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GENERALIZED FREDHOLM OPERATORS AND THE BOUNDARY OF THE MAXIMAL GROUP OF INVERTIBLE OPERATORS

G. W. TREESE AND E. P. KELLY, JR.

ABSTRACT. Let V denote an infinite dimensional Banach space over the complex field and let $G[V]$ denote the subset of bounded operators on V with the property that the null space has a closed complement and the range is closed, where the null space and range are proper subspaces of V. Necessary and sufficient conditions for $T \in G[V]$ to be in the boundary, $\mathfrak{B}$, of the maximal group, $\mathfrak{M}$, of invertible operators are determined. As a result, $\mathfrak{B} \cap G[V]$ is the set of products of operators in $\mathfrak{M}$ and operators in $\mathcal{P}$, where $\mathcal{P}$ is the set of projections other than the identity operator and null operator.

1. Introduction. Let $\mathfrak{A}$ denote a Banach algebra, let $\mathfrak{M}$ denote the maximal group of invertible elements of $\mathfrak{A}$ and let $\mathfrak{B}$ denote the boundary of $\mathfrak{M}$. The problem to which this paper is addressed is the determination of necessary and sufficient conditions for an element of $\mathfrak{A}$ to be an element of $\mathfrak{B}$. Rhoades **[9]** considered this problem in the setting in which $\mathfrak{A}$ is the Banach algebra of conservative, infinite, triangular matrices. Sufficient conditions for such matrices to be in $\mathfrak{B}$ were given. In **[7]**, Kelly and Hogan considered this problem in the setting in which $\mathfrak{A}$ is the Banach algebra of bounded linear operators defined on an infinite dimensional Banach space V that leave a closed subspace of V invariant. Again, only sufficient conditions for such an operator to be in $\mathfrak{B}$ were given. This problem was solved by Feldman and Kadison **[5]**, if $\mathfrak{A}$ is the ring of operators on a Hilbert space. Herein, $\mathfrak{A}$ will denote the Banach algebra $B[V]$, of bounded linear operators defined on an infinite dimensional Banach space V over the complex field.

The terminology generalized Fredholm operator was used by Caradus **[4]** for those operators in $B[V]$ which have the property that both the null space and the range are closed complemented subspaces of V. This set of operators contains the Fredholm operators and, following Caradus, is denoted by $GF[V]$. It is well known **[4]**, **[8]** that this set of operators is the same as the set of operators in $B[V]$ which have generalized inverses in the sense that T in $B[V]$ is said to have a generalized inverse S in $B[V]$ if and only if $TST = T$ and $STS = S$. Also, by way of reference, in a fundamental paper, Atkinson

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[1] called such operators, i.e., those with generalized inverses, relatively regular. In [1] it was observed that the set of relatively regular operators includes those operators which have inverses in $B[V]$, those operators which have right or left inverses in $B[V]$ and the projection operators. Moreover, in [1] it was shown that generalized nilpotent operators and those completely continuous operators which are not finite-dimensional are not relatively regular.

Now since projections other than the identity and generalized nilpotent operators are both types of operators in $\mathcal{B}$ [7], it is natural to ask what relation exists between $\mathcal{B}$ and $GF[V]$. Denote by $G[V]$ the subset of $B[V]$ consisting of those operators which have the property that the null space has a closed complement and the range is closed, where the null space and range are proper closed subspaces of V. Necessary and sufficient conditions for T to be in $\mathcal{B} \cap G[V]$ will be proven, and as a corollary it is shown that $\mathcal{B} \cap G[V]$ is the subset of $B[V]$ consisting of those operators which are products of invertible operators and projections other than the identity operator and null operator.

2. Necessary and sufficient conditions for T to be in $\mathcal{B} \cap G[V]$. For $T \in G[V]$, let $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null space of T and the range of T, respectively, and let V_1 denote a closed complement of $\mathcal{N}(T)$. Then T_1, the restriction of T to V_1, is a linear homeomorphism from V_1 onto $\mathcal{R}(T)$. The main part of the proof of the following theorem is the construction of an extension of T_1 to an invertible operator in $\mathcal{M}$. To that end, $\mathcal{S}$ will denote a set of extensions of T_1 with certain properties. $\mathcal{S}$ will be shown to be nonempty and will be partially ordered. Then, a maximal element of $\mathcal{S}$ will be shown to be the desired extension.

