PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 63, Number 1, March 1977

AN APPLICATION OF A THEOREM OF R. E. ZINK

H. E. WHITE, JR.[1]

ABSTRACT. In §1 we discuss a measure theoretic analogue of Blumberg's theorem; in §2 we discuss a topological analogue of the Saks-Sierpinski theorem.

1. In this section we discuss a measure theoretic analogue of Blumberg's theorem [7, 1.2]. Suppose $(X, \mathfrak{S}, \mu)$ is a totally finite measure space [2, p. 73], and μ^* is the outer measure engendered by μ. We consider the following statement.

1.1. For every real valued function f defined on X, there is a subset D of X such that $\mu^*(D) = \mu(X)$ and $f|D$ is measurable $(\mathfrak{S} \cap D)$, where $\mathfrak{S} \cap D = \{S \cap D: S \in \mathfrak{S}\}$.

Actually 1.1 is equivalent to a special case of Blumberg's theorem. Let $(X, \mathfrak{S}_c, \mu_c)$ denote the completion of $(X, \mathfrak{S}, \mu)$. By [3, p. 88], there is a topology $\mathfrak{T}(\mu_c)$ on X such that (a) if $U \in \mathfrak{T}(\mu_c)$ and $U \neq \varnothing$, then $U \in \mathfrak{S}_c$ and $\mu_c(U) > 0$; and (b) if $A \in \mathfrak{S}_c$, then there is U in $\mathfrak{T}(\mu_c)$ such that $U \subset A$ and $\mu_c(A \sim U) = 0$. It is easily verified that 1.1 holds for $(X, \mathfrak{S}, \mu)$ if and only if Blumberg's theorem holds for $(X, \mathfrak{T}(\mu_c))$.

In [7, 2.1] it is shown that

1.2. every subset of the closed unit interval I of cardinality $< 2^{\aleph_0}$ has Lebesgue measure zero,

then 1.1 is false for $(I, \mathfrak{B}, m)$, where m denotes Lebesgue measure on the collection $\mathfrak{B}$ of Borel subsets of I. In this section we show that the following statement is a consequence of [9, Theorem 9].

1.3. THEOREM. *Suppose* 1.1 *is false for* $(I, \mathfrak{B}, m)$. *Then* 1.1 *is false for every separable, nonatomic measure space* $(X, \mathfrak{S}, \mu)$.

REMARK. It follows that, in this case, there is $f: X \to R$ such that, if D is a subset of X for which $f|D$ is measurable $(\mathfrak{S} \cap D)$, then $\mu^*(D) = 0$.

If $2^{\aleph_0} = \aleph_1$, then clearly 1.2 holds; therefore, in this case, 1.1 is false for every separable, nonatomic measure space. However, there is a weaker statement–called Martin's axiom–which implies that 1.2 is true [5, §4].

Presented to the Society May 17, 1976; received by the editors May 29, 1976.

AMS (MOS) subject classifications (1970). Primary 28A20.

Key words and phrases. Totally finite nonatomic separable measure space, Saks-Sierpinski theorem.

[1]During the period in which this research was done, the author held a Visiting Postdoctoral Fellowship from the Institute for Medicine and Mathematics.

MARTIN'S AXIOM. If $\mathcal{C}$ is any collection of fewer than $2^{\aleph_0}$ dense open subsets of a compact Hausdorff space which satisfies the countable chain condition, then $\cap \mathcal{C} \neq \varnothing$.

1.4. COROLLARY. *Suppose Martin's axiom holds. If* $(X, \mathfrak{T})$ *is a quasi-regular* [4, *p*. 157] *Baire space which admits a separable, nonatomic category measure* [4, *p*. 156] μ, *then Blumberg's theorem does not hold for* $(X, \mathfrak{T})$.

This corollary follows from 1.3 because Blumberg's theorem holds for $(X, \mathfrak{T})$ if and only if 1.1 holds for $(X, \mathbb{S}, \mu)$, where $\mathbb{S}$ denotes the collection of all subsets of $(X, \mathfrak{T})$ having the property of Baire.

