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THERE EXIST ARBITRARILY MANY DIFFERENT DISK KNOTS WITH THE SAME EXTERIOR

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ABSTRACT. We prove that, for $n \geq 5$, exteriors of disk knots of D^n in D^{n+2} can be exteriors of arbitrarily many different disk knots.

1. Introduction. In [**H-S**], we showed that there are at least three different disk knots of D^4 in D^6 with the same exterior, and at least six different disk knots of D^n in D^{n+2} with the same exterior for $n \geq 5$. We improve the latter result here by showing that there exist arbitrarily large classes of inequivalent disk knots with the same exterior.

Let Y denote the bounded exterior of a smooth n-disk knot. The *indeterminacy index* $\varsigma(Y)$ is the number of inequivalent n-disk pairs having exteriors diffeomorphic to Y. We prove the following

THEOREM. *Let $n \geq 5$. Given a positive integer N, there exists an n-disk knot exterior Y with $\varsigma(Y) \geq N$.*

This answers Question 1 in [**H-S**] in the affirmative.

2. The construction. For convenience, we work in the smooth category, although the results hold in the locally flat PL category as well. An *n-disk knot* is a manifold pair $(D^{n+2}, f(D^n))$ where $f\colon D^n \to D^{n+2}$ denotes a proper embedding in which the submanifold $f(D^n)$ intersects ∂D^{n+2} transversely. The *exterior* Y of an n-disk knot is the complement in D^{n+2} of a trivial open 2-disk bundle neighborhood of the submanifold $f(D^n)$. Two disk knots are *equivalent* if they are diffeomorphic as unoriented pairs, i.e., if there is a diffeomorphism of D^{n+2} onto itself which sends one submanifold onto the other (disregarding orientations).

The construction used in [**H-S**] to show that $\varsigma(Y)$ can be as large as six was a modification of an example of Kato [**Ka**, Theorem 4.9]. We recall the construction and further modify it as follows.

Let G be any finitely presented group with $H_1(G;\mathbf{Z}) = H_2(G;\mathbf{Z}) = 0$. If $n + 1 \geq 6$, then Kervaire [**Ke**] has shown that there is a contractible manifold M^{n+1} with $\pi_1(\partial M) = G$. Suppose also that G has weight one (i.e., G has an element, called a *weight element*, whose normal closure is all of G). Then form the manifold $Y = S^1 \times M^{n+1}$. We see that Y is a disk knot exterior by attaching a 2-handle h^2 to Y along the path $tg \in \pi_1(\partial Y) = J \times G$, where g is a weight element of G, and t is a generator of the infinite cyclic factor J. In this case, tg is a weight

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element of $\pi_1(\partial Y)$, and $(Y \cup h^2, \text{cocore}(h^2))$ is an n-disk knot. If now $J \times G$ has many different weight elements, this gives rise to different handle attachments, and possibly inequivalent n-disk knots.

To help measure the inequivalency, we call two elements a, b in a group H *algebraically distinct* if the orbit of the set $\{a, a^{-1}\}$ under all automorphisms of H is disjoint from the orbit of the set $\{b, b^{-1}\}$. Then any two algebraically distinct weight elements of $J \times G$ give rise to inequivalent disk knots in this construction. Thus, the proof of the theorem is reduced to finding a suitable class of groups to use in the role of the group G.

3. The special linear groups. In [**H-S**], we used the group $G = \langle a, b | a^5 = b^3 = (ab)^2 \rangle = SL(2,5)$ to obtain a group $J \times G$ with three algebraically distinct weight elements, and observed that $J \times G \times G \times G$ contains at least six algebraically distinct weight elements. Here, we use $SL(2,p)$ for p a prime, $p \geq 5$.

Recall that $SL(2,p)$ is its own commutator subgroup (see, e.g., [**D**, pp. 38–40]), so $H_1(SL(2,p)) = 0$. And, as Gordon [**G**] points out, $H_2(SL(2,p)) = 0$ [**S**, p. 95, Corollary 2]. Furthermore, $Z(SL(2,p)) = \{I, -I\}$ where $Z(G)$ denotes the center of the group G and I denotes the identity matrix; and, any noncentral element of $SL(2,p)$ is a weight element (e.g. [**R**, p. 159]). Thus, any element of the form $tg \in J \times SL(2,p)$, where t generates J and g is not in the center of $SL(2,p)$, is a weight element of $J \times SL(2,p)$. Now let $[a]$ denote the matrix

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in SL(2,p)$$

and let $\overline{[a]}$ denote the equivalence class of $[a]$ in the group $PSL(2,p)$. Since any automorphism of $J \times SL(2,p)$ induces one on

$$\frac{J \times SL(2,p)}{Z(J \times SL(2,p))} \cong \frac{SL(2,p)}{Z(SL(2,p))} \cong PSL(2,p),$$

we can show that there are algebraically distinct weight elements in $J \times SL(2,p)$ by showing that their projections in $PSL(2,p)$ are algebraically distinct. But the order of $\overline{[a]} \in PSL(2,p)$ is the same as the order of a in the multiplicative group $\mathbf{Z}_p^*$ of the field with p elements $\mathbf{Z}_p$; and, the order of $\overline{[a]}$ is the same as the order of $\overline{[a]}^{-1}$. Moreover, since $\mathbf{Z}_p^*$ is cyclic, given any divisor of its order $p-1$, there is an element in $\mathbf{Z}_p^*$ of that order. The Theorem then follows once it is noted that $\limsup\{\tau(p-1) | p \text{ prime}\} = +\infty$ where $\tau(p-1)$ denotes the number of divisors of $p-1$. But this follows from Dirichlet's Theorem, which implies that for any integer k, there is a prime of the form $1 + km$.

F. Gonzalez-Acuña [**G-A**] has pointed out that it follows from Huppert [**H**, Seite 646, Satz 25.7] that $H_2(SL(2, 2^{p-1})) = 0$ for p prime, $p \geq 5$. Also, $H_1(SL(2, 2^{p-1})) = 0$ since $SL(2, 2^{p-1})$ is simple. Dirichlet's Theorem can also be used here to show that $\limsup\{\tau(2^{p-1} - 1) | p \text{ prime}\} = +\infty$.

Thus, either of the classes of groups $SL(2,p)$, $SL(2, 2^{p-1})$ for p prime, $p \geq 5$, can be used in the role of G in the construction. This completes the proof of the theorem.

As in [**H-S**], any of the above n-disk knot exteriors can be modified by taking the boundary connected sum with an n-disk knot having arbitrarily prescribed Alexander polynomial in a single dimension k $(2 \leq k \leq n-1)$ and trivial Alexander polynomial elsewhere [Su]. This produces an infinite class of n-disk knot exteriors, each having indeterminacy index at least that of the original n-disk knot exterior. Thus we have the following

COROLLARY. *Let $n \geq 5$. Given a positive integer N, there exist infinitely many homeomorphically distinct n-disk knot exteriors, each having indeterminacy index greater than N.*

NOTE. We have recently learned that F. Gonzalez-Acuña and S. Plotnick (independently) have produced examples with $\varsigma(Y) = +\infty$ (private communications).

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