THEOREM. *If $T \in G[V]$ and if V_1 is a closed complement of $\mathcal{N}(T)$, then the following are equivalent*:

(i) *there is a sequence $\{\tilde{M}_n\}$ in $\mathcal{M}$, such that $\{\tilde{M}_n|_{V_1}\}$ converges to $T|_{V_1}$ and the sequence $\{\|\tilde{M}_n^{-1}\|\}$ is bounded*;

(ii) *there is an invertible $S' \in \mathcal{M}$, such that $S'|_{V_1} = T|_{V_1}$*;

(iii) $T \in \mathcal{B}$;

(iv) $T \in GF[V]$ *with $\mathcal{N}(T)$ linearly homeomorphic to a closed complement of $\mathcal{R}(T)$.*

PROOF. ((i) implies (ii).) Define a new norm $\|\cdot\|'$ on $V = V_1 \oplus \mathcal{N}(T)$ by $\|x\|' = \|x_1\| + \|x_0\|$, where $x = x_1 + x_0$ with $x_1 \in V_1$ and $x_0 \in \mathcal{N}(T)$. Since $\|\cdot\|'$ is equivalent to the original norm, there is a positive real number α, such that $\alpha\|x\|' \leqslant \|x\|$ or $\alpha[\|x_1\| + \|x_0\|] \leqslant \|x\|$.

Because the sequence $\{\|\tilde{M}_n^{-1}\|\}$ is bounded, there is a positive number λ, such that $\|\tilde{M}_n^{-1}\|^{-1} \geqslant \lambda$. Hence, $\lambda\|x\| \leqslant \|\tilde{M}_n^{-1}\|^{-1}\|x\| \leqslant \|\tilde{M}_n(x)\|$ for all $x \in V$ and for all $\tilde{M}_n$. Let ε and δ denote positive real numbers such that $2\delta < \varepsilon < \alpha\lambda/3$ and let M be a member of the given sequence $\{\tilde{M}_n\}$ such that

$\|M|_{V_1} - T|_{V_1}\| < \varepsilon$. Denote by N_1 the subset of $\mathcal{N}(T)$ which contains all elements of $\mathcal{N}(T)$ with norm one.

It will now be shown that:

$$(*) \qquad \begin{array}{l} \text{if } x_0 \in N_1, \text{ then there is a } y \in V, \text{ such that} \\ \|y - M(x_0)\| < \varepsilon \text{ and } \|y - z\| \geqslant \delta \text{ for all } z \in \mathcal{R}(T). \end{array}$$

Case 1. $M(x_0) \notin \mathcal{R}(T)$. Now, $\rho = \inf\{\|M(x_0) - z\|: z \in \mathcal{R}(T)\} > 0$. If $\rho \geqslant \delta$, let $y = M(x_0)$. If $\rho < \delta$, then by [**10**, Lemma 1.7, p. 86], there is $u \in V$ such that $\|u\| = 1$ and $\|u - z\| > \beta$, for all $z \in \mathcal{R}(T)$, where $2\delta/\varepsilon < \beta < 1$. Let γ denote a real number such that $1 - \varepsilon < \gamma < 1 - 2\delta/\beta$. Then let $y = (1 - \gamma)u + M(x_0)$.

Case 2. $M(x_0) \in \mathcal{R}(T)$. Let β and γ denote real numbers such that $\delta/\varepsilon < \beta < 1$ and $1 - \varepsilon < \gamma < 1 - \delta/\beta$. Let u be defined as in Case 1 and, again, let $y = (1 - \gamma)u + M(x_0)$. Then it follows that $(*)$ is true.

Let $\mathcal{S}$ denote the set of one-to-one, bounded, linear, extensions, S, of $T_1 = T|_{V_1}$ such that if $x \in N_1 \cap \mathrm{dom}(S)$ and $z \in \mathcal{R}(T)$, then $\|S(x) - M(x)\| \leqslant \varepsilon$ and $\|S(x) - z\| \geqslant \delta$. It will now be shown that $\mathcal{S} \neq \varnothing$. Let $x_0 \in N_1$ and let y, which depends on x_0, be as in $(*)$. Denote by $\langle x_0 \rangle$ the subspace of V generated by x_0. Define S: $V_1 \oplus \langle x_0 \rangle \to \mathcal{R}(T) \oplus \langle y \rangle$ by $S(x_1 + \omega x_0) = T_1(x_1) + \omega y$, where $x_1 \in V_1$ and ω is a complex number. Then $S \in \mathcal{S}$.

