PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 1, September 1982

RIEMANNIAN METRICS INDUCED BY TWO IMMERSIONS

M. DO CARMO AND M. DAJCZER

ABSTRACT. We consider the situation where a Riemannian manifold M^n can be isometrically immersed into spaces $N^{n+1}(c)$ and $N^{n+q}(\tilde{c})$ with constant curvatures $c < \tilde{c}$, $q \leq n-3$, and show that this implies the existence, at each point $p \in M$, of an umbilic subspace $U_p \subset T_pM$, for both immersions, with $\dim U_p \geq n-q$. In particular, if M^n can be isometrically immersed as a hypersurface into two spaces of distinct constant curvatures, M^n is conformally flat.

1. Introduction.

(1.1) An n-dimensional Riemannian manifold M^n is *conformally flat* if, for each point $p \in M$, there exists a conformal diffeomorphism of a neighborhood of p onto an open set of the euclidean space R^n. Let $x\colon M^n \to \overline{M}^k$ be an immersion of a differentiable manifold M^n into a Riemannian manifold $\overline{M}^k$, and let $\alpha\colon T_pM \times T_pM \to (T_pM)^\perp$ be the second fundamental form of x at $p \in M$; here $(T_pM)^\perp$ is the orthogonal complement of $dx_p(T_pM)$ in $T_{x(p)}\overline{M}$. We say that $U_p \subset T_pM$ is an *umbilic subspace* of x at p if $\langle \alpha(X,Y), \xi\rangle = \lambda\langle X,Y\rangle$, $\lambda = \text{const}$, for all $X \in U_p$, all $\xi \in (T_pM)^\perp$ and all $Y \in T_pM$, where $\langle\ ,\ \rangle$ denotes both the Riemannian metric on $\overline{M}$ and the Riemannian metric on M induced by x. We will denote by $\overline{M}^k(x)$ a k-dimensional Riemannian manifold with constant sectional curvature c. It is well known that if $n \geq 4$ and $x\colon M^n \to \overline{M}^{n+1}(c)$ is an immersion, the metric induced on M^n by x is conformally flat iff, for each $\in M$, there exists an umbilic subspace $U_p \subset T_pM$ with $\dim U_q \geq n-1$. We will prove the following local theorem.

(1.2) THEOREM. *Let M^n be a Riemannian manifold. Assume that M^n can be isometrically immersed in both $\overline{M}^{n+1}(c)$ and $\overline{M}^{n+1}(\tilde{c})$, $\tilde{c} > c$, $q \leq n-3$. Then, for each $p \in M$, there exists an umbilic subspace $U_p \subset T_pM$ of both immersions with* $\dim U_p \geq n-q$.

(1.3) COROLLARY. *Let M^n, $n \geq 4$, be a Riemannian manifold. Assume that M^n can be isometrically immersed in both $\overline{M}^{n+1}(c)$ and $\overline{M}^{n+1}(\tilde{c})$, $\tilde{c} \neq c$. Then M^n is conformally flat.*

(1.4) REMARK. Corollary (1.3) is, in a certain sense, the strongest restriction that can be expected under its hypothesis. In fact, it can be shown [1] that if an immersion $x\colon M^n \to \overline{M}^{n+1}(\tilde{c})$ of a differentiable manifold M^n induces on M^n a general conformally flat Riemannian metric, then, with such a metric, M^n can be isometrically immersed into a $\overline{M}^{n+1}(c)$, for any $c < \tilde{c}$.

Received by the editors December 1, 1980.
1980 *Mathematics Subject Classification*. Primary 53C25; Secondary 53A20.
Key words and phrases. Isometric immersions, umbilic subspaces, conformally flat Riemannian manifolds.

(1.5) REMARK. The bound on the dimension of U_p given by Theorem (1.2) is sharp as shown by the following example. Let $n > 3$ and let $M^n = S_1^{n-2} \times R^2$ be the Riemannian product of the sphere S_1^{n-2} of curvature one with the euclidean space R^2. As the first immersion $x_1: M^n \to R^{n+1}$, take the product immersion of the canonical embedding $i_{n-1}: S_1^{n-2} \subset R^{n-1}$ with the identity map $R^2 \subset R^2$. To define the second immersion, consider the map $f: R^2 \to R^4$ obtained by composing the immersion of R^2 as a flat torus in S_1^3 with $i_4: S_1^3 \subset R^4$. By taking the product immersion $i_{n-1} \times f: S_1^{n-2} \times R^2 \to R^{n+3}$, it is easily checked that $i_{n-1} \times f(S^{n-2} \times R^3)$ is contained in a sphere of radius, say, $1/\sqrt{\tilde{c}}$, of R^{n+3}. This gives an immersion $x_2: M^n \to S_{\tilde{c}}^{n+2}$, and clearly there exists, for all $p \in M$, an umbilic subspace $U_p = T_p(S_1^{n-2})$ of both immersions, with $\dim U_p = n - 2$.

