THE HARDY CLASS OF A SPIRAL-LIKE FUNCTION AND ITS DERIVATIVE

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ABSTRACT. A determination is made of the Hardy classes to which a spiral-like univalent function and its derivative belong. An estimate for the size of the Taylor coefficients is deduced.

Let $f(z)$ be analytic in D, the unit disc $|z|<1$, and let α be a real number such that $|\alpha|<\pi/2$. If $f(0)=0, f'(0)\neq 0$, and if

$$\operatorname{Re}[e^{i\alpha}zf'(z)/f(z)] > 0, \qquad z \in D,$$

then $f(z)$ is univalent [5] and is said to be spiral-like. Under these conditions we have

$$e^{i\alpha}zf'(z)/f(z) = Q(z),$$

where $\operatorname{Re} Q(z)>0$ and $Q(0)=e^{i\alpha}$. Defining $P(z)=Q(z)\sec\alpha-i\tan\alpha$, we may write

$$zf'(z)/f(z) = e^{-i\alpha}[P(z)\cos\alpha + i\sin\alpha], \tag{1}$$

where $\operatorname{Re} P(z)>0$, $P(0)=1$.

If $f'(0)=1$,[2] if $f(z)$ satisfies (1), and if $\operatorname{Re} P(z)\geqq\rho\geqq 0$, $z\in D$, we shall say that $f(z)$ belongs to the class $S_{\alpha,\rho}$ [3]. In particular, with $\alpha=0$, $S_{0,\rho}$ coincides with the class of normalized starlike functions of order ρ. The relationship between $S_{\alpha,\rho}$ and $S_{0,\rho}$ is indicated in the following lemma.

LEMMA 1. *$f(z)\in S_{\alpha,\rho}$ if and only if there is a $g(z)\in S_{0,\rho}$ such that*

$$[f(z)/z]^{\exp(i\alpha)} = [g(z)/z]^{\cos\alpha}, \tag{2}$$

where the branches are chosen so that each side of the equation has the value 1 *when* $z=0$.

PROOF. If $g(z)\in S_{0,\rho}$ then $zg'(z)/g(z)=P(z)$, where $\operatorname{Re} P(z)\geqq\rho$, $P(0)=1$. If $f(z)$ is defined by (2) then, differentiating logarithmically and multiplying by $ze^{-i\alpha}$, we obtain (1), which shows that $f(z)\in S_{\alpha,\rho}$.

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[2] This normalization is made only for convenience. The conclusion of Theorem 2 and its corollary remain valid for the classes $S_{\alpha,\rho}$, when defined without normalization.

Conversely, if $f(z) \in S_{\alpha,\rho}$ and if we define $g(z)$ by

$$g(z)/z = [f(z)/z]^{1+i \tan \alpha},$$

then a similar calculation shows that $g(z) \in S_{0,\rho}$.

For a real $\lambda > 0$, we say that a function $h(z)$, analytic in D, belongs to the class H^λ if

$$\int_{-\pi}^{\pi} | h(re^{i\theta}) |^\lambda d\theta < K$$

for $0 \leqq r < 1$, where K is a constant depending on $h(z)$ and λ.

The following theorem is equivalent to Theorem 6 in [1].

THEOREM 1. *If $g(z) \in S_{0,\rho}$ is not of the form*

$$g(z) = z(1 - ze^{i\tau})^{2\rho-2}$$

for some real τ, then

(i) *there exists $\delta = \delta(g) > 0$ such that $g(z)/z \in H^{(1+\delta)/2(1-\rho)}$; and*

(ii) *there exists $\epsilon = \epsilon(g) > 0$ such that $g'(z) \in H^{(1+\epsilon)/(3-2\rho)}$.*

The object of this note is to extend this theorem to the classes $S_{\alpha,\rho}$. To do this we require some further lemmas.

LEMMA 2. *If $g(z) \in S_{0,0}$, then*

$$| \arg(g(z)/z) | < \pi, \qquad z \in D,$$

where the principal value of the argument is taken.

LEMMA 3. *If $Q(z)$ is analytic and* Re $Q(z) > 0$ *in D, then $Q(z) \in H^\lambda$ for all $\lambda < 1$.*

LEMMA 4. *If $h(z) \in H^\lambda$, $0 < \lambda < 1$, and $h(z) = \sum_0^\infty a_n z^n$, then*

$$a_n = o(n^{(1/\lambda)-1}).$$

Lemma 2 is in [4], Lemma 4 in [2]. Lemma 3 is well known.

The following theorem contains Theorem 1.