Partially order $\mathcal{S}$ by $S_1 \leqslant S_2$ if and only if $\mathrm{dom}(S_1) \subseteq \mathrm{dom}(S_2)$ and S_2 is an extension of S_1. Let $\mathcal{C}$ be a chain in $\mathcal{S}$ and let $D_0 = \bigcup_{S \in \mathcal{C}} \mathrm{dom}(S)$. Define S_0: $D_0 \to V$ by $S_0(x) = S(x)$ if $x \in \mathrm{dom}(S)$ and $S \in \mathcal{C}$. To see that S_0 is well defined, suppose that $x \in \mathrm{dom}(S_1) \cap \mathrm{dom}(S_2)$ for some $S_1, S_2 \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, it may be assumed without loss of generality that $S_1 \leqslant S_2$. Hence, $x \in \mathrm{dom}(S_1) \cap \mathrm{dom}(S_2) = \mathrm{dom}(S_1)$, and so $S_1(x) = S_2(x) = S_0(x)$, which implies that S_0 is well defined. If $x, y \in D_0$, then $x, y \in \mathrm{dom}(S)$ for some $S \in \mathcal{S}$. For any complex number ω,

$$S_0(\omega x + y) = S(\omega x + y) = \omega S(x) + S(y) = \omega S_0(x) + S_0(y).$$

Therefore, S_0 is linear. If $S_0(x) = \theta$, then $S(x) = \theta$, which implies that $x = \theta$. Thus, S_0 is one-to-one on D_0. Since each S in $\mathcal{C}$ is an extension of T_1, S_0 is also an extension of T_1 in addition to being an extension of each S in $\mathcal{C}$. If $x_0 \in N_1 \cap \mathrm{dom}(S_0)$, then $x_0 \in N_1 \cap \mathrm{dom}(S)$ for some $S \in \mathcal{C}$. Hence, $\|S_0(x_0) - M(x_0)\| = \|S(x_0) - M(x_0)\| \leqslant \varepsilon$, and for $z \in \mathcal{R}(T)$, $\|S_0(x_0) - z\| = \|S(x_0) - z\| \geqslant \delta$. If $x = x_1 + x_0 \in \mathrm{dom}(S_0)$, where $x_1 \in V_1$ and $x_0 \in \mathcal{N}(T)$, then $x, x_1, x_0 \in \mathrm{dom}(S)$ for some $S \in \mathcal{C}$. Hence,

$$\begin{aligned} \|M(x) - S_0(x)\| &\leqslant \|M(x_1) - S_0(x_1)\| + \|M(x_0) - S_0(x_0)\| \\ &\leqslant \|M|_{V_1} - T|_{V_1}\| \, \|x_1\| + \varepsilon\|x_0\| \\ &\leqslant \varepsilon\|x_1\| + \varepsilon\|x_0\| = \varepsilon\big[\,\|x_1\| + \|x_0\|\,\big] \leqslant \|x\|\varepsilon/\alpha. \end{aligned}$$

Thus,

$$\|S_0(x)\| \leqslant \|M(x)\| + \|x\|\varepsilon/\alpha \leqslant [\|M\| + \varepsilon/\alpha]\|x\|,$$

which implies that S_0 is bounded. Therefore, $S_0 \in \mathcal{S}$, and since $S \leqslant S_0$, for all $S \in \mathcal{C}$, the chain $\mathcal{C}$ has an upper bound. Then, by Zorn's Lemma, $\mathcal{S}$ has a maximal element, say S'. Let D' denote the domain of S'.

Now it will be proven that:

(**) there is a positive number ξ, such that $\xi\|x\| \leqslant \|S'(x)\|$, for all $x \in D'$.