(1.6) REMARK. As the proof of Theorem (1.2) will show, we actually obtain that $\dim U_p \geq n - l$, where $l \leq q$ is the dimension of the first normal space at p of the second immersion.

By imposing further restrictions on the immersions of Corollary (1.3) we can characterize those Riemannian metrics that will arise there.

(1.7) CORLLLARY. *Let M^n be as in the hypothesis of Corollary* (1.3). *Assume further that the first immersion has constant mean curvature H and that the second immersion has constant mean curvature $\tilde{H}$. Then M^n is either a space of constant curvature $M^n(a)$ or a Riemannian product $M^{n-1}(a) \times R$.*

PROOF OF (1.7). Let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM such that $e_1, \ldots, e_{n-1}$ is a basis for the umbilic subspace $U_p \subset T_pM$ of both immersions given by Theorem (1.2). Relative to such a basis, let λ, μ (resp. β, γ) be the eigenvalues of the second fundamental form of the first (resp. second) immersion. From Gauss' equations, we easily see that the fact that H and $\tilde{H}$ are constant implies that λ and μ are also constant. The result follows from Ryan [**3**, p. 373].

2. Proof of Theorem (1.2).

(2.1) We will use the theory of flat bilinear forms as developed by J. D. Moore in [**2**, pp. 459–465]. We recall that, given vector spaces V, W, a W-valued bilinear form $\beta: V \times V \to W$ is *flat* relative to a real inner product $(\, , \,): W \times W \to R$ if

$$(\beta(x,z), \beta(y,w)) - (\beta(x,w), \beta(y,z)) = 0,$$

for all $x, y, z, w \in V$. The *nullity* $N(\beta)$ of β is

$$N(\beta) = \{n \in V; \beta(x,n) = 0, \text{ for all } x \in V\},$$

and β is *null* if

$$(\beta(x,y), \beta(z,w)) = 0, \quad \text{all } x,y,z,w \in V.$$

Now let $x_1: M^n \to \overline{M}^{n+1}(c)$ and $x_2: M^n \to \overline{M}^{n+q}(\tilde{c})$ be the two immersions referred to in the statement, and denote by $\langle \, , \, \rangle$, $\langle \, , \, \rangle_1$, $\langle \, , \, \rangle_2$ the Riemannian inner products of M^n, $\overline{M}^{n+1}(c)$, $\overline{M}^{n+q}(\tilde{c})$, respectively. Fix throughout the proof a point $p \in M$ and let

$$\alpha_1: T_pM \times T_pM \to (T_p(M))_1^{\perp}, \quad \alpha_2: T_pM \times T_pM \to (T_pM)_2^{\perp}$$

be the second fundamental forms of x_1, x_2, respectively. Let N_i be the first normal space of x_i, i.e.,

$$N_i = \text{span}\{\eta \in (T_pM)_i^{\perp}; \eta = \alpha_i(X,Y), X, Y \in T_pM\}, \qquad i = 1,2.$$

Set $W = N_1 \oplus R \oplus N_2$, and define a Lorentzian inner product $(\ ,\ )$ in W by requiring that $(\ ,\ ) = -\langle\ ,\ \rangle_1$ in N_1, $(\ ,\ ) = \langle\ ,\ \rangle_2$ in N_2, and that the direct summands of W are pairwise orthogonal. Define a bilinear form $\beta\colon T_pM \times T_pM \to W$ by

$$\beta(X,Y) = \alpha_1(X,Y) + \sqrt{\tilde{c}-c}\langle X,Y\rangle\varsigma + \alpha_2(X,Y), \qquad X,Y \in T_pM,$$

where η is a generator of R with $(\eta,\eta) = 1$.

It follows from Gauss' equations for x_1 and x_2 that β is flat and, since $\tilde{c} - c > 0$, $N(\beta) = 0$. Notice that α_1 is not zero; otherwise, the inner product $(\ ,\ )$ in W is positive definite, hence (Moore [**2**, p. 463, Corollary 1])

$$0 = \dim N(\beta) \geq \dim V - \dim W \geq n - (q+2) \geq 1,$$

which is a contradiction.

Let N be a vector that generates N_1 with $\langle N,N\rangle_1 = -(N,N) = 1$.

