

# ON THE RELATION BETWEEN THE ABEL-TYPE AND BOREL-TYPE METHODS OF SUMMABILITY

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**ABSTRACT.** In recent papers, R. Jajte and B. L. R. Sawyer have proved that under certain conditions a series summable by a Borel-type or absolute Borel-type method of summability is also summable by the Abel or absolute Abel method of summability to the same sum. In the present paper, the Abel method is replaced by the more general Abel-type method, giving similar results for ordinary, strong and absolute summability.

**1. Introduction.** It is known that the Abel method and the Borel exponential method of summability are not equivalent, but that under certain conditions, both methods sum the same series to the same sum [4]. This was recently extended in one direction, to the conditions under which a series summable by a Borel-type method is also summable by the Abel method [5]. The last result was extended to absolute summability by one of the present authors [9]. The object of this paper is to replace the Abel method by the more general Abel-type method and give results for ordinary, strong and absolute summability.

**2. Definitions.** Suppose throughout that  $\sigma$ ,  $a_n$  ( $n=0, 1, \dots$ ) are arbitrary complex numbers, that  $\lambda > -1$ , that  $\alpha > 0$  and that  $\beta$  is real. Let  $N$  be any nonnegative integer greater than  $1 - \beta/\alpha$ .

Define

$$s_n = \sum_{r=0}^n a_r; \quad s_{-1} = 0.$$

*2.1 Definitions of the Borel-type methods of summability.* Define

$$s(x) = \sum_{n=N}^{\infty} (s_n x^{\alpha n + \beta - 1}) / \Gamma(\alpha n + \beta); \quad S_{\alpha, \beta}(x) = \alpha e^{-x} s(x).$$

This series is assumed convergent for all  $x \geq 0$ .

**ORDINARY SUMMABILITY [1].** If  $S_{\alpha, \beta}(x) \rightarrow \sigma$  as  $x \rightarrow \infty$ , then  $s_n \rightarrow \sigma(B, \alpha, \beta)$ .

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STRONG SUMMABILITY [3]. If

$$\int_0^x e^t |S_{\alpha, \beta-1}(t) - \sigma|^p dt = o(e^x) \quad (p \geq 1)$$

as  $x \rightarrow \infty$ , then  $s_n \rightarrow \sigma [B, \alpha, \beta]_p$ .

ABSOLUTE SUMMABILITY [3]. If  $S_{\alpha, \beta}(x) \rightarrow \sigma$  as  $x \rightarrow \infty$  and  $S_{\alpha, \beta}(x)$  is of bounded variation with respect to  $x$  in the range  $[0, \infty)$ , then  $s_n \rightarrow \sigma [B, \alpha, \beta]$ .

2.2 *Definitions of the Abel-type methods of summability.* Define

$$E_n^\lambda = \binom{\lambda + n}{n} = \frac{(\lambda + 1)(\lambda + 2) \cdots (\lambda + n)}{n!} \quad \text{for } n = 0, 1, \dots,$$

$$E_0^\lambda = 1, \quad E_n^\lambda = 0 \quad \text{for } n = -1, -2, \dots,$$

$$\sigma_\lambda(y) = (1 + y)^{-\lambda-1} \sum_{n=0}^\infty E_n^\lambda s_n \left(\frac{y}{1 + y}\right)^n, \quad y > 0.$$

ORDINARY SUMMABILITY [2]. If the series defining  $\sigma_\lambda(y)$  is convergent for all  $y > 0$  and  $\sigma_\lambda(y) \rightarrow \sigma$  as  $y \rightarrow \infty$ , then  $s_n \rightarrow \sigma (A_\lambda)$ .

STRONG SUMMABILITY [6]. If the series defining  $\sigma_\lambda(y)$  is convergent for all  $y > 0$ , and

$$\int_0^y |\sigma_\lambda(t) - \sigma|^p dt = o(y) \quad (p \geq 1) \quad \text{as } y \rightarrow \infty,$$

then  $s_n \rightarrow \sigma [A_{\lambda-1}]_p$ .