If not, then there is a sequence $\{y_n\}$ in D' such that $\|y_n\| = 1$ and $\|S'(y_n)\| < 1/n$. Thus, $\lim_n[S'(y_n)] = \theta$, where θ denotes the zero element of V. Let $y_n = y_{n1} + y_{n0}$, where $y_{n1} \in V_1$ and $y_{n0} \in \mathfrak{N}(T)$. For $y_{n0} \neq \theta$,

$$1/n > \|S'(y_n)\| = \|y_{n0}\| \, \|S'(y_{n0}/\|y_{n0}\|) + S'(y_{n1}/\|y_{n0}\|)\|$$

$$= \|y_{n0}\| \, \|S'(y_{n0}/\|y_{n0}\|) - T(-y_{n1}/\|y_{n0}\|)\| \geqslant \|y_{n0}\|\delta,$$

since $y_{n0}/\|y_{n0}\| \in N_1 \cap \operatorname{dom}(S')$ and $T(-y_{n1}/\|y_{n0}\|) \in \mathfrak{R}(T)$. Hence, $\lim_n[y_{n0}] = \theta$. S' bounded implies that $\lim_n[S'(y_{n0})] = \theta$. This, along with the fact that $\lim_n[S'(y_n)] = \theta$ implies that $\lim_n[T_1(y_{n1})] = \lim_n[S'(y_{n1})] = \theta$. Since T_1 is a homeomorphism defined on V_1, $\lim_n[y_{n1}] = \theta$. Thus, $\lim_n[y_n] = \theta$, which contradicts the fact that $\|y_n\| = 1$. Hence, (**) is true. Now, S' can be extended to the closure of D' be defining S'': $\operatorname{cl}(D') \to V$ by $S''(x) = \lim_n[S'(x_n)]$, where $x_n \in D'$ and $\lim_n[x_n] = x$. S'' is a bounded linear extension of both T_1 and S', with $\|S'\| = \|S''\|$, [**2**, p. 279]. If $x \in \operatorname{cl}(D')$ and $x_n \in D'$ with $\lim_n[x_n] = x$, then $\lim_n[S'(x_n)] = S''(x)$ and $\xi\|x_n\| \leqslant \|S'(x_n)\|$. Thus, $\xi\|x\| \leqslant \|S''(x)\|$. By [**11**, Theorem 3.1-B, p. 86], the inverse of S'' exists and is bounded on its domain, $\mathfrak{R}(S'')$. It follows that $S'' \in \mathcal{S}$. Since $S' \leqslant S''$ and S' is a maximal element of $\mathcal{S}$, then $S' = S''$ and D' is closed. The inverse of S' being bounded implies $\mathfrak{R}(S')$ is closed. If $x = x_1 + x_0 \in D'$, where $x_1 \in V_1$ and $x_0 \in \mathfrak{N}(T)$, then

$$\|M(x) - S'(x)\| \leqslant \|M(x_1) - T_1(x_1)\| + \|M(x_0) - S'(x_0)\|$$

$$< \varepsilon[\|x_1\| + \|x_0\|] \leqslant \|x\|\varepsilon/\alpha,$$

which compares $\mathfrak{R}(S')$ and $M(D')$ and will be used later.

Consider the following possibilities:

(a) D' and $\mathfrak{R}(S')$ are proper closed subspaces of V,

(b) $D' = V$ and $\mathfrak{R}(S') \neq V$,

(c) $D' \neq V$ and $\mathfrak{R}(S') = V$,

(d) $D' = V$ and $\mathfrak{R}(S') = V$.

If D' and $\mathfrak{R}(S')$ are proper subspaces of V, then S' can be extended as T_1 was extended in the proof that $\mathcal{S} \neq \varnothing$, which contradicts the maximality of S'.