Assertion. There exists a unit vector $\eta_0 \in R \oplus N_2$ *such that*

$$(\alpha_1(X,Y),N) = (\sqrt{\tilde{c}-c}\langle X,Y\rangle\varsigma + \alpha_2(X,Y),\eta_0). \tag{2.2}$$

To prove the assertion, we use the fact that (Moore [**2**, p. 464, Corollary 3]) W has a direct sum decomposition $W = W_1 \oplus W_2$ such that the restrictions of $(\ ,\ )$ to W_1 and W_2 are nondegenerate, and if β_1 and β_2 are the W_1- and W_2-components of β, respectively, then β_1 is null and $\dim N(\beta_2) \geq \dim T_pM - \dim W_2$.

It follows that β_1 is not zero; otherwise, $\beta = \beta_2$ and

$$0 = \dim N(\beta) = \dim N(\beta_2) \geq n - 1.$$

Since β_1 is null, the restriction of $(\ ,\ )$ to W_1 is Lorentzian. Thus $\dim W_1 \geq 2$, and we can choose bases $e_1,\dots,e_k$ of W_1 and $\delta_0,\dots,\delta_{l+1}$ of $R \oplus N_2$, such that

$$e_1 = \cosh\varphi N + \sinh\varphi\delta_1, \qquad e_2 = \delta_2,\dots,e_k = \delta_k.$$

Thus, by writing

$$\beta_1(X,Y) = -(\beta(X,Y),e_1)e_1 + \sum_{\alpha=2}^{k}(\beta(X,Y),e_\alpha)e_\alpha$$

we obtain that the condition for β_1 to be null is

$$(\beta(X,Y),e_1)(\beta(Z,W),e_1) = \sum_{\alpha=2}^{k}(\beta(X,Y),e_\alpha(\beta(Z,W),e_\alpha), \tag{2.3}$$

$X,Y,Z,W \in T_pM$. Define linear maps B and D_α of T_pM, $\alpha = 2,\dots,k$, by

$$\langle BX,Y\rangle = (\beta(X,Y),e_1), \qquad \langle D_\alpha X,Y\rangle = (\beta(X,Y),e_\alpha).$$

Then (2.3) is equivalent to

$$\langle BX,Y\rangle\langle BZ,W\rangle = \sum_\alpha\langle D_\alpha X,Y\rangle\langle D_\alpha Z,W\rangle. \tag{2.4}$$

We need the following lemma of linear algebra.

(2.5) LEMMA. *Let V be a finite-dimensional real vector space with a positive definite inner product $\langle\ ,\ \rangle$ and let B and D_α, $\alpha = 2,\dots,k$, be selfadjoint linear maps of V such that (2.4) holds for all $X,Y,Z,W \in V$. Then there exist real numbers c_α with $\sum_\alpha c_\alpha^2 = 1$ such that $B = \sum_\alpha c_\alpha D_\alpha$.*

PROOF OF THE LEMMA. Let $Z_1, \dots, Z_n$, $n = \dim V$, be a basis of V that diagonalizes B, i.e., $\langle BZ_i, Z_j\rangle = \lambda_i \delta_{ij}$, $i, j = 1, \dots, n$. Then by (2.4)

$$\langle BZ_i, Z_j\rangle^2 = \sum_\alpha \langle D_\alpha Z_i, Z_j\rangle^2.$$

It follows that $Z_1, \dots, Z_n$ diagonalizes each D_α. Define γ_i^α by $D_\alpha Z_i = \gamma_i^\alpha Z_i$. Then, again by (2.4),

$$\lambda_i \lambda_j = \sum_\alpha \gamma_i^\alpha \gamma_j^\alpha. \tag{2.6}$$

We first notice form (2.6) that if some $\lambda_i = 0$ then $\gamma_i^\alpha = 0$ for all α. Next, set $\gamma_i = (\gamma_i^1, \dots, \gamma_i^k) \in R^k$ and notice that (3.6) means that, in the usual inner product $\cdot$ of R^k,

$$|\gamma_i \cdot \gamma_j|^2 = \|\gamma_i\|^2 \|\gamma_j\|^2.$$

It follows that

$$\frac{\gamma_i^\alpha}{\gamma_j^\alpha} = \frac{\gamma_i^\alpha \gamma_j^\alpha}{(\gamma_j^\alpha)^2} = \frac{\gamma_i^\beta \gamma_j^\beta}{(\gamma_j^\beta)^2} = \frac{\sum_\alpha \gamma_i^\alpha \gamma_j^\alpha}{\sum_\alpha (\gamma_j^\alpha)^2} = \frac{\lambda_i}{\lambda_j},$$

hence, by setting $\gamma_i^\alpha / \lambda_i = \gamma_j^\alpha / \lambda_j = c_\alpha$, we obtain from (2.6) that $\sum_\alpha c_\alpha^2 = 1$. Finally, since $(c_\alpha \gamma_i^\alpha)/\lambda_i = c_\alpha^2$, we obtain that $\lambda_i = \sum_\alpha c_\alpha \gamma_i^\alpha$ and this proves Lemma (2.5).