THEOREM 2. *If $f(z) \in S_{\alpha,\rho}$ is not of the form*

(3) $\qquad f(z) = z(1 - ze^{i\tau})^{-a}, \qquad a = 2(1-\rho)(\cos\alpha - i \sin\alpha)\cos\alpha$

for some real τ, then

(i) *there exists $\delta = \delta(f) > 0$ such that*

$$f(z)/z \in H^\mu, \qquad \mu = (1+\delta)\sec^2\alpha/2(1-\rho);$$

and

(ii) *there exists* $\epsilon=\epsilon(f)>0$ *such that*

$$f'(z)\in H^\nu,\qquad \nu=(1+\epsilon)/(1+2(1-\rho)\cos^2\alpha).$$

PROOF. (i) By Lemma 1, there is a function $g(z)\in S_{0,\rho}$ such that

$$f(z)/z=[g(z)/z]^{\cos^2\alpha-i\sin\alpha\cos\alpha}.$$

Taking moduli we obtain

$$|f(z)/z|=|g(z)/z|^{\cos^2\alpha}\exp[\sin\alpha\cos\alpha\arg(g(z)/z)],$$

so

$$|f(z)/z|^\mu=|g(z)/z|^{(1+\delta)/2(1-\rho)}\exp[(1+\delta)\tan\alpha/2(1-\rho)\arg(g(z)/z)].$$

By Lemma 2, since $S_{0,\rho}\subset S_{0,0}$, the exponential factor is bounded, and the conclusion follows immediately from part (i) of Theorem 1.

(ii) Writing $e^{i\alpha}zf'(z)/f(z)=Q(z)$, and applying Lemma 3, we see that

$$zf'(z)/f(z)\in H^\lambda,\qquad \text{all }\lambda<1. \tag{4}$$

Next, writing

$$f'(z)=(f(z)/z)\,(zf'(z)/f(z)),$$

and apply Hölder's inequality with conjugate indices p, q to $|f'(z)|^\lambda$, with $z=re^{i\theta}$ we obtain

$$\int_{-\pi}^{\pi}|f'(z)|^\lambda d\theta\leqq\left(\int_{-\pi}^{\pi}\left|\frac{f(z)}{z}\right|^{\lambda p}d\theta\right)^{1/p}\left(\int_{-\pi}^{\pi}\left|\frac{zf'(z)}{f(z)}\right|^{\lambda q}d\theta\right)^{1/q}=I_1\cdot I_2,$$

say. By part (i), I_1 is bounded if we choose λ, p so that $\lambda p=\mu$, and by (4), I_2 is bounded if we make the further restriction $\lambda q<1$. This is achieved by taking $\lambda=\nu$ provided that

$$\nu=\frac{1+\epsilon}{1+2(1-\rho)\cos^2\alpha}<\frac{1+\delta}{1+\delta+2(1-\rho)\cos^2\alpha},$$

which holds if ϵ is sufficiently small.

From part (ii) of the theorem and Lemma 4 we deduce the following:

COROLLARY. *If* $f(z)\in S_{\alpha,\rho}$ *is not of the form* (3) *for some real* τ, *and if*

$$f(z)=\sum_0^\infty a_nz^n,$$

then there exists $\eta=\eta(f)>0$ *such that*

$$a_n = o(n^{2(1-\rho)\cos^2\alpha-1-\eta}).$$

In conclusion we remark that for the function (3) we have

$$a_n = \frac{\Gamma(a+n)}{\Gamma(n+1)\Gamma(a)} e^{ni\tau},$$

$$|a_n| \sim |n^{a-1}| / |\Gamma(a)| = n^{2(1-\rho)\cos^2\alpha-1} / |\Gamma(a)|.$$

This shows that (3) is indeed exceptional in the corollary, and therefore also in part (ii) of Theorem 2; a computation shows also that for the function (3) we have $f(z)/z \in H^\lambda$ if and only if $\lambda < \frac{1}{2}\sec^2\alpha/(1-\rho)$, so that this function is also exceptional in part (i) of Theorem 2.

A weaker consequence of the corollary, well known for starlike functions, is that *for all spiral-like functions* $f(z)$ *with the exception only of those of the form* $z(1-ze^{i\tau})^{-2}$, *we have* $a_n = o(n^{1-\eta})$ *for some* $\eta = \eta(f) > 0$.

Added in proof (June 16, 1970). Application of Theorem 2 and of Theorem E of [1] yields the following result for functions of the class C_α [6].

If $f(z) \in C_\alpha$ *is not of the form*

$$f(z) = e^{-i\tau}(1 - ze^{i\tau})^{-a+1}/(a-1) + c, \quad a = 2(\cos\alpha - i\sin\alpha)\cos\alpha$$

for some complex c *and real* τ, *then there exists* $\delta = \delta(f) > 0$ *such that*

(i) $f'(z) \in H^\beta$, $\beta = \frac{1}{2}(1+\delta)\sec^2\alpha$;

(ii) *if* $|\alpha| < \pi/4$ *then* $f(z) \in H^\gamma$, $\gamma = (1+\delta)\sec^2\alpha/[2-(1+\delta)\sec^2\alpha]$.

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