PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 2, May 1978

# QUASI-SIMILARITY OF WEAK CONTRACTIONS

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ABSTRACT. Let $T$ be a completely nonunitary (c.n.u.) weak contraction (in the sense of Sz.-Nagy and Foiaş). We show that $T$ is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. As a corollary, two c.n.u. weak contractions are quasi-similar to each other if and only if their $C_0$ parts and $C_{11}$ parts are quasi-similar to each other, respectively. We also completely determine when c.n.u. weak contractions and $C_0$ contractions are quasi-similar to normal operators.

Recall that a contraction $T$ on the Hilbert space $H$ is called a weak contraction if its spectrum $\sigma(T)$ does not fill the open unit disc $D$ and $1 - T^*T$ is of finite trace. Contained in this class are all contractions $T$ with finite defect index $d_T \equiv \dim \operatorname{rank}(1 - T^*T)^{1/2}$ and with $\sigma(T) \neq \bar{D}$ (cf. [**9**, p. 323]).

Assume that $T$ is a weak contraction which is also completely nonunitary (c.n.u.), that is, $T$ has no nontrivial reducing subspace on which $T$ is a unitary operator. For such a contraction, Sz.-Nagy and Foiaş obtained a $C_0$-$C_{11}$ decomposition and then found a variety of invariant subspaces which furnish its spectral decomposition (cf. [**9**, Chapter VIII]). In this note we are going to supplement other interesting properties of such contractions. We show that a c.n.u. weak contraction is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. Although the proof is not difficult, some of its interesting applications justify the elaboration here. An immediate corollary is that two such contractions are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other. This is, in turn, used to show that two quasi-similar weak contractions have equal spectra. Another interesting consequence is that a c.n.u. weak contraction is quasi-similar to a normal operator if and only if its $C_0$ part is. The latter can be shown to be equivalent to the condition that its minimal function is a Blaschke product with simple zeros, thus completely settling the question when a c.n.u. weak contraction is quasi-similar to a normal operator.

Before we start to prove our main theorem, we provide some background work for our notations and terminology. The main reference is [**9**].

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Presented to the Society September 9, 1976; received by the editors October 1, 1976.

*AMS (MOS) subject classifications* (1970). Primary 47A45.

*Key words and phrases*. Completely nonunitary weak contractions, quasi-similarity, $C_0$ parts, $C_{11}$ parts, normal operators.

[1]This research was done while the author was visiting Indiana University during the summer of 1976.

Let $T$ be an arbitrary contraction on $H$. Let $H_0 = \{h \in H: T^n h \to 0\}$, $H_0' = \{h \in H: T^{*n} h \to 0\}$, $H_1' = H \ominus H_0$ and $H_1 = H \ominus H_0'$. Note that $H_0$ and $H_0'$ are invariant for $T$ and $T^*$, respectively. Consider the triangulations of $T$ with respect to the orthogonal decompositions $H = H_0 \oplus H_1'$ and $H = H_1 \oplus H_0'$:

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & Y \\ 0 & T_0' \end{bmatrix}.$$

The triangulations are of type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix},$$

respectively (cf. [**9**, p. 73]). Recall that a contraction $T$ is of class $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n h \to 0$ (resp. $T^{*n} h \to 0$) as $n \to \infty$ for all $h$ and $T$ is of class $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n h \not\to 0$ (resp. $T^{*n} h \not\to 0$) as $n \to \infty$ for all $h \neq 0$. $T$ is of class $C_{00}$ if $T \in C_{0\cdot} \cap C_{\cdot 0}$ and of class $C_{11}$ if $T \in C_{1\cdot} \cap C_{\cdot 1}$. A c.n.u. contraction $T$ is said to be of class $C_0$ if there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$. In this case we can choose $u$ to be a minimal inner function in the sense that $u$ is an inner function such that $u(T) = 0$ and $u$ divides (in $H^\infty$) every other function $v \in H^\infty$ for which $v(T) = 0$. Such a function is called a minimal function for $T$ and is denoted by $m_T$. If $T$ is a c.n.u. weak contraction, then in the previous triangulations $T_0$ is of class $C_0$ and $T_1$ is of class $C_{11}$, called the $C_0$ part and the $C_{11}$ part of $T$ (cf. [**9**, p. 331]). Note that in this case we have $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$ (cf. [**9**, p. 332]). For arbitrary operators $T_1$, $T_2$ on $H_1$, $H_2$, respectively, $T_1 \prec T_2$ denotes that $T_1$ is a quasi-affine transform of $T_2$, that is, there exists a linear one-to-one and continuous transformation $S$ from $H_1$ onto a dense linear manifold in $H_2$ (called quasi-affinity) such that $ST_1 = T_2 S$. $T_1$ and $T_2$ are quasi-similar if $T_1 \prec T_2$ and $T_2 \prec T_1$.