ABSOLUTE SUMMABILITY [7]. If the series defining  $\sigma_\lambda(y)$  is convergent for all  $y > 0$ ,  $s_n \rightarrow \sigma (A_\lambda)$  and  $\sigma_\lambda(y)$  is of bounded variation with respect to  $y$  in the range  $[0, \infty)$ , then  $s_n \rightarrow \sigma |A_\lambda|$ .

*Note.* The condition that

$$(C) \quad \sum_{n=0}^\infty a_n x^n \quad \text{is convergent for all } x \text{ in } (-1, 1)$$

will be a condition for each main theorem of this paper. It can easily be shown that the condition (C) implies that the series defining  $\sigma_\lambda(y)$  is convergent for all  $y > 0$ .

**3. Preliminary results.** The following lemmas are required:

LEMMA 1. *If*

$$F(w) = \int_a^b g(w, u)f(u)du \quad (w \geq h > 0, -\infty < a < b < \infty),$$

and

$$\int_h^\infty |d_w g(w, u)| \leq g(u) \quad (a < u < b),$$

then

$$\int_h^\infty |dF(w)| \leq \int_a^b g(u) |f(u)| du.$$

PROOF. Let  $h = w_0 < w_1 < \dots < w_m$ , then for all  $m$

$$\begin{aligned} \sum_{k=0}^{m-1} |F(w_{k+1}) - F(w_k)| &\leq \int_a^b \sum_{k=0}^{m-1} |g(w_{k+1}, u) - g(w_k, u)| |f(u)| du \\ &\leq \int_a^b g(u) |f(u)| du. \end{aligned}$$

Hence

$$\int_h^\infty |dF(w)| \leq \int_a^b g(u) |f(u)| du.$$

LEMMA 2. Suppose that  $m$  is a positive integer and that

$$\mu_n = \prod_{i=1}^m \frac{b_i n + c_i}{d_i n + e_i} s_n \quad (n = 0, 1, \dots),$$

where  $b_i, c_i, d_i$  and  $e_i$  are all real with  $\prod_{i=1}^m d_i \neq 0$  and  $\prod_{i=1}^m (d_i n + e_i) \neq 0$ . Then

$$\mu_n \rightarrow \prod_{i=1}^m (b_i/d_i) \sigma(A_\lambda) \text{ (or } |A_\lambda| \text{) whenever } s_n \rightarrow \sigma(A_\lambda) \text{ (or } |A_\lambda| \text{)}.$$

PROOF. That  $s_n \rightarrow \sigma(A_\lambda)$  (or  $|A_\lambda|$ ) implies  $s_n/(n+k) \rightarrow 0$  ( $A_\lambda$ ) (or  $|A_\lambda|$ ), where  $k$  is real, is known ([2] or [8]). It follows that

$$\frac{bn + c}{dn + e} s_n = \frac{b}{d} s_n + \frac{cd - be}{d^2 n + de} s_n \rightarrow \frac{b}{d} \sigma(A_\lambda) \text{ (or } |A_\lambda| \text{)},$$

for all real  $b, c, d$  and  $e$  with  $d \neq 0$ . The conclusion then follows immediately by repeatedly using the above result.

Define

$$v_n = \frac{\Gamma(\alpha n + \beta + \lambda) \Gamma(n + 1)}{\Gamma(\alpha n + \beta) \Gamma(n + \lambda + 1)} s_n \quad \text{for } n = 0, 1, \dots$$

We then have the following two lemmas:

LEMMA 3. If  $s_n \rightarrow \sigma$  ( $B, \alpha, \beta$ ) and condition (C) holds, then for all  $\lambda > -1$ ,  $\nu_n \rightarrow \alpha^\lambda \sigma$  ( $A_\lambda$ ).