(2.7) To complete the proof of the assertion, we set for convenience

$$\bar{\alpha}(X, Y) = \sqrt{\tilde{c} - c}\langle X, Y\rangle \varsigma + \alpha_2(X, Y),$$

and notice that Lemma (2.5) implies that

$$-\cosh \varphi(\alpha_1(X, Y), N) + \sinh \varphi(\bar{\alpha}(X, Y), \delta_1) = \sum_{\alpha=2}^k c_\alpha(\bar{\alpha}(X, Y), \delta_\alpha).$$

Thus

$$(\alpha_1(X, Y), N) = \left(\bar{\alpha}(X, Y), \frac{\sum_{\alpha=2}^k c_\alpha \delta_\alpha - \sinh \varphi \delta_1}{-\cosh \varphi}\right) = (\bar{\alpha}(X, Y), \eta_0),$$

where

$$\eta_0 = \frac{\sinh \varphi \delta_1 - \sum_{\alpha=2}^k c_\alpha \delta_\alpha}{\cosh \varphi}$$

is easily seen to have norm one. This proves the assertion.

(2.8) We now complete the proof of Theorem (1.2). Choose an orthonormal basis $\bar{\eta}_1, \eta_2, \dots, \eta_l$ of N_2 so that

$$\eta_0 = \sin \theta \eta + \cos \theta \bar{\eta}_1. \tag{2.9}$$

Let $\eta_1 = \cos \theta \eta - \sin \theta \bar{\eta}_1$ and set

$$R^l = \operatorname{span}\{\eta_1, \dots, \eta_l\}.$$

Define a bilinear form $\gamma\colon T_pM \times T_pM \to R^l$ by

$$\gamma(X, Y) = (\sqrt{\tilde{c} - c}\langle X, Y\rangle \eta + \alpha_2(X, Y), \eta_1)\eta_1 + \sum_{j=2}^l \langle \alpha_2(X, Y), \eta_j\rangle_2 \eta_j.$$

By Gauss' equations for x_1 and x_2, γ is flat relative to the (Riemannian) inner product obtained by restricting (,) to R^l. It follows from Moore ([**2**, p. 463, Corollary 1]) thadim $N(\gamma) \geq n - l \geq n - q$. On the other hand, $X \in N(\gamma)$ if and only if, for all $Y \in T_pM$, both conditions below are satisfied:

$$(2.10) \qquad \begin{cases} \text{(i)} & \cos\theta\sqrt{\tilde{c}-c}\langle X,Y\rangle - \sin\theta\langle\alpha_2(X,Y),\bar{\eta}_1\rangle_2 = 0, \\ \text{(ii)} & \langle\alpha_2(X,Y),\eta_j\rangle_2 = 0, \qquad j \geq 2. \end{cases}$$

Notice that by (2.2) and (2.9)

$$(2.11) \qquad (\alpha_1(X,Y),N) = \sqrt{\tilde{c}-c}\langle X,Y\rangle\sin\theta + \langle\alpha_2(X,Y),\bar{\eta}_1\rangle_2\cos\theta.$$

It follows by (2.10) (i) that $\sin\theta \neq 0$, and by (2.11) that we can assume that $\cos\theta \neq 0$ (otherwise the whole T_pM is an umbilic subspace). Thus $X \in N(\gamma)$ if and only if, for all $Y \in T_pM$,

$$\alpha_2(X,Y) = \operatorname{cotg}\theta\sqrt{\tilde{c}-c}\langle X,Y\rangle\bar{\eta}_1,$$

and by (2.11) this is equivalent to

$$\alpha_1(X,Y) = \frac{\sqrt{\tilde{c}-c}}{\cos\theta}\langle X,Y\rangle.$$

Therefore, $N(\gamma) \subset T_pM$ is an umbilic subspace of both x_1 and x_2 with $\dim N(\gamma) \geq n - q$. This proves Theorem (1.2).

REFERENCES

1. M. do Carmo and M. Dajczer, *General conformally flat hypersurfaces* (preprint).
2. J. D. Moore, *Submanifolds of constant positive curvature*. I, Duke Math. J. **44** (1977), 449–484.
3. P. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Tôhoku Math. J. **21** (1969), 363-388.

INSTITUTO DE MATEMÁTICA PURA E APLICADA (IMPA), ESTRADA DONA CASTORINA, 110, CEP 22.460, RIO DE JANEIRO, RJ, BRAZIL