If $D' = V$ and $\mathfrak{R}(S') \neq V$, then since $\mathfrak{R}(S')$ is closed, by [**10**, p. 86], there is $y \in V$ such that $\|y\| = 1$ and $\|y - z\| > \beta$, for all $z \in \mathfrak{R}(S')$, where $\varepsilon/(\alpha\lambda - 2\varepsilon) < \beta < 1$. Let $x = M^{-1}(y)$. Then $\beta < \|M(x) - S'(x)\| \leqslant \|x\|\varepsilon/\alpha$. Hence, $\|x\| > \alpha\beta/\varepsilon$. $\|M(x) - S'(x)\| < \|x\|\varepsilon/\alpha$ and $\lambda\|x\| \leqslant$

$\|M(x)\|$ imply that $\|S'(x)\| \geqslant \|x\|(\lambda - \varepsilon/\alpha)$. Hence

$$\|x\|(\lambda - \varepsilon/\alpha) \leqslant \|M(x) - S'(x)\| + \|M(x)\| \leqslant \|x\|\varepsilon/\alpha + 1.$$

Therefore, $\lambda - \varepsilon/\alpha \leqslant \varepsilon/\alpha + \|x\|^{-1} < \varepsilon/\alpha + \varepsilon/\alpha\beta$ or $\beta < \varepsilon/(\alpha\lambda - 2\varepsilon)$, which contradicts the choice of β, namely, $\varepsilon/(\alpha\lambda - 2\varepsilon) < \beta$.

If $D' \neq V$ and $\mathcal{R}(S') = V$, let β' denote a real number such that $\varepsilon/(\alpha\lambda - \varepsilon) < \beta' < 1$. Since D' is closed, $M(D')$ is also closed. Hence, there is $y' \in V$ such that $\|y'\| = 1$ and $\|y' - z\| > \beta'$ for all $z \in M(D')$. Let $x' = S'^{-1}(y')$. Then $\beta' < \|M(x') - S'(x')\| \leqslant \|x'\|\varepsilon/\alpha$ or $\|x'\| > \alpha\beta'/\varepsilon$. Now,

$$\lambda\|x'\| < \|M(x')\| \leqslant \|M(x') - S'(x')\| + \|S'(x')\| \leqslant \|x'\|\varepsilon/\alpha + 1.$$

Hence, $\lambda \leqslant \varepsilon/\alpha + \|x'\|^{-1} < \varepsilon/\alpha + \varepsilon/\alpha\beta'$ or $\beta' < \varepsilon/(\alpha\lambda - \varepsilon)$, which contradicts the choice of β'.

Therefore $D' = V$ and $\mathcal{R}(S') = V$ is the only possibility and $S' \in \mathfrak{M}$.

((ii) implies (iii).) Let P denote the projection of V onto V_1 along $\mathfrak{N}(T)$ and let I denote the identity operator on V. Then $S_n = S'P + n^{-1}S'(I - P) \in \mathfrak{M}$ and $\lim_n[S_n] = T$. Therefore, $T \in \mathfrak{B}$.

((iii) implies (i).) Define another norm $\|\cdot\|''$ on V by $\|x\|'' = \max\{\|x_1\|, \|x_0\|\}$, where $x = x_1 + x_0$ with $x_1 \in V_1$ and $x_0 \in \mathfrak{N}(T)$. Let V'' denote V with norm $\|\cdot\|''$. Then the function ϕ, defined from V'' onto V by $\phi(x) = x$, is a linear homeomorphism. The three spaces, $B[V]$, $B[V'', V]$ and $B[V, V'']$ are linearly homeomorphic with $M \in B[V]$ corresponding to $M\phi \in B[V'', V]$ and $\phi^{-1}M \in B[V, V'']$. Hence,

$$\|M\phi\| = \sup\{\|M(x)\|: \|x\|'' = 1\}, \quad \text{and}$$

$$\left\|(M\phi)^{-1}\right\|^{-1} = \inf\{\|M(x)\|: \|x\|'' = 1\}.$$

For $M \in \mathfrak{M}$, let $m_1 = \inf\{\|M(x_1)\|: x_1 \in V_1 \text{ and } \|x_1\| = 1\}$ and $m_0 = \inf\{\|M(x_0)\|: x_0 \in \mathfrak{N}(T) \text{ and } \|x_0\| = 1\}$. Define $\tilde{M} \in \mathfrak{M}$ by $\tilde{M} = MP + (m_1/m_0)M(I - P)$. Then,

$$\begin{aligned}\left\|(\tilde{M}\phi)^{-1}\right\|^{-1} &= \inf\{\|\tilde{M}(x)\|: \|x\|'' = 1\}\\ &= \inf\{\|M(x_1)\|: x_1 \in V_1 \text{ and } \|x_1\| = 1\}.\end{aligned}$$