LEMMA 4. If  $s_n \rightarrow \sigma$  ( $B, \alpha, \beta$ ) and condition (C) holds, then for all  $\lambda > -1$ ,  $\nu_n \rightarrow \alpha^\lambda \sigma$  ( $A_\lambda$ ).

PROOF OF LEMMA 3. Let

$$J(t) = \int_0^\infty e^{-u/t} u^\lambda S_{\alpha, \beta}(u) du, \quad \text{where } 0 < t < \infty.$$

It follows by hypothesis that

$$J(t) = \alpha \Gamma(\lambda + 1) t^{\lambda+1} \left( \frac{t}{1+t} \right)^{\beta-1} \left( \frac{1+y}{1+t} \right)^{\lambda+1} \sigma_\lambda^*(y),$$

where  $t$  and  $y$  are related by  $(t/(1+t))^\alpha = y/(1+y)$ , and

$$\sigma_\lambda^*(y) = (1+y)^{-\lambda-1} \sum_{n=N}^\infty E_n^\lambda \nu_n \left( \frac{y}{1+y} \right)^n.$$

Since  $S_{\alpha, \beta}(u) \rightarrow \sigma$  as  $u \rightarrow \infty$ , it follows from Lemma 1 in [4] that

$$J(t)/\Gamma(\lambda + 1) t^{\lambda+1} \rightarrow \sigma \quad \text{as } t \rightarrow \infty.$$

Furthermore  $(1+y)/(1+t) \rightarrow 1/\alpha$  as  $y \rightarrow \infty$ , and  $t \rightarrow \infty$  if and only if  $y \rightarrow \infty$ . Hence

$$\sigma_\lambda^*(y) \rightarrow \alpha^\lambda \sigma \quad \text{as } y \rightarrow \infty.$$

This completes the proof.

PROOF OF LEMMA 4. Let  $J(t)$  and  $\sigma_\lambda^*(y)$  be defined as in the proof of the above lemma. We see that

$$\sigma_\lambda^*(y) = (1/\alpha \Gamma(\lambda + 1)) A_1(t) A_2(t) A_3(t),$$

where

$$A_1(t) = J(t)/t^{\lambda+1}, \quad A_2(t) = ((1+t)/t)^{\beta-1}$$

and

$$A_3(t) = \left[ \frac{1 - (t/(1+t))^\alpha}{1 - t/(1+t)} \right]^{\lambda+1}.$$

For  $h > 0$ , it is easy to show that  $A_2(t)$  and  $A_3(t)$  are of bounded variation with respect to  $t$  in  $[h, \infty)$ , whereas that  $A_1(t)$  is of bounded variation in  $[h, \infty)$  follows by hypothesis and Lemma 1. Thus  $\sigma_\lambda^*(y)$  is of bounded variation with respect to  $y$  in  $[g, \infty)$ , where  $g$ , greater

than zero, is dependent on  $h$ . Thus this lemma follows from the definition and the previous lemma.

**4. Theorems.** All the following theorems are under the condition (C).

**THEOREM 1.** *If  $s_n \rightarrow \sigma(B, \alpha, \beta)$ , then  $s_n \rightarrow \sigma(A_\lambda)$  for all  $\lambda > -1$ .*

**PROOF.** Since  $s_n \rightarrow \sigma(A_\lambda)$  implies  $s_n \rightarrow \sigma(A_\mu)$  for all  $\lambda > \mu > -1$  [2], we may assume that  $\lambda$  is an integer, and so by Lemma 3,

$$\nu_n = \frac{(\alpha n + \beta + \lambda - 1)(\alpha n + \beta + \lambda - 2) \cdots (\alpha n + \beta + 1)}{(n + \lambda)(n + \lambda - 1) \cdots (n + 1)} s_n \rightarrow \alpha^\lambda \sigma(A_\lambda).$$

Therefore by Lemma 2,  $s_n \rightarrow \sigma(A_\lambda)$  for all  $\lambda > -1$ .

