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COMPLEMENTARY COMPONENTS OF POLYNOMIAL HULLS[1]

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ABSTRACT. Let X be a compact subset of the unit sphere S in $\mathbf{C}^n$, $n > 1$. It is shown that if a point z is not in the polynomially convex hull of X, then there is a complementary component U of X in S such that z is not in the hull of $S \setminus U$.

Alexander [1] has demonstrated several consequences of Browder's Theorem that $H^i(X) = 0$ when X is a polynomially convex subset of $\mathbf{C}^n$ and $i \geqslant n$. One of these is that the polynomially convex hull of a compact subset of a sphere (in $\mathbf{C}^n$, $n \geqslant 2$) has the same number of complementary components inside the sphere as the set itself has on the sphere. Here we use this result to further elaborate the role of complementary components in determining the hull of such a set.

Fix $n \geqslant 2$ and let $S = \{z \in \mathbf{C}^n\colon |z| = (\Sigma|z_j|^2)^{1/2} = 1\}$, $B = \{z \in \mathbf{C}^n\colon |z| < 1\}$. For a compact subset X of $\mathbf{C}^n$, let $\hat{X}$ denote the polynomially convex hull of X. For $Y \subseteq \mathbf{C}^n$, $\overline{Y}$ is the closure of Y and ∂Y is the boundary of Y relative to $\mathbf{C}^n$. If $Y \subseteq S$, $\partial_S Y$ will denote the boundary of Y relative to S.

Our main result is:

THEOREM. *If $U_1, \ldots, U_N$ ($1 \leqslant N \leqslant \infty$) are disjoint open subsets of S, then*

$$(S \setminus \cup U_j)\hat{} = \cap (S \setminus U_j)\hat{}.$$

Thus in order to compute the hull of a subset X of S, it is sufficient to be able to describe the hulls of subsets of S with connected complements, for we can now write $\hat{X} = \cap (S \setminus U_j)\hat{}$, taking $U_1, U_2, \ldots$ to be the components of $S \setminus X$. Before proceeding to the proof we mention some immediate consequences.

COROLLARY 1. *Suppose that $\Delta_1, \ldots, \Delta_N$ ($1 \leqslant N \leqslant \infty$) are disjoint open "spherical caps", i.e., for each j there is a $z_j \in S$ and an ε_j, $0 < \varepsilon_j < 2$, with $\Delta_j = \{z \in S\colon |z - z_j| < \varepsilon_j\}$. Then*

$$(S \setminus \cup \Delta_j)\hat{} = \textit{linear convex hull}\,(S \setminus \cup \Delta_j).$$

PROOF. For each j, $(S \setminus \Delta_j)\hat{} =$ linear convex hull $(S \setminus \Delta_j)$.

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COROLLARY 2. *If X is a compact subset of S and if none of the complementary components of X extend over more than a hemisphere of S, then* $0 \in \hat{X}$.

PROOF. If H is an open hemisphere of S, then $0 \in (S \setminus H)\hat{}$.

COROLLARY 3. *If T is a compact totally disconnected subset of S, then* $(S \setminus T)\hat{} = \bar{B} \setminus T$.

PROOF. Let $x \in B$. Choose $\varepsilon > 0$ so that for any spherical cap

$$\Delta = \{\eta \in S: |\eta - z| < \varepsilon\}$$

with $z \in S$, $x \in (S \setminus \Delta)\hat{}$. Cover T by such sets $\Delta_1, \ldots, \Delta_N$. Since T is totally disconnected, there are disjoint open subsets of S, say $U_1, \ldots, U_N$, with $U_j \subseteq \Delta_j$ for each j and with U_j still covering T. Then $X = S \setminus \bigcup U_j$ is a compact subset of $S \setminus T$, and from the theorem $x \in \hat{X}$. Thus $x \in (S \setminus T)\hat{}$.

The main result needed to prove the theorem is the following lemma, which makes essential use of Alexander's result.

LEMMA. *If X is a compact subset of S, then* $\partial\hat{X} \cap B \subseteq [\partial_S X]\hat{}$.

PROOF. If the lemma is false, then there is a compact set $X \subseteq S$, a point $x \in \partial\hat{X} \cap B$, and a polynomial p with

$$p(x) = 1 + \delta, \qquad \delta > 0;$$
$$|p| \leqslant 1 \quad \text{on } \partial_S X.$$

We will show that this leads to a contradiction.

Let $K = \{z \in S \mid |p(z)| \leqslant 1 + \delta/2\}$, and note that $\partial_S X \subseteq K$. Let V be the component of $B \setminus \hat{K}$ which contains x. From the proof of Theorem 3 in [1], it follows that there is a unique component U of $S \setminus K$ for which $U \cap \bar{V} \neq \varnothing$. Since U is connected and $U \subseteq S \setminus K \subseteq S \setminus \partial_S X$, either $U \subseteq X$ or $U \subseteq S \setminus X$. We consider these two possibilities separately.

Case I. Suppose $U \subseteq X$. Let $z \in V$. Let $W = \{\eta \in \mathbf{C}^n: p(\eta) = p(z)\}$. By the maximum principle for analytic varieties, we have that for all polynomials q, $|q(z)| \leqslant \max_{W \cap \partial V}|q|$. But $\partial V \subseteq \bar{U} \cup \{\eta \in B: |p(\eta)| = 1 + \delta/2\}$, whence $|q(z)| \leqslant \max_{\bar{U}}|q| \leqslant \max_X|q|$. Thus $x \in V \subseteq \hat{X}$; but this contradicts $x \in \partial\hat{X}$.

Case II. Suppose $U \subseteq S \setminus X$. Let $L = \hat{X} \cap \bar{V}$, and let $m = \max_L |p|$. Since $x \in L$, $m \geqslant 1 + \delta$. Choose $y \in L$ such that $|p(y)| = m$, and let

$$T = \{z \in L: p(z) = p(y)\}.$$

Observe that $T \subseteq \hat{X} \cap V$, since $|p| \geqslant 1 + \delta$ on T while $\partial V \subseteq \bar{U} \cup \{z \in B: |p(z)| = 1 + \delta/2\}$ and $\hat{X} \cap \bar{U} \subseteq \partial_S X$ where $|p| \leqslant 1$. But then T is a local peak set for $P(\hat{X})$, hence a peak set for $P(\hat{X})$, which contradicts $T \subseteq V \subseteq B$.

PROOF OF THEOREM. Observe that it suffices to prove the theorem for the case $N < \infty$, since the infinite case can be obtained from the finite one by taking a decreasing intersection. It is trivial that $(S \setminus \bigcup U_j)\hat{} \subseteq \bigcap (S \setminus U_j)\hat{}$. To prove the reverse inclusion it is evidently sufficient to show that $x \in \partial[\bigcap(S \setminus U_j)\hat{}] \cap B$ implies $x \in (S \setminus \bigcup U_j)\hat{}$. Since $N < \infty$, there is some j

for which $x \in \partial[(S \setminus U_j)\hat{}] \cap B$, so by the lemma $x \in [\partial_S(S \setminus U_j)]\hat{}$. Since the U_j are disjoint, $\partial_S(S \setminus U_j) \subseteq S \setminus \bigcup U_j$, so $x \in (S \setminus \bigcup U_j)\hat{}$ as desired.

References

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