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ON PRECOMPACT (QUASI-) UNIFORM STRUCTURES

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ABSTRACT. The following are shown:

(1) The subbasis theorem on precompactness does not hold even in uniform spaces.

(2) The supremum of (even finitely many) precompact quasi-uniform structures is not necessarily precompact.

(3) A compact quasi-uniform space is not necessarily totally bounded.

These results contradict corresponding assertions in the literature.

1. **Introduction.** By definition, a quasi-uniform space is a space fulfilling the axioms of a uniform space with the possible exception of the symmetry axiom.

Each topological space may be considered as a quasi-uniform space and vice versa.

Two of the most useful notions in such spaces are those of "total boundedness" and "precompactness". A (quasi-) uniform space $(X, \mathfrak{U})$ is called totally bounded iff for each entourage $V \in \mathfrak{U}$ there exists a finite family of subsets $A_1, \ldots, A_n$ of X such that

$$\bigcup\{A_k : k = 1, \ldots, n\} = X \quad \text{and} \quad \bigcup\{A_k \times A_k : k = 1, \ldots, n\} \subset V.$$

$(X, \mathfrak{U})$ is called precompact iff for each entourage $V \in \mathfrak{U}$ there exists a finite subset $\{x_1, \ldots, x_p\}$ of X such that $\bigcup\{V(x_k) : k = 1, \ldots, p\} = X$.

A uniform space is totally bounded iff it is precompact. A totally bounded quasi-uniform space is precompact, but the converse does not hold, in general.

The above results may be found for example, in [2], [3].

2. **Remarks on subbase and supremum of (quasi-) uniform structures.** According to the subbase theorems [2, p. 205], [3, p. 49] in the above given definition of total boundedness in a (quasi-) uniform space $(X, \mathfrak{U})$, we may replace the part "for each entourage $V \in \mathfrak{U}$" by the expression "for each entourage V of a given subbase $\mathfrak{U}_s$ of the (quasi)-uniformity $\mathfrak{U}$".

[3, p. 49] and [1, Lemma 3, p. 302] assert that similar subbase theorems hold also in precompact (quasi)-uniform spaces. This, however, is not true in

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general, even in the case of uniform spaces, as the following counterexample shows:[1]

Let X be the set of the reals. Then, the family $\mathfrak{U}_s = \{V, W\}$, where $V = \{(x,y): x,y \leqslant a \text{ or } x,y \geqslant a\}$, $W = \{(x,y): x = y \text{ or } x < a < y \text{ or } y < a < x\}$, $a \in R$, is a subbase for the discrete uniformity on X, which, of course, is not precompact since it contains the diagonal. Nevertheless, if we take $t_1 < a < t_2$, then $V(t_1, t_2) = X = W(t_1, a, t_2)$.

In fact, the equality $(S_1 \cap \cdots \cap S_m)(A) = S_1(A) \cap \cdots \cap S_m(A)$ used in [1, Lemma 3, p. 302] does not hold in general (see the above counterexample). However, although this false equality has been also used in [1, Lemma 4, p. 303] its result that "the least upper bound of precompact uniform structures is precompact" is really true, since for uniform structures precompactness is equivalent to total boundedness and the subbasis theorem [2, p. 205] holds for total boundedness. On the other hand, the corresponding proposition on quasi-uniform structures is not true in general, as the counterexample below shows (thus contradicting Theorem 4.10.vi.b of [3, p. 50]).

Let X be as before; then the set $V_{a,b} = \{(x,y): x = y \text{ or } a < x < b\}$, where $a, b \in X$, $a < b$, is a base for a precompact quasi-uniformity $\mathfrak{Q}_{a,b}$ on X. However, the supremum of the family $\{\mathfrak{Q}_{0,1}, \mathfrak{Q}_{1,2}\}$ is the discrete uniformity which is not precompact. We note that any of the above quasi-uniform spaces $(X, \mathfrak{Q}_{a,b})$ is compact but not totally bounded. Thus, the assertion of [4, p. 368] that "every compact quasi-uniform space is totally bounded" is not true.

Finally, we note that despite the above results, the product theorem on precompact quasi-uniform structures [3, Theorem 4.10.iv.b, p. 50] is true. This can be shown using the fact that a quasi-uniform space is precompact if and only if every ultrafilter in the space is a Cauchy filter [5, Theorem 1.1, p. 79].

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[1] Meanwhile, the author has been informed that a somewhat similar counterexample has been constructed by Earl Ray McMahan in his unpublished Master's thesis, *Completions and compactifications of uniform structures*, Southern Illinois University, Carbondale, Illinois, 1966.