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WEAK LAWS FOR DEPENDENT SUMS

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ABSTRACT. A general weak law of large numbers for sums $S_n=X_1+\cdots+X_n$ is proved. That is, without assuming the existence of any moments, and allowing any sort of dependence structure, conditions are given for $S_n/n\to 0$ in probability; the conditions are not necessary. However they are sufficient for a much stronger statement, namely that $S_{\nu_n}/\nu_n\to 0$ in probability in many cases where positive, integer-valued random variables $\nu_n\to\infty$.

Introduction. Let $\{X_i\}$ be a sequence of random variables on a probability space $(\Omega, \mathscr{A}, p)$ with partial sums $S_n=X_1+\cdots+X_n$. For reals $0<a_n\to\infty$, $X_n =^p o(a_n)$ means that $X_n/a_n\to 0$ in probability. The weak law of large numbers holds when $S_n =^p o(n)$.

When the X_n are independent the weak law is well understood. Indeed, Kolmogoroff [**1**] gave a set of conditions, (C), that are necessary and sufficient for $S_n =^p o(n)$. However without independence, the results are less striking. The best one seems to be that $f(S_n)\equiv E(S_n^2/(n^2+S_n^2))\to 0$, $n\to\infty$, is necessary and sufficient for the weak law, a fact that states the obvious; $d(X, Y)\equiv f(X-Y)$ defines a metric for the (equivalence classes of almost surely equal) random variables on Ω whose topology is that of convergence in probability. Because it is difficult to know when $d(S_n, 0)\to 0$, $n\to\infty$, this paper explicates the weak law for dependent sums.

It turns out that a conditional version of (C) is sufficient but not necessary for the weak law; this is Proposition 1. A corollary is that when integer random variables $\nu_n>0$ are sufficiently well-behaved, then also $S_{\nu_n} =^p o(\nu_n)$, a weak law for random sums; the ν_n need not be independent of the X_n.

This work began with the proof of the corollary and depended on the truth of Proposition 1. However the obvious sources ([**2**] and [**4**]) made no mention of the weak law in the present context, a surprising fact that necessitated proving this simple result.

The results. For each $n\geq 1$ let $\mathscr{F}_n\subseteq\mathscr{F}_{n+1}\subseteq\mathscr{A}$ be a sigma field of subsets of Ω containing the Borel field $B(S_n)$ generated by S_n and put

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$\mathscr{F}_0 \equiv \{\varnothing, \Omega\}$. For $A \in \mathscr{A}$, I_A denotes the characteristic function of A and $\bar{A}=\Omega\backslash A$. Finally let $E_{j,n}=\{\omega \in \Omega : |X_j| \leqq n\}$. We prove the following result.

PROPOSITION 1. *$S_n =^p o(n)$ if*

$$\sum_{j=1}^{n} p(\bar{E}_{j,n}) \to 0, \tag{1}$$

$$\sum_{j=1}^{n} E(X_j I_{E_{j,n}} \mid \mathscr{F}_{j-1}) \equiv \sum_{j=1}^{n} c_j =^p o(n), \tag{2}$$

$$\sum_{j=1}^{n} E[(X_j I_{E_{j,n}} - c_j)^2] = o(n^2). \tag{3}$$

PROOF. Put $X'_j = X_j I_{E_{j,n}}$, $Y_j = X'_j - E(X'_j | \mathscr{F}_{j-1})$, $S'_j = X'_1 + \cdots + X'_j$ and $T_j = Y_1 + \cdots + Y_j$. Since $B(S'_j) \subseteq \mathscr{F}_j$, $\{T_i, \mathscr{F}_i\}$ is a martingale.

Given an integer $n>0$ and numbers ε, $\delta>0$, let $A_n=\{\omega \in \Omega : |S_n| > \delta n\}$, and $A'_n = \{\omega \in \Omega : |S'_n| > \delta n\}$. Clearly

$$\begin{aligned} p(A_n) &= p(A_n \cap \{S_n = S'_n\}) + p(A_n \cap \{S_n \neq S'_n\}) \\ &= p(A'_n) + p(A_n \cap \{S_n \neq S'_n\}) \leqq p(A'_n) + p(S_n \neq S'_n) \\ &\leqq p(A'_n) + \sum_{j=1}^{n} p(\bar{E}_{j,n}) \end{aligned} \tag{4}$$

and, by (1), the second term can be made less than $\varepsilon/3$ for n large enough. Now

$$\begin{aligned} p(A'_n) &= p(|T_n + S'_n - T_n| > \delta n) \\ &\leqq p(|T_n| > \delta n/2) + p(|S'_n - T_n| > \delta n/2) \end{aligned}$$

and since $S'_n - T_n = \sum_{j=1}^{n} c_j =^p o(n)$ by (2),

$$p(A'_n) < p(|T_n| > \delta n/2) + \varepsilon/3 \tag{5}$$

for large enough n.

