p \in \mathfrak{G}^*$ and M is a linear subspace of $\mathfrak{G}^*$, which is invariant under the coadjoint representation. The quotient representation of the coadjoint representation in the quotient space $\mathfrak{G}^*/M$ is unipotent, consequently the orbits in $\mathfrak{G}^*/M$ are closed. Thus $k(\Omega_\rho) \subseteq \mathfrak{G}^*/M$ is closed, where $k: \mathfrak{G}^* \to \mathfrak{G}^*/M$ is the canonical mapping. By $0 \in \overline{\Omega_\pi - \Omega_\rho}$ we follow $k(p) \in k(\Omega_\rho)$, then $k(p) = k(q)$ with $q \in \Omega_\rho$, then $q \in p + M = \Omega_\pi$.

Conversely, assuming that Ω_π is not a linear variety, we have to show that there exists an orbit $\Omega \neq \Omega_\pi$ with $0 \in \overline{\Omega_\pi - \Omega}$. By assumption, there are points $p, q \in \Omega_\pi$ such that the line segment $[p,q]$ joining p and q is not contained in Ω_π. Let $\operatorname{Exp} \mathbf{R} X$, $X \in L(\mathfrak{G}^*)$, be a one-parameter subgroup of the coadjoint group of G with $(\operatorname{Exp} X)p = q$. Because X is a nilpotent endomorphism, there is a greatest natural

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number n such that $X^n p \neq 0$. Let $m \leqslant n$ be the smallest number such that $\operatorname{Exp} \mathbf{R} Xp + F \subseteq \Omega_\pi$, F being the linear hull of the vectors $X^{m+1}p, X^{m+2}p, \ldots, X^n p$ in $\mathfrak{G}^*$. (The linear subspace F of $\mathfrak{G}^*$ can be regarded as the "flat part" of Ω_π along $\operatorname{Exp} \mathbf{R} Xp$.) In view of $[p,q] \not\subseteq \Omega_\pi$ we conclude $m > 1$. Thus we can find $\alpha, \beta \in \mathbf{R}$ and $f \in F$ such that the point $q := (\operatorname{Exp} \alpha X)p + \beta X^m p + f$ does not belong to Ω_π. Let Ω be the coadjoint orbit of q. The subspace F of $\mathfrak{G}^*$ is invariant under X. Therefore we can form the endomorphism X_F, associated with X, of the quotient space $\mathfrak{G}^*/F$. In order to see that $0 \in \overline{\Omega_\pi - \Omega}$ it is enough to show that

$$0 \in \overline{k(\operatorname{Exp} \mathbf{R} Xp) - k(\operatorname{Exp} \mathbf{R} Xq)} = \overline{\operatorname{Exp} \mathbf{R} X_F k(p) - \operatorname{Exp} \mathbf{R} X_F k(q)}$$

in $\mathfrak{G}^*/F$, $k : \mathfrak{G}^* \to \mathfrak{G}^*/F$ being the canonical mapping. Let us define the sequence

$$x_\nu := (\operatorname{Exp} \gamma_\nu X_F) k(p) \in \operatorname{Exp} \mathbf{R} X_F k(p)$$

with $\gamma_\nu := (\nu^m + m!\beta)^{1/m}$ and the sequence

$$x'_\nu := (\operatorname{Exp}(\nu - \alpha) X_F) k(q) \in \operatorname{Exp} \mathbf{R} X_F k(q).$$

Then

$$\begin{aligned} x_\nu - x'_\nu &= \sum_{r=0}^{m} \frac{\gamma_\nu^r - \nu^r}{r!} X_F^r k(p) - \beta X_F^m k(p) \\ &= \sum_{r=0}^{m-1} \frac{\gamma_\nu^r - \nu^r}{r!} X_F^r k(p) \end{aligned}$$

is a null sequence, because $\gamma_\nu^r - \nu^r$ is a null sequence in $\mathbf{R}$ for $r < m$.

COROLLARY. *$\pi \in \hat{G}$ is square integrable modulo its kernel if and only if any element $\rho \in \hat{G}$, for which* 1 *is weakly contained in $\pi \otimes \bar{\rho}$, must be equal to π.*

Maybe the assertion of this corollary remains true for more general locally compact groups.

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