PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

A NOTE ON MAXIMAL INEQUALITIES[1]

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ABSTRACT. D. L. Burkholder and S. Sawyer extended in two apparently different directions a theorem of E. M. Stein involving maximal inequalities as necessary conditions for almost everywhere convergence. We will unify these theorems by showing that the like conclusions follow from certain conditions which are implied by each of Burkholder's and Sawyer's hypotheses.

Throughout this note we will use the following notation. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with $\mu(\Omega)=1$, and $\mathscr{D}$ be the collection of all sequences $f=(f_1, f_2, \cdots)$ with each f_n a measurable function. For $f \in \mathscr{D}$ define $f^*=\sup_n|f_n|$. If $f, g \in \mathscr{D}$, we write $f\sim g$ (respectively $f^*\sim g^*$) if f and g (respectively f^* and g^*) have the same distribution.

Let us state the two theorems.

BURKHOLDER'S THEOREM [1, THEOREM 2]. *Let $0<p<\infty$. Suppose that $\mathscr{C}\subset\mathscr{D}$ satisfies the following condition:*

If $f_k=(f_{k1}, f_{k2}, \cdots)\in\mathscr{C}$, $k=1,2,\cdots$, then there are sequences $g_k=(g_{k1}, g_{k2}, \cdots)\in\mathscr{D}$, $k=1,2,\cdots$, such that

(i) *the g_k's are independent,*

(ii) $f_k^*\sim g_k^*$, $k=1,2,\cdots$,

(iii) *if $\{a_k\}$ is a real number sequence satisfying $\sum_k |a_k|^p=1$, then the series $\sum_k a_k g_{kn}$ converges almost everywhere, $n=1,2,\cdots$, and there is an $h\in\mathscr{C}$ and a $\theta>0$ such that $h\sim\{\theta \sum_k a_k g_{kn}\}$.*

Then $\mu(f^*<\infty)>0$, $\forall f\in\mathscr{C}$ implies that there is an absolute constant K such that

$$\mu(f^* > \lambda) \leqq K/\lambda^p, \qquad \forall\lambda > 0, f\in\mathscr{C}.$$

Before we state Sawyer's theorem we need the following definition. A collection $\mathscr{E}$ of measure-preserving transformations on Ω is called an ergodic family if whenever $A\in\mathscr{A}$ satisfies $A=E^{-1}A$ (modulo null sets) for every $E\in\mathscr{E}$, then either $\mu A=0$ or $\mu A=1$.

Received by the editors January 24, 1972.

AMS 1970 *subject classifications*. Primary 42A36, 46E30.

Key words and phrases. Maximal inequalities, sequences of operators, almost everywhere convergence.

[1] This note is part of a doctoral dissertation written under the direction of Professor Richard A. Hunt at Purdue University.

SAWYER'S THEOREM [2, THEOREM 1]. *Let $\{T_n\}$ be a sequence of linear operators from $L^p(\Omega)$ $(1 \leqq p \leqq 2)$ into the set of measurable functions on Ω such that each T_n is continuous in measure, and $\mathscr{E}$ some ergodic family of measure-preserving transformations on Ω which "commute" with $\{T_n\}$ in the following sense*:

$$T^*g(x) \geqq T^*f(Ex), \qquad \forall f \in L^p,\ E \in \mathscr{E},$$

where $g(x)=f(Ex)$ and $T^*f(x)=\sup_n |T_nf(x)|$. Then

$$\mu(T^*f < \infty) > 0, \qquad \forall f \in L^p,$$

implies that there is an absolute constant K such that

$$\mu(T^*f > \lambda) \leqq (K/\lambda^p)\int_\Omega |f|^p\,d\mu, \qquad \forall \lambda > 0,\ f \in L^p. \tag{1}$$

REMARK. Sawyer's result contains Stein's theorem [3, Theorem 1], in which Ω is a homogeneous space of a compact group, and $\{T_n\}$ a sequence of operators which commute with translations.