1.5. EXAMPLES. (1) Suppose $\mathfrak{D}$ denotes the density topology on the real line R [7, 2.1]. Then the Stone-Čech compactification of $(R, \mathfrak{D})$ satisfies the hypothesis of 1.4 because $(R, \mathfrak{D})$ does [3, p. 91].

(2) Suppose $(S, \mathfrak{U})$ denotes the Stone space of $\mathcal{L}/\mathfrak{N}$, where $\mathcal{L}$ is the collection of all Lebesgue measurable subsets of I and $\mathfrak{N} = \{A \in \mathcal{L}: m(A) = 0\}$. Then $(S, \mathfrak{U})$ satisfies the hypothesis of 1.4 [3, p. 91].

REMARK. It is proven in [6] that if $2^{\aleph_0} = \aleph_1$, then Blumberg's theorem does not hold for $(S, \mathfrak{U})$. The preceding provides a measure theoretic proof of this statement.

PROOF OF 1.3. We shall show that 1.3 follows easily from the next statement, which is a result from [9].

1.6. THEOREM (R. E. ZINK). *Suppose* $(X, \mathbb{S}, \mu)$ *is a separable, nonatomic, totally finite measure space such that* $\mu(X) = 1$. *Then there is a function* T: $X \to I$ *such that* (a) $T^{-1}[\mathfrak{B}] \subset \mathbb{S}$; (b) *if* $B \in \mathfrak{B}$, *then* $\mu(T^{-1}[B]) = m(B)$; *and* (c) if $S \in \mathbb{S}$, *then there is* B *in* $\mathfrak{B}$ *such that* $\mu(S \triangle T^{-1}[B]) = 0$, *where* "$\triangle$" *denotes symmetric difference.*

To prove 1.3, we assume that 1.1 holds for some separable, nonatomic, totally finite measure space $(X, \mathbb{S}, \mu)$. We assume $\mu(X) = 1$, and that T is as described in 1.6. Suppose f_0: $I \to R$, and let $f = f_0 \circ T$. Then there is a subset D_1 of X such that $\mu^*(D_1) = 1$ and $f|D_1$ is measurable $(\mathbb{S} \cap D_1)$. Because 1.6 (c) holds, there is a subset of D of D_1 such that $\mu^*(D) = 1$ and $f|D$ is measurable $T^{-1}[\mathfrak{B}] \cap D$. Then $f|T^{-1}[T[D]]$ is measurable $(T^{-1}[\mathfrak{B}] \cap T^{-1}[T[D]])$, and $f_0|T[D]$ is measurable $\mathfrak{B} \cap T[D]$. Because 1.6 (b) holds, $m^*(T[D]) \geqslant \mu^*(D) = 1$; hence 1.1 holds for $(I, \mathfrak{B}, m)$.

Questions. (1) Is it consistent with ZF + AC that 1.1 holds for $(I, \mathfrak{B}, m)$?

(2) Does 1.3 remain true if the word "separable" is deleted from the hypothesis?

2. In [9], it is shown that the following statement, known as the Saks-Sierpinski theorem, holds for every totally finite measure space $(X, \mathbb{S}, \mu)$.

2.1. For every real valued function f defined on X, there is a function g: $X \to R$ which is measurable $(\mathbb{S})$ such that, for every positive number ε,

$$\mu^*\big(\{x: |f(x) - g(x)| < \varepsilon\}\big) = \mu(X).$$

A topological analogue of 2.1 provides another characterization of Baire spaces.

2.2. THEOREM. *The following statements are equivalent for any topological space* $(X, \mathfrak{T})$.

(a) $(X, \mathfrak{T})$ *is a Baire space.*

(b) *For every real valued function* f *defined on* X, *there is a function* g *such that*: (i) *domain* g *is a dense* G_δ *subset of* X; (ii) g *is continuous*; *and* (iii) *for every positive number* ε, *the set* $\{x \in \text{domain } g\colon |f(x) - g(x)| < \varepsilon\}$ *is dense in* X.