Our main theorem is the following:

THEOREM 1. *Let $T$ be a c.n.u. weak contraction on $H$. Let $T_0$ and $T_1$ be the $C_0$ part and $C_{11}$ part of $T$. Then $T$ is quasi-similar to $T_0 \oplus T_1$.*

PROOF. Let $S$: $H_0 \oplus H_1 \to H$ be defined by $S(h_0 \oplus h_1) = h_0 + h_1$. Certainly $T$ is a continuous linear transformation. Since $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$, it is easily seen that $S$ is a quasi-affinity such that $S(T_0 \oplus T_1) = TS$. Thus $T_0 \oplus T_1 \prec T$. Note that $T^*$ is also a c.n.u. weak contraction and $T_0'^*$ and $T_1'^*$ are the $C_0$ and $C_{11}$ parts of $T^*$ (cf. [**9**, p. 332]). As above, we have $T_0'^* \oplus T_1'^* \prec T^*$. Hence $T \prec T_0' \oplus T_1'$, and $T_0 \oplus T_1 \prec T \prec T_0' \oplus T_1'$. Let $V$ be the quasi-affinity from $H_0 \oplus H_1$ to $H_0' \oplus H_1'$ such that $V(T_0 \oplus T_1) = (T_0' \oplus T_1')V$. Since $T_0$ and $T_1'$ are of class $C_{0\cdot}$ and $C_{1\cdot}$, respectively, it is easily seen that $VH_0 \subseteq H_0'$. Say,

$$V = \begin{bmatrix} V_0 & Z \\ 0 & V_1 \end{bmatrix}$$

is the corresponding triangulation. An easy calculation shows that $ZT_1 = T_0'Z$. Since $T_1$ is of class $C_{11}$ and $T_0'$ is of class $C_{00}$, we must have $Z = 0$ (cf. [4, Lemma 4.4]). Thus $V_0$ and $V_1$ are quasi-affinities satisfying $V_0T_0 = T_0'V_0$ and $V_1T_1 = T_1'V_1$. Hence $T_0 \prec T_0'$ and $T_1 \prec T_1'$. It follows from the uniqueness of the Jordan model for $C_0$ contractions that $T_0$ and $T_0'$ are quasi-similar to each other (cf. [2]). To show that $T_1$ is quasi-similar to $T_1'$, note that $T_1$ and $T_1'$, being $C_{11}$ contractions, are quasi-similar to unitary operators, say $U_1$ and $U_1'$, respectively. We have $U_1 \prec U_1'$. By a theorem of Douglas [5], $U_1$ and $U_1'$ are unitarily equivalent. Hence $T_1$ is quasi-similar to $T_1'$, and $T$ is quasi-similar to $T_0 \oplus T_1$.

An immediate corollary of Theorem 1 is

Corollary 1. *Let $T_1$ and $T_2$ be c.n.u. weak contractions. Then $T_1$ and $T_2$ are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other.*

Proof. The sufficiency follows immediately from Theorem 1. The necessity can be proved by a similar argument as in Theorem 1.

In particular, for c.n.u. contractions with scalar-valued characteristic functions, we have

Corollary 2. *For $j = 1, 2$, let $T_j$ be a c.n.u. contraction with the scalar-valued characteristic function $\psi_j \not\equiv 0$. Let $\psi_j = \psi_{ji}\psi_{je}$ be the canonical factorization into the product of its inner part $\psi_{ji}$ and outer part $\psi_{je}$, and let $E_j = \{e^{it}: |\psi_j(e^{it})| < 1\}$. Let*

$$T_j = \begin{pmatrix} T_{j1} & X_j \\ 0 & T_{j2} \end{pmatrix}$$

*be the triangulation of type*

$$\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}, \qquad j = 1, 2.$$

*Then the following are equivalent:*

(i) *$T_1$ is quasi-similar to $T_2$;*

(ii) *$T_{11}$ is quasi-similar to $T_{21}$ and $T_{12}$ is unitarily equivalent to $T_{22}$;*

(iii) *$\psi_{1i} = \psi_{2i}$ and $E_1$ and $E_2$ differ by a set of zero Lebesgue measure.*

Proof. Since $T_1$ and $T_2$ are c.n.u. weak contractions, the equivalence of (i) and (ii) follows from Corollary 1. Note that $T_{j1}$ is quasi-similar to the multiplication by $e^{it}$ on the space $L^2(E_j)$ and $T_{j2}$ is unitarily equivalent to the compression of the shift $S(\psi_{ji})$ on $H^2 \ominus \psi_{ji}H^2, j = 1, 2$. Thus the equivalence of (ii) and (iii) follows immediately.

