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# REMARK ON MEASURABLE GRAPH THEOREMS

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ABSTRACT. A theorem generalizing some known measurable graph theorems is proved.

1. **Notations.** Two measurable spaces $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ are isomorphic if there exists a bimeasurable one-to-one function from $X_1$ onto $X_2$. A measurable space is called a standard measurable space if it is isomorphic with $(I, \mathcal{B}(I))$, the unit interval and its Borel $\sigma$-field. We call it a Blackwell space if it is isomorphic with some space $(A, \mathcal{B}(A))$, where $A$ is an analytic subset of $I$ and $\mathcal{B}(A)$ is the system of all intersections of $A$ with sets in $\mathcal{B}(I)$. We claim that $(X, \mathcal{B}(X))$ is a standard measurable space if $X$ is any nondenumerable complete separable metric space and $\mathcal{B}(X)$ its Borel $\sigma$-field. If $(X, \mathcal{A}, \mu)$ is a measure space, then we write throughout the paper $\mathcal{A}_\mu$ for the completion of $\mathcal{A}$ with respect to $\mu$.

2. **Main results.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, $f$ a function $f\colon X \to Y$, $\mathrm{gr}(f)$ the graph of $f$ and $\mathcal{A} \otimes \mathcal{B}$ the product $\sigma$-field. It is well known that $(\mathcal{A}, \mathcal{B})$-measurability of the function implies $\mathrm{gr}(f) \in \mathcal{A} \otimes \mathcal{B}$ if $\mathcal{B}$ is countably separated (see e.g. [5, I.3.4 and 5]). Concerning the reverse implication we shall prove in this paper two notes:

(A) *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be Blackwell spaces and $\mu$ be a measure on $\mathcal{A}$. If $\mathrm{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$, then $f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable.*

(B) *Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $(Y, \mathcal{B})$ a Blackwell space. If $\mathrm{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$, then $f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable.*

In [3] Buckley has proved a result corresponding to (A) on the more restrictive assumption that the spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are complete separable metric spaces with their Borel $\sigma$-fields. In the most important case of a $\sigma$-finite measure, Buckley's result is known from Bierlein's measurable graph theorem in [2], where $(Y, \mathcal{B})$ is assumed to be a standard measurable space and $(X, \mathcal{A}, \mu)$ is any $\sigma$-finite measure space. In (B) Bierlein's assumption on $(Y, \mathcal{B})$ is generalized to the assumption of a Blackwell space.

Now (A) and (B) may be applied to give a more general version of Buckley's theorem, of the main part of Bierlein's measurable graph theorem and of the known fact that $\mathrm{gr}(f) \in \mathcal{A} \otimes \mathcal{B}$ is equivalent to $(\mathcal{A}, \mathcal{B})$-measura-

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bility of $f$ if both spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are Blackwell spaces (cf. [5, II.4.3]). Since $\mathcal{B}$ is countably separated if $(Y, \mathcal{B})$ is a Blackwell space, the following theorem is a consequence of (A) and (B).

MEASURABLE GRAPH THEOREM. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a Blackwell space and $f$ a function $f\colon X \to Y$. If either the measure $\mu$ is $\sigma$-finite or the space $(X, \mathcal{A})$ is a Blackwell space, then the equivalence holds:*

*$f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable if and only if* $\mathrm{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$.

3. **Proofs.** The proof of (A) is based on the following lemma. The proof of the lemma is omitted as a standard one.