**THEOREM 2.** *If  $s_n \rightarrow \sigma(B, \alpha, \beta)$ , then  $s_n \rightarrow \sigma[A_\lambda]_p$  for all  $p \geq 1, \lambda > -1$ .*

**PROOF.** We need that [6], if  $\lambda > -1$  and  $p \geq 1$ , then  $s_n \rightarrow \sigma[A_\lambda]_p$  if and only if  $s_n \rightarrow \sigma(A_\lambda)$  and

$$\int_0^x \left| y \frac{d}{dy} \sigma_\lambda(y) \right|^p dy = o(x), \quad \text{as } x \rightarrow \infty.$$

Since

$$y \frac{d}{dy} \sigma_\lambda(y) = (\lambda + 1)[\sigma_{\lambda+1}(y) - \sigma_\lambda(y)] = o(y)$$

as  $y \rightarrow \infty$ , by Theorem 1. It follows immediately that

$$\int_0^x \left| y \frac{d}{dy} \sigma_\lambda(y) \right|^p dy = o(x),$$

as  $x \rightarrow \infty$ . This completes the proof of this theorem.

Since  $s_n \rightarrow \sigma[B, \alpha, \beta]_q$  implies  $s_n \rightarrow \sigma(B, \alpha, \beta)$  for all  $q \geq 1$  [3], the following is a corollary of the previous theorem.

**THEOREM 3.** *If  $s_n \rightarrow \sigma[B, \alpha, \beta]_q$ , then  $s_n \rightarrow \sigma[A_\lambda]_p$  for all  $p \geq 1, \lambda > -1, q \geq 1$ .*

Finally, we give the corresponding result for absolute summability.

**THEOREM 4.** *If  $s_n \rightarrow \sigma |B, \alpha, \beta|$ , then  $s_n \rightarrow \sigma |A_\lambda|$  for all  $\lambda > -1$ .*

**PROOF.** Since  $s_n \rightarrow \sigma |A_\lambda|$  implies  $s_n \rightarrow \sigma |A_{\lambda+\delta}|$  for all  $\delta > 0$  [7], we may assume that  $\lambda$  is an integer, and then the rest of the proof is similar to that of Theorem 1, except that Lemma 3 is replaced by Lemma 4.

5. **Remark.** This section is presented for the sake of completeness. Let

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} \sum_{n=N}^{\infty} \frac{a_n t^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)} dt,$$

$$U_{\lambda}(y) = \lambda \int_0^y (1+t)^{-\lambda-1} \sum_{n=0}^{\infty} E_n^{\lambda} a_n \left( \frac{t}{1+t} \right)^n dt.$$

Both series are assumed to be convergent for all  $t \geq 0$ . We note that the "B'" methods and the "A'" methods are obtained by formally replacing  $S_{\alpha,\beta}(x)$  by  $A_{\alpha,\beta}(x)$ ,  $\sigma_{\lambda}(y)$  by  $U_{\lambda}(y)$  and  $\sigma$  by  $\sigma - s_{N-1}$ , respectively.

The following results are known ([3] and [8] respectively).

5.1.  $s_n \rightarrow \sigma$  ( $B', \alpha, \beta$ ) (or  $[B', \alpha, \beta]_q$ , or  $|B', \alpha, \beta|$ ) if, and only if,  $s_n \rightarrow \sigma$  ( $B, \alpha, \beta+1$ ) (or  $[B, \alpha, \beta+1]_q$ , or  $|B, \alpha, \beta+1|$ ).

5.2.  $s_n \rightarrow \sigma$  ( $A_{\lambda}$ ) (or  $[A_{\lambda}]_p$ , or  $|A_{\lambda}|$ ) if, and only if,  $s_n \rightarrow \sigma$  ( $A_{\lambda-1}$ ) (or  $[A_{\lambda-1}]_p$ , or  $|A_{\lambda-1}|$ ).

With these, it is easy to show that in Theorems 1-4, either  $B$  or  $A$ , or both, may be replaced by  $B'$  or  $A'$ , or both, respectively.

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