(c) *For every real valued function* f *defined on* X, *there is a function* h *defined on* X *such that*: (i) h *is Borel measurable*; *and* (ii) *for every positive number* ε, *there is a dense subset* D_ε *of* X *such that* $h|D_\varepsilon$ *is continuous and* $D_\varepsilon \subset \{x\colon |f(x) - h(x)| < \varepsilon\}$.

PROOF. Obviously (b) implies (c). If [7, 1.5] is modified by replacing Y by R in (3), then the resulting statement is true, and its proof is very similar to the proof of [7, 1.5]. It follows from this modified statement that (c) implies (a). Finally, the proof of the Saks-Sierpinski theorem, which is given in [9, §4], when translated into topological terms, establishes that (a) implies (b), provided the following statement is substituted for Lemma B of [9].

LEMMA B$'$. *Suppose* f *is a real valued function defined on the Baire space* X, *and* Y *is a dense subspace of* X *such that* Y *is a Baire space. For every positive number* ε, *there is a function* f_ε *such that*: (i) domain f_ε *is a dense open subset of* X; (ii) f_ε *is continuous*; *and* (iii) *if* $X_\varepsilon = \{x \in \text{domain } f_\varepsilon\colon |f(x) - f_\varepsilon(x)| < \varepsilon\}$, *then* $X_\varepsilon \cap Y$ *is dense in* X *and* $X_\varepsilon \cap Y$ *is a Baire space.*

The proof of Lemma B$'$ is quite similar to the proof that (1) implies (3) of [7, 1.5], and is omitted.

REMARKS. (1) It is clear from the proof of 2.2 that:

(a) The function g (resp. h) can be chosen so that for every positive number ε, the set $\{x \in \text{domain } g\colon |f(x) - g(x)| < \varepsilon\}$ (resp. D_ε) is a dense Baire subspace of X.

(b) If f is bounded, then function g (resp. h) can be chosen to be bounded.

(c) If X is completely regular and satisfies the countable chain condition, then h can be chosen to be a Baire function.

(2) The Saks-Sierpinski theorem, when μ is complete, is a special case of 2.2 (apply 2.2(c) to the Baire space $(X, \mathfrak{T}(\mu))$).

(3) Suppose $(X, \mathfrak{T})$ is an extremally disconnected [1, p. 22] Baire space. It follows from 2.2(b) and [1, p. 96] that for every bounded, real valued function f defined on X, there is a continuous function $g\colon X \to R$ such that, for every positive number ε, the set $\{x\colon |f(x) - g(x)| < \varepsilon\}$ is dense in X. In particular, the preceding statement holds for the space $(S, \mathfrak{U})$ of 1.5(2). However, if $2^{\aleph_0} = \aleph_1$, then there is a bounded real valued function f_0 defined on S such

that $\{x: f_0(x) = g(x)\}$ is nowhere dense in S for every continuous real valued function g defined on S.

References

1. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

2. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950. MR **11**, 504.

3. J. C. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1971.

4. ______, *Spaces that admit a category measure*, J. Reine Angew. Math. **205** (1960/61), 156–170. MR **25** #4054; **26**, p. 1543.

5. F. Tall, *The density topology* (preprint).

6. W. A. R. Weiss, *A solution to the Blumberg problem*, Bull. Amer. Math. Soc. **81** (1975), 957–958.

7. H. E. White, Jr., *Topological spaces in which Blumberg's theorem holds*, Proc. Amer. Math. Soc. **44** (1974), 454–462. MR **49** #6130.

8. ______, *Some Baire spaces for which Blumberg's theorem does not hold*, Proc. Amer. Math. Soc. **51** (1975), 477–482.

9. R. E. Zink, *On the structure of measure spaces*, Acta Math. **107** (1962), 53–71. MR **25** #4060.

671 S. Eureka Avenue, Columbus, Ohio 43204