LEMMA. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $(Y, \mathcal{B})$ a measurable space. If $G \in \mathcal{A}_\mu \otimes \mathcal{B}$, then there exists a set $N_0 \in \mathcal{A}$ such that $\mu(N_0) = 0$ and $G \setminus (N_0 \times Y) \in \mathcal{A} \otimes \mathcal{B}$.*

PROOF OF (A). Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be Blackwell spaces and $\mathrm{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$. Then, by the Lemma, $X$ is the union of two disjoint sets $X_0 \in \mathcal{A}$ and $N_0 \in \mathcal{A}$ such that

$$\mu(N_0) = 0 \quad \text{and} \quad \mathrm{gr}(f) \setminus (N_0 \times Y) \in \mathcal{A} \otimes \mathcal{B}.$$

Let $\mathcal{A} \cap X_0$ be the $\sigma$-field of all intersections of $X_0$ with sets of $\mathcal{A}$. Since, by definition, $(X_0, \mathcal{A} \cap X_0)$ is a Blackwell space, too, we can apply the measurable graph theorem for Blackwell spaces (see [5, II.4.3]) to the function $f_0$: $X_0 \to Y$ defined as the restriction of $f$ on $X_0$. Since $\mathrm{gr}(f_0) = \mathrm{gr}(f) \setminus (N_0 \times Y)$ is an element of $(\mathcal{A} \cap X_0) \otimes \mathcal{B}$, by this theorem, $f_0$ is an $(\mathcal{A} \cap X_0, \mathcal{B})$-measurable function. Then we have for any $B \in \mathcal{B}$,

$$\begin{aligned} f^{-1}(B) &= \left(f^{-1}(B) \cap X_0\right) \cup \left(f^{-1}(B) \cap N_0\right) \\ &= f_0^{-1}(B) \cup \left(f^{-1}(B) \cap N_0\right) \in \mathcal{A}_\mu, \end{aligned}$$

because $f_0^{-1}(B) \in \mathcal{A}$, $f^{-1}(B) \cap N_0$ is a subset of $N_0$ and $\mu(N_0) = 0$. This shows that $f$ is $(\mathcal{A}_\mu, \mathcal{B})$-measurable, and (A) is proved.

To prove (B) it is possible to apply a projection theorem (II.2.2 in [5] is a suitable one) and Choquet's theorem on capacities (see [4, Theorem 1]) as done by Bierlein [2] in the case of a standard measurable space $(Y, \mathcal{B})$. However, to give a short proof, we claim that (B) is an immediate consequence of the following generalization of Aumann's selection theorem (see [1, p. 17]) proved in [6] (cf. Theorem 3, and for the difference in formulation see [5, III.2.3]):

SELECTION THEOREM. *Let $(X, \mathcal{A})$ be a measurable space having the property that $\mathcal{A}$ equals its universal completion, $(Y, \mathcal{B})$ be a Blackwell space and $C \in \mathcal{A} \otimes \mathcal{B}$. If the projection of $C$ on $X$ is all of $X$, then there exists an $(\mathcal{A}, \mathcal{B})$-measurable function $s\colon X \to Y$ such that $(x, s(x)) \in C$ for all $x \in X$.*

REMARKS. 1. That our theorem generalizes Buckley's theorem [3] and the main part of Bierlein's measurable graph theorem [2] is evident. But our

theorem generalizes also the known result that for Blackwell spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, $\mathrm{gr}(f) \in \mathcal{A} \otimes \mathcal{B}$ is equivalent to $(\mathcal{A}, \mathcal{B})$-measurability of the function $f$. This can be seen if we define the measure $\mu$ on $\mathcal{A}$ by $\mu = 0$ for the void set and $\mu = \infty$ for all other sets in $\mathcal{A}$.

2. That neither in (A) nor in (B) the assumption on $(Y, \mathcal{B})$ to be a Blackwell space is superfluous can be seen by the following simple example: $X = Y = I$ (the unit interval), $\mathcal{A}$ the Borel sets in $I$, $\mathcal{B}$ the $\sigma$-field generated by the Borel sets and one set $V$ being nonmeasurable with respect to Lebesgue measure, and $\mu$ the Lebesgue measure on $\mathcal{A}$. If $f$ is the identity on $I$, then $\mathrm{gr}(f) \in \mathcal{A}_\mu \otimes \mathcal{B}$, but the function is not $(\mathcal{A}_\mu, \mathcal{B})$-measurable, since $f^{-1}(V) \notin \mathcal{A}_\mu$.

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