The equivalence of (i) and (iii) in Corollary 2 is compatible with the result of Kriete [7] that $T_1$ is similar to $T_2$ if and only if $\psi_1/\psi_2, \psi_2/\psi_1 \in H^\infty$ and $E_1$ and $E_2$ differ by a set of zero Lebesgue measure.

Corollary 3. *Let $T_1$ and $T_2$ be c.n.u. weak contractions. If $T_1$ and $T_2$ are* *quasi-similar to each other, then* $\sigma(T_1) = \sigma(T_2)$.

PROOF. For $j = 1, 2$, let $T_{j0}$ and $T_{j1}$ be the $C_0$ part and $C_{11}$ part of $T_j$. By Corollary 1, $T_{10}$ and $T_{11}$ are quasi-similar to $T_{20}$ and $T_{21}$, respectively. Since the spectrum of a $C_0$ contraction is completely determined by its minimal function [9, p. 126], and $T_{10}$ and $T_{20}$ have the same minimal function, we have $\sigma(T_{10}) = \sigma(T_{20})$.

To show that $\sigma(T_{11}) = \sigma(T_{21})$, let $U_j$ be the residual part of the minimal unitary dilation of $T_{j1}, j = 1, 2$ (cf. [9, p. 61]). Note that $T_{j1}$ is quasi-similar to $U_j$ and $\sigma(T_{j1})$ lies entirely on the unit circle (cf. [9, pp. 75, 328]). It follows that $\sigma(T_{j1}) = \sigma(U_j)$ (cf. [9, pp. 311–312]). By Douglas' theorem [5], $U_1$ and $U_2$ are quasi-similar implies they are unitarily equivalent. Thus $\sigma(T_{11}) = \sigma(U_1) = \sigma(U_2) = \sigma(T_{21})$. Since $\sigma(T_j) = \sigma(T_{j0}) \cup \sigma(T_{j1})$ [9, p. 332], we have $\sigma(T_1) = \sigma(T_2)$, completing the proof.

We remark that the proof can be modified to show that quasi-similar weak contractions (not necessarily c.n.u.) have equal spectra. This result is not new. It also follows from the facts that weak contractions are decomposable [6] and quasi-similar decomposable operators have equal spectra [3]. However, our proof seems more direct.

In the remaining part of this note we are concerned with the question when a c.n.u. weak contraction is quasi-similar to a normal operator. The next theorem reduces the problem to the $C_0$ part of the c.n.u. weak contraction.

THEOREM 2. *Let T be a c.n.u. weak contraction on H. Let*

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix}$$

*be the triangulation of type*

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$$

*on the (orthogonal) decomposition* $H = H_0 \oplus H_1'$. *Then T is quasi-similar to a normal operator if and only if* $T_0$ *is.*

PROOF. The sufficiency follows trivially from Theorem 1. To prove the necessity, we may assume that $T$ is quasi-similar to a normal operator $N$ on the space $K$ with $\|N\| \leqslant \|T\| \leqslant 1$ (cf. [1, Proof of the sufficiency part of Theorem]). Let $K = K_1 \oplus K_2$ be the direct sum of reducing subspaces for $N$ such that $N_1 \equiv N|K_1$ is c.n.u. and $N_2 \equiv N|K_2$ is unitary. Let $S$ be the quasi-affinity from $H$ to $K$ such that $ST = NS$. Since $T_0$ is of class $C_{0\cdot}$ and $N_2$ is of class $C_{11}$, it is easily seen that $SH_0 \subseteq K_1$. Note that $\overline{SH_0}$ is an invariant subspace for $N_1$. Let $N_1' = N_1|\overline{SH_0}$. Then $S_1 \equiv S|H_0$ is a quasi-affinity from $H_0$ to $\overline{SH_0}$ satisfying $S_1 T_0 = N_1' S_1$. Since $T_0$ is of class $C_0$, so is $N_1'$ (cf. [9, p. 125]). By the uniqueness of the Jordan model for $C_0$ contractions, we have $T_0$ is quasi-similar to $N_1'$ (cf. [2]). Since $N_1'$ is subnormal and $\sigma(N_1')$ has planar area zero (cf. [**9**, p. 126]), it follows from Putnam's theorem [**8**] that $N_1'$ is normal. This completes the proof.