We will utilize the common features of the proofs of these theorems to develop a hypothesis which is implied by those of Burkholder and Sawyer and which leads to the common conclusion. To this end we will introduce a collection $\mathscr{C} \subset \mathscr{D}$. Our hypothesis, given in terms of conditions on $\mathscr{C}$, will enable us to construct an example of an $h \in \mathscr{C}$ such that $h^* = \infty$ almost everywhere unless there exists an absolute constant K such that

$$\mu(f^* > \lambda) \leqq K/\lambda^p, \qquad \forall f \in \mathscr{C},\ \lambda > 0. \tag{2}$$

Fix p, $0<p<\infty$, and suppose there is no constant K such that (2) holds. We can then find a positive number sequence $\{\lambda_k\}$ and a sequence $\{f_k=(f_{k1}, f_{k2}, \cdots)\}$ in $\mathscr{C}$ such that

$$\mu(f_k^* > \lambda_k) > k^{p+2}/\lambda_k^p, \qquad k = 1, 2, \cdots.$$

Let $a_k=k/\lambda_k$, then $\mu(a_kf_k^*>k)>k^2a_k^p$. By repeating elements in the sequences $\{f_k\}$, $\{a_k\}$, we obtain new sequences, which we again call $\{f_k\}$, $\{a_k\}$, and a positive number sequence $\{R_k\}$ such that

$$\sum_k \mu(a_kf_k^* > R_k) = \infty, \quad \sum_k a_k^p < \infty \quad \text{and} \quad \lim_k R_k = \infty. \tag{3}$$

If $(a_kf_k^*>R_k)$, $k=1, 2, \cdots$, were independent, by the Borel-Cantelli Lemma, we would have

$$\mu\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}(a_kf_k^* > R_k)\right) = 1, \tag{4}$$

and we could continue with our construction. In general, however, (4) may not hold for $\{f_k\}$. To overcome this difficulty we associate $\{f_k\}$ with a sequence $\{g_k\}$ for which (4) does hold. Precisely, for every sequence $\{f_k\}$, $\{a_k\}$, $\{R_k\}$ satisfying (3), we need to have a sequence $\{g_k=(g_{k1}, g_{k2}, \cdots)\}$ in $\mathscr{D}$ such that

$$\text{(a)} \qquad \mu\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}(a_k g_k^* > R_k)\right) = 1.$$

From this it follows that

$$\limsup_{k\to\infty} a_k g_k^*(x) = \infty, \quad \text{almost every } x.$$

We now continue with our construction. Our next condition is motivated by the following result about Rademacher functions $r_k(t)$. (For definition see [**1**, p. 78].)

LEMMA 1 [**2**, LEMMA 1]. *If $\{a_{nk}\}$ is a sequence of real numbers such that* $\sum_k a_{nk}^2<\infty$, $n=1, 2, \cdots$, *and* $\limsup_{k\to\infty} \sup_n |a_{nk}|=\infty$, *then*

$$\sup_n \left|\sum_k a_{nk} r_k(t)\right| = \infty \quad \textit{almost everywhere.}$$

Thus, we will also want the g_k's to satisfy

(b) for almost every $t \in (0, 1)$, $\sum_k r_k(t)a_k g_{kn}(x)$ converges almost everywhere (x), $n=1, 2, \cdots$, or equivalently,

$$\sum_k a_k^2 g_{kn}(x)^2 < \infty, \quad \text{almost every } x,\ n = 1, 2, \cdots.$$

Then by the above lemma, we will obtain, for almost every t,

$$\text{(5)} \qquad \sup_n \left|\sum_k r_k(t)a_k g_{kn}(x)\right| = \infty, \quad \text{almost every } x.$$

For each t where (5) holds, we have a sequence $u_t=\{\sum_k r_k(t)a_k g_{kn}\}$ with $u_t^*(x)=\infty$ for almost every x. We would be done if $u_t \in \mathscr{C}$, or even if there exists an $h_t \in \mathscr{C}$ with $h_t^* \sim \theta_t u_t^*$ for some $\theta_t>0$. Hence, we require

(c) for every t in some set of positive measure, there is an $h_t \in \mathscr{C}$ and a $\theta_t>0$ such that

$$h_t^* \sim \theta_t \sup_n \left|\sum_k r_k(t)a_k g_{kn}\right|.$$

This completes our construction. Collecting results we obtain the following

THEOREM. *Let $0<p<\infty$. Suppose $\mathscr{C}\subset\mathscr{S}$ satisfies the following condition*:

For any sequence $\{f_k=(f_{k1}, f_{k2}, \cdots)\}$ in $\mathscr{C}$ and any positive number *sequences* $\{a_k\}$, $\{R_k\}$ *such that*

$$\sum_k \mu(a_k f_k^* > R_k) = \infty, \quad \sum_k a_k^p < \infty \quad \textit{and} \quad \lim_k R_k = \infty,$$

there exists a sequence $\{g_k = (g_{k1}, g_{k2}, \cdots)\}$ *in* $\mathscr{D}$ *such that*

(a) $\mu(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} (a_k g_k^* > R_k)) = 1$,

(b) *for almost every* $t \in (0, 1)$, $\sum_k r_k(t) a_k g_{kn}(x)$ *converges almost everywhere* (x), $n = 1, 2, \cdots$,

(c) *for every* t *in some set of positive measure, there is an* $h_t \in \mathscr{C}$ *and a* $\theta_t > 0$ *such that* $h_t^* \sim \theta_t \sup_n |\sum_k r_k(t) a_k g_{kn}|$.

Then $\mu(f^* < \infty) > 0$, $\forall f \in \mathscr{C}$, *implies that there is an absolute constant* K *such that*

$$\mu(f^* > \lambda) \leqq K/\lambda^p, \qquad \forall \lambda > 0, f \in \mathscr{C}.$$

Now we will proceed to show that Burkholder's and Sawyer's hypotheses both imply that of this theorem. The former is clear. In order to prove the latter we need the following

LEMMA 2 [2, LEMMA 2]. *Assume* $\mathscr{E}$ *is an ergodic family. Then, if* $\{A_k\}$ *is a sequence of measurable sets such that* $\sum_k \mu(A_k) = \infty$, *there exists a sequence* $\{E_k\}$ *in* $\mathscr{E}$ *such that*

$$\mu\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_k^{-1} A_k\right) = 1.$$

As we only have to establish (1) for $\int_\Omega |f|^p \, d\mu \leqq 1$, we let $\mathscr{C} = \{(T_1 f, T_2 f, \cdots) : \int_\Omega |f|^p \, d\mu \leqq 1\}$. Suppose $\{f_k\}$ is a sequence in L^p such that $\int_\Omega |f_k|^p \, d\mu \leqq 1$ and $\{a_k\}$, $\{R_k\}$ positive number sequences such that

$$\sum_k \mu(a_k T^* f_k > R_k) = \infty, \quad \sum_k a_k^p < \infty \quad \text{and} \quad \lim_k R_k = \infty.$$

From Lemma 2 we obtain a sequence $\{E_k\}$ in $\mathscr{E}$ such that

$$\mu\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} \{x : a_k T^* f_k(E_k x) > R_k\}\right) = 1.$$

Define $g_k(x) = f_k(E_k x)$, $k = 1, 2, \cdots$. By the commutativity of $\{T_n\}$ with $\mathscr{E}$ we get

$$\mu\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} (a_k T^* g_k > R_k)\right) = 1,$$

hence (a) holds.

The main step in proving (b) and (c) is the following:

$$\begin{aligned}\int_\Omega\int_0^1\left|\sum_{K_1}^{K_2} r_k(t)a_kg_k(x)\right|^p dt\,dx &\leqq \int_\Omega\left(\int_0^1\left|\sum_{K_1}^{K_2} r_k(t)a_kg_k(x)\right|^2 dt\right)^{p/2} dx\\ &= \int_\Omega\left(\sum_{K_1}^{K_2}|a_kg_k(x)|^2\right)^{p/2} dx\\ &\leqq \int_\Omega \sum_{K_1}^{K_2} a_k^p\,|g_k(x)|^p\,dx = \sum_{K_1}^{K_2} a_k^p\\ &\to 0 \quad \text{as } K_1, K_2 \to \infty.\end{aligned}$$

Note that we have used the fact that $1 \leqq p \leqq 2$ in the above inequalities. The above L^p convergence of $\sum_k r_k(t)a_kg_k(x)$ and the continuity in measure of T_n imply that, for almost every t, $\sum_k r_k(t)a_kT_ng_k(x)$ converges almost everywhere (x), $n=1, 2, \cdots$, and there is a $\theta_t > 0$ such that

$$\left\{\theta_t \sum_k r_k(t)a_kT_ng_k\right\} \in \mathscr{C}.$$

(See [**2**, p. 167], [**3**, p. 150].)

REMARK. In a similar manner we can show that there is a correspondence between [**1**, Theorem 1] and the first part of [**2**, Theorem 2].

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