OSCILLATION CRITERIA FOR NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

KURT KREITH

ABSTRACT. Oscillation criteria are established for nonlinear matrix differential equations of the form $[R(t)U']'+F(t,U,U')=0$. These criteria are more general than some similar ones of E. C. Tomastik insofar as they do not require F to be positive definite.

In [1] Tomastik derives oscillation criteria for nonlinear matrix differential equations of the form

$$[R(t)U']' + F(t, U, U')U = 0 \tag{1}$$

under the hypothesis that the matrix F is positive definite. The purpose of this note is to present an oscillation criterion for (1) which does not require such an assumption.

As in [1] we consider a "prepared" solution $U(t)$ of (1) satisfying

$$U^*(t)R(t)U'(t) = U^{*\prime}(t)R(t)U(t), \tag{2}$$

and say that (1) is *oscillatory* on $[a, \infty)$ if the determinant of every prepared solution has arbitrarily large zeros. Our assumptions regarding the coefficient matrices are as in [1]. In particular, R and F are to be sufficiently regular, symmetric and real $n\times n$ matrices, and $R(t)$ is to be positive definite for all t.

The oscillation criterion to be derived below depends on a comparison of solutions of (1) and an equation of the same type,

$$[P(t)V']' + G(t, V, V')V = 0. \tag{3}$$

LEMMA 1. *Let $U(t)$ be a prepared matrix solution of* (1) *such that* $\det U(t)\neq 0$ *on some interval* $[b, c]$, *and let* $S(t)=R(t)U'(t)U^{-1}(t)$. *If $V(t)$ satisfies* (3) *on* $[b, c]$, *then*

$$\begin{aligned}[V^*PV' - V^*SV]_{t=b}^{t=c} = {} & \int_b^c V^*(F - G)V\,dt + \int_b^c V^{*\prime}(P - R)V'dt \\ & + \int_b^c (V' - U'U^{-1}V)^*R(V' - U'U^{-1}V)\,dt.\end{aligned} \tag{4}$$

PROOF. If $\det U(t)\neq 0$, then a direct computation using (2) and the

---

Received by the editors May 8, 1969.

AMS 1969 *subject classifications*. Primary 3442, 3445.

Key words and phrases. Oscillation criteria, prepared solution, Picone-type identity, Sturm-Liouville equation, Leighton oscillation criterion.

fact that $U^{-1\prime} = -U^{-1}U'U^{-1}$ yields the following Picone-type identity

$$(V^*PV' - V^*RU'U^{-1}V)' = V^*(PV')' - V^*(RU')'U^{-1}V + V^{*\prime}(P-R)V' + (V' - U'U^{-1}V)^*R(V' - U'U^{-1}V).$$

Substituting (1) and (3) for the first two terms on the right side of this equation and integrating, (4) follows readily.

LEMMA 2. *Suppose* $V(t)$ *is a nontrivial solution of* (3) *satisfying*

(i) $V^*(t)[F(t, U, U') - G(t, U, U')]V(t)$ *is positive semidefinite for* $b \leqq t \leqq c$ *and all values of* U *and* U',

(ii) $V^{*\prime}(t)[P(t) - R(t)]V'(t)$ *is positive semidefinite for* $b \leqq t \leqq c$,

(iii) $V(b) = V(c) = 0$.

If $U(t)$ *is a prepared solution of* (1), *then* det $U(t)$ *has a zero in* $[b, c]$.

PROOF. If det $U(t) \neq 0$ in $[b, c]$, then (4) holds and our hypotheses assure that the left side of (4) is 0 while each term on the right side of (4) is positive semidefinite. Furthermore, the last term on the right side of (4) is zero if and only if $V' - U'U^{-1}V \equiv 0$ on $[b, c]$, and this requires $V'(b) = 0$. By the uniqueness theorem for matrix systems, $V(b) = V'(b) = 0$ implies $V(t) \equiv 0$, contradicting the hypotheses and showing that det $U(t) = 0$ for some t in $[b, c]$.

Our oscillation criteria for (1) now follow easily by comparing (1) with the linear matrix equation

$$[p(t)IV']' + g(t)IV = 0. \tag{3'}$$

Let J be a nonzero matrix with zeros and ones down the main diagonal and zeros elsewhere.

THEOREM. *If the Sturm-Liouville equation* $(p(t)v')' + g(t)v = 0$ *is oscillatory at* $t = \infty$, *and if for some real* a *and some* J

(i) $J[F(t, U, U') - g(t)I]J$ *is positive semidefinite for* $t \geqq a$ *and all values of* U *and* U',

(ii) $J[p(t)I - R(t)]J$ *is positive semidefinite for* $t \geqq a$, *then* (1) *is oscillatory on* $[a, \infty)$.

PROOF. Let $v(t)$ be a nontrivial solution of $(pv')' + gv = 0$ which is oscillatory at ∞ and define $V(t) = v(t)J$. Then $V(t)$ satisfies (3′), and we can find arbitrarily large pairs of numbers (b, c) satisfying $c > b > a$ for which $V(b) = V(c) = 0$. Furthermore

$$V^*[F - gI]V = v^2J[F - gI]J \quad \text{and} \quad V^{*\prime}[pI - R]V' = v'^2J[pI - R]J$$

so that conditions (i) and (ii) of Lemma 2 are satisfied on $[a, \infty)$. By Lemma 2, equation (1) is oscillatory on $[a, \infty)$.

In order to apply this Theorem, it is useful to recall the Leighton oscillation criterion: if

$$\int^{\infty} \frac{1}{p_1(t)}\, dt = \int^{\infty} g_1(t)\, dt = \infty, \tag{5}$$

then $(p_1 v')' + g_1 v = 0$ is oscillatory at $t = \infty$. Consider now the system (1) where $n = 2$ and

$$0 < R(t) \leqq \begin{pmatrix} p_1(t) & 0 \\ 0 & p_2(t) \end{pmatrix}, \qquad F(t, U, U') \geqq \begin{pmatrix} g_1(t) & 0 \\ 0 & g_2(t) \end{pmatrix}.$$

According to the Theorem with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

if (5) is satisfied then (1) is oscillatory. This result does not follow from [1].

References

1. E. C. Tomastik, *Oscillation of nonlinear matrix differential equations of second order*, Proc. Amer. Math. Soc. **19** (1968), 1427–1431. MR **38** #372.

2. E. L. Ince, *Ordinary differential equations*, Longmans, Green, London, 1926; reprint, Dover, New York, 1944. MR **6**, 65.

University of California, Davis, California 95616