Notice that Theorem 2 is compatible with the result that $T$ is similar to a normal operator if and only if $T_0$ is similar to a normal operator and $T_1'$ is similar to a unitary operator. This is true even for an arbitrary c.n.u. contraction (cf. [**10**, Theorem 3]).

Since the $C_0$ part of a c.n.u. weak contraction is a $C_0$ contraction, the next theorem furnishes the complete solution to the previously posed question.

THEOREM 3. *Let $T$ be a $C_0$ contraction on the space $H$ with the minimal function $m_T$. Then $T$ is quasi-similar to a normal operator if and only if $m_T$ is a Blaschke product with simple zeros.*

PROOF. *Necessity*. Let $T$ be quasi-similar to the normal operator $N$ on the space $K$ and let $S$ be the quasi-affinity from $H$ to $K$ such that $ST = NS$. As before we may assume that $\|N\| \leqslant \|T\| \leqslant 1$ (cf. [**1**]). Now we show that $N$ must be c.n.u. Indeed, for any $k \in K$ and $\varepsilon > 0$, let $h \in H$ be such that $\|k - Sh\| < \varepsilon$. Since $ST^nh = N^nSh \to 0$ as $n \to \infty$, we have $\|N^nSh\| < \varepsilon$ for all $n \geqslant N_0$. Hence

$$\begin{aligned}\|N^nk\| &\leqslant \|N^nk - N^nSh\| + \|N^nSh\| \leqslant \|N\|^n\,\|k - Sh\| + \|N^nSh\| \\ &< \varepsilon + \varepsilon = 2\varepsilon \quad \text{for all } n \geqslant N_0.\end{aligned}$$

This shows that $N^nk \to 0$ for all $k \in K$ and hence $N$ is c.n.u. Since $N$ is quasi-similar to a $C_0$ contraction, $N$ is also a $C_0$ contraction with the same minimal function $m_N = m_T$ (cf. [**9**, p. 125]). Let $m_T = Bs$, where

$$B(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|}\left(\frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i\lambda}\right)^{n_i}$$

is a Blaschke product and $s$ is a singular function. Note that $\lambda_i$ is a characteristic value of $N$ with index $n_i$ (cf. [**9**, p. 135]). Since $N$ is a normal operator, $n_i = 1$ for all $i$. Let $K_i$ be the corresponding eigenspace. Then $\bigvee_i K_i$ reduces $N$ and the normal operator $N_1 \equiv N|(\bigvee_i K_i)^\perp$ has no eigenvalue. Hence the minimal function of the $C_0$ contraction $N_1$ must be $s$ (cf. [**9**. p. 129]). It follows that $\sigma(N_1)$ is contained in the unit circle, and thus $N_1$ is a unitary operator. Since $N$ is c.n.u., we must have $(\bigvee_i K_i)^\perp = \{0\}$ and $K = \bigvee_i K_i$. Hence $m_T = B$ is a Blaschke product with simple zeros (cf. [**9**, p. 135]).

*Sufficiency*. Assume that $m_T$ is a Blaschke product with simple zeros, say,

$$m_T(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|}\;\frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i\lambda},$$

where the distinct $\lambda_i$'s satisfy $|\lambda_i| < 1$ and $\sum_i(1 - |\lambda_i|) < \infty$. For each $i$ let $H_i = \{h \in H\colon (T - \lambda_i)h = 0\}$. Then $T|H_i$ is a normal operator and the system $\{H_i\}_{i=1}^\infty$ of invariant subspaces satisfies

$$H = H_i \dotplus \bigvee_{j \neq i} H_j \quad \text{for each } i, \quad \text{and} \quad \bigcap_i \Big(\bigvee_{j > i} H_j\Big) = \{0\}$$

(cf. [**9**, pp. 135, 131]). That is, $\{H_i\}_{i=1}^{\infty}$ is a basic system of invariant subspaces for $T$. By a result of Apostol [**1**], $T$ is quasi-similar to a normal operator, completing the proof.

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