By Tchebycheff $p(|T_n| > \delta n/2) \leqq 4E(T_n^2)/(n\delta)^2$ and since $\{T_j, \mathscr{F}_j\}$ is a martingale, $E(T_n^2) = \sum_{j=1}^{n} E(Y_j^2)$. Hence, because of (3) and the preceding remark,

$$p(|T_n| > \delta n/2) < \varepsilon/3 \tag{6}$$

for large enough n. Combining (4), (5), (6), $p(A_n) < \varepsilon$ for all sufficiently large n, which, because ε, δ are arbitrary, proves the asserted proposition.

REMARK. Let $\{S_i\}$ be a sequence of independent, symmetrically distributed random variables and define $X_1 = S_1$, $X_{n+1} = S_{n+1} - S_n$, $n \geqq 1$. The following examples show that none of the conditions of the proposition are necessary.

First, let $p(S_n=n/\log(n+1)\}=\frac{1}{2}-\frac{1}{2}n^{-1}$ and $p\{S_n=e^{n+1}\}=\frac{1}{2}n^{-1}$, $n\geqq 1$, so that $S_n =^p o(n)$. However $\sum_{i=1}^{n} p\{\bar{E}_{i,n}\}\geqq\sum (4i+1)/(i^2+1)>1$ for large n, the latter sum extending over $[\log n]+1\leqq i\leqq n$. Thus (1) fails; $\{S_n\}$ *need not be tail equivalent to sums of truncates for the weak law to hold.*

Next, let $p\{S_n/n=1/\log(n+1)\}=\frac{1}{2}$ so that $S_n=o(n)$ almost surely. However when $j>7$, $|X_j|<j$ so that $X_jI_{E_{j,n}}=X_j$ when $1\leqq j\leqq n$ and $n>7$. Therefore

$$\sum_{i=1}^{n} E(X_iI_{E_{i,n}} \mid \mathscr{F}_{i-1}) = \sum_{i=1}^{n} E(X_i \mid \mathscr{F}_{i-1}) = -\sum_{i=1}^{n-1} S_i.$$

Since

$$W_n \equiv n^{-2}\sum_{i=1}^{n} E(S_i^2) = n^{-2}\sum_{i=1}^{n}\left(\frac{i}{\log(i+1)}\right)^2 \geqq K > 0 \quad \text{for large } n,$$

$-\sum_{i=1}^{n-1} S_i =^p o(n)$ is false, $W_n\to 0$ being necessary. Thus (2) fails; *centering truncates at conditional means need not, on the average, have a negligible effect for the weak law of large numbers to hold.*

Finally, the same example shows that (3) is also not necessary for the weak law. Under the above notation, n^2W_n is equal to the expression on the left-hand side of (3). By the foregoing this is not $o(n^2)$, since $W_n\geqq K>0$.

Let $\nu_n>0$ be integer-valued random variables on $(\Omega, \mathscr{A}, p)$ not necessarily mutually independent nor independent of the X_n. In view of the preceding proposition it is natural to expect that $S_{\nu_n} =^p o(\nu_n)$, as long as $\nu_n\to\infty$ in some appropriate way. The following result, motivated by work in [3], shows that this is indeed the case.