Since $T_1 = T|_{V_1}$ is a linear homeomorphism from V_1 onto $\mathcal{R}(T)$, T_1^{-1} exists. $T \in \mathfrak{B}$ implies that there is a sequence $\{M_n\}$ in $\mathfrak{M}$ such that $\lim_n[M_n] = T$. Let ε denote a positive real number less than $\|T_1^{-1}\|^{-1}$. Then there is a positive real number N, such that if $n \geqslant N$, then $\|T(x) - M_n(x)\| < \varepsilon\|x\|$ for all $x \in V$. Then

$$\|T_1^{-1}\|^{-1} - \varepsilon \leqslant \left\|(\tilde{M}_n\phi)^{-1}\right\|^{-1} = \|\phi^{-1}\tilde{M}_n^{-1}\|^{-1} \quad \text{for } n \geqslant N.$$

Thus, $\{\phi^{-1}\tilde{M}_n^{-1}\}$ is a sequence in $B[V, V'']$ such that $\{\|\phi^{-1}\tilde{M}_n^{-1}\|\}$ is bounded and, hence, the corresponding sequence in $B[V]$, namely $\{\tilde{M}_n^{-1}\}$, is such that $\{\|\tilde{M}_n^{-1}\|\}$ is bounded. Since $\tilde{M}_n|_{V_1} = M_n|_{V_1}$, $\lim_n[\tilde{M}_n|_{V_1}] = T_1$.

((ii) implies (iv).) $S' \in \mathfrak{M}$ implies that $V = S'(V_1) \oplus S'(\mathfrak{N}(T)) = \mathcal{R}(T)$

$\oplus S'(\mathfrak{N}(T))$. Thus, $T \in GF[V]$ and $S'(\mathfrak{N}(T))$ is a closed complement of $\mathfrak{R}(T)$ that is homeomorphic to $\mathfrak{N}(T)$.

((iv) implies (iii).) This is Corollary 2.2 of [**6**]. Thus, the proof of the theorem is complete.

Let $\mathfrak{P}$ denote the set of bounded projections on V, other than the identity operator and the null operator, let $\mathfrak{M}\mathfrak{P} = \{MP\colon M \in \mathfrak{M} \text{ and } P \in \mathfrak{P}\}$ and let $\mathfrak{P}\mathfrak{M} = \{PM\colon P \in \mathfrak{P} \text{ and } M \in \mathfrak{M}\}$.

COROLLARY. $\mathfrak{B} \cap G[V] = \mathfrak{M}\mathfrak{P} = \mathfrak{P}\mathfrak{M}$.

PROOF. This follows from the observation that for T as in the hypothesis of the theorem, (ii) can be stated as:

(ii)′ there is $S \in \mathfrak{M}$ such that $T = SP$, where P is the projection of V onto V_1 along $\mathfrak{N}(T)$.

3. Comments. The equivalence of (ii)′ and (iv) is a special case of Theorem 1 of [**3**]. In [**3**], it is noted that in order to extend T_1, a linear homeomorphism between $\mathfrak{N}(T)$ and a closed complement of $\mathfrak{R}(T)$ is needed. If the above theorem is viewed as a theorem about generalized Fredholm operators, then the fact that the operator is also in $\mathfrak{B}$ is sufficient to produce this homeomorphism even though it is only assumed that the operator is in $G[V]$. If the above theorem is viewed as a theorem about operators in $\mathfrak{B}$, it provides the fact that $\mathfrak{B} \cap G[V]$ is the same as $\{T \in GF[V]\colon \mathfrak{N}(T) \text{ is linearly homeomorphic to a closed complement of } \mathfrak{R}(T)\}$.

As noted in [**4**], the hypothesis that $T \in G[V]$ is satisfied if V is a Hilbert space and if $\mathfrak{R}(T)$ is closed. The sufficient condition for $T \in \mathfrak{B}$ given in [**7**] is $V = \mathfrak{R}(T) \oplus \mathfrak{N}(T)$, which is a special case for $T \in GF[V]$.

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DEPARTMENT OF MATHEMATICS, LOUISIANA TECH UNIVERSITY, RUSTON, LOUISIANA 71272