COROLLARY. *Let* $0<a_n\to\infty$ *be reals and* $\nu_n>0$ *be integer-valued random variables on* $(\Omega, \mathscr{A}, p)$ *that satisfy*

$$\nu_n/a_n \xrightarrow{\mathscr{L}} F, \qquad F(0) = 0, \tag{7}$$

where $\xrightarrow{\mathscr{L}}$ *denotes convergence in law and* F *is a distribution function. Then* $S_{\nu_n} =^p o(\nu_n)$ *under the conditions of the proposition.*

PROOF. $S_n =^p o(n)$ by the proposition. Let ε, $\delta>0$ be given and choose $0<x<y<\infty$, x, y continuity points of F, so that $F(y)-F(x)>1-\varepsilon/6$. Defining

$$B_n = \{\omega\in\Omega : \nu_n/a_n \in (x, y]\},$$

and noting that both $|F(x)-p\{\nu_n/a_n\leqq x\}|<\varepsilon/12$ and $|F(y)-p\{\nu_n/a_n\leqq y\}|<\varepsilon/12$ are true for large n because of (7), $p(B_n)=p\{\nu_n/a_n\leqq y\}-F(y)+F(y)-F(x)+F(x)-p\{\nu_n/a_n\leqq x\}>F(y)-F(x)-\varepsilon/6>1-\varepsilon/3$ for large n.

Consequently, defining $D_n = \{\omega \in \Omega : |S_{\nu_n}| \geqq \delta\nu_n\}$,

(8) $$p(D_n) < \varepsilon/3 + p(D_n \cap B_n)$$

for large n.

Let $j_n = [x\dot{a}_n]$, $k_n = [ya_n]+1$ ($[\cdot]$ is the greatest integer function on $R \to N$), and note that $\nu_n \in [j_n, k_n]$ on B_n. Hence, the inclusion

$$D_n \cap B_n \subseteq B_n \cap (\{|S_{j_n}| \geqq \delta\nu_n/2\} \cup \{|S_{\nu_n} - S_{j_n}| \geqq \delta\nu_n/2\})$$

holds which, along with (8), means that

$$p(D_n) < \varepsilon/3 + p(B_n \cap \{|S_{j_n}| \geqq \delta\nu_n/2\}) + p(B_n \cap \{|S_{\nu_n} - S_{j_n}| \geqq \delta\nu_n/2\}).$$

When n is large, the middle term is $<\varepsilon/3$ because $S_n =^p o(n)$ and $\nu_n \geqq j_n$ on B_n; i.e., $p(D_n) < 2\varepsilon/3 + p(B_n \cap \{|S_{\nu_n} - S_{j_n}| > \delta\nu_n/2\})$.

Using the notation of the proof of the proposition, fix n and put

$$X'_j = X_j I_{E_{j,k_n}}, \qquad Y_j = X'_j - E(X'_j \mid \mathscr{F}_{j-1}),$$

$$S'_j = X'_1 + \cdots + X'_j, \qquad T_j = Y_1 + \cdots + Y_j, \qquad j = 1, \cdots, k_n.$$

On B_n, $\{|S_{\nu_n} - S_{j_n}| \geqq \delta\nu_n/2\} \subseteq \{\bigcup_* (\bar{E}_{j,k_n})\} \cup \{|S'_{\nu_n} - S'_{j_n}| \geqq \delta\nu_n/2\}$, $\bigcup_*$ being the union over $j \in [j_n, k_n]$, and $\{|S'_{\nu_n} - S'_{j_n}| \geqq \delta\nu_n/2\} \subseteq \{|S'_{\nu_n} - T_{\nu_n} + T_{j_n} - S'_{j_n}| \geqq \delta\nu_n/4\} \cup \{|T_{\nu_n} - T_{j_n}| \geqq \delta\nu_n/4\}$. Combining these inclusions with the last statement of the preceding paragraph, and using (1) and (2), there results, for large enough n,

$$p(D_n) < 8\varepsilon/9 + p(B_n \cap \{|T_{\nu_n} - T_{j_n}| \geqq \delta\nu_n/4\}).$$

Since $B_n \cap \{|T_{\nu_n} - T_{j_n}| \geqq \delta\nu_n/4\} \subseteq \{\sup(|T_i - T_{j_n}| \geqq \delta j_n/4, j_n \leqq i \leqq k_n)\}$, Kolmogoroff's inequality gives

$$p(D_n) < 8\varepsilon/9 + 16E(T_{k_n} - T_{j_n})^2/(\delta^2 j_n^2).$$

Finally, since $j_n^{-1} \leqq 4yx^{-1}k_n^{-1}$ and because $\{T_j, \mathscr{F}_j\}$ is a martingale, (3) applies to show $p(D_n) < \varepsilon$ for large n, as required for $S_{\nu_n} =^p o(\nu_n)$.

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