COMPONENT FUNCTORS

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Introduction. Many concrete categories $\mathcal{A}$ admit, besides the usual forgetful functor, another such functor $V: \mathcal{A} \to \mathcal{S}$ (where $\mathcal{S}$ is the category of sets) which may reasonably be termed a "component functor." For example, let M be a monoid and let $\mathcal{A}$ be the category of sets on which M acts on the left. If M acts on A, let $\sim$ be the equivalence relation generated by $a \sim ma$ for $a \in A$, $m \in M$. Then the component functor V takes A to the set $A/\sim$ of connected components of A under the action of M.

Many such functors are especially interesting in that they have right adjoints $U: \mathcal{S} \to \mathcal{A}$ which are *monadic* (i.e., triplable). In this paper we show that this situation will arise whenever $\mathcal{A}$ is *comonadic* over $\mathcal{S}$, and exhibit some typical examples.

1. **Statement of the theorem.** Let $\boldsymbol{F} = (F, \epsilon, \delta)$ be a comonad over $\mathcal{S}$. We recall that this means that $F: \mathcal{S} \to \mathcal{S}$ is a functor and that $\epsilon: F \to 1_{\mathcal{S}}$ and $\delta: F \to F^2 = F \circ F$ are natural transformations such that $F\epsilon \circ \delta = \epsilon F \circ \delta = 1_F$ and $F\delta \circ \delta = \delta F \circ \delta$. An algebra over $\boldsymbol{F}$ is a pair (A, ξ) consisting of a set A and a map $\xi: A \to FA$ such that $\epsilon_A \circ \xi = 1_A$ and $F(\xi) \circ \xi = \delta_A \circ \xi$. A homomorphism $f: (A, \xi) \to (B, \rho)$ is a map $f: A \to B$ such that $\rho \circ f = F(f) \circ \xi$. Let $\mathcal{A}$ be the category of algebras over $\boldsymbol{F}$.

We note that $\mathcal{A}$ possesses coproducts, which are formed by disjoint union just as in the category of sets. For example, the coproduct of (A, ξ) and (B, ρ) is $(A \coprod B, \omega)$ where the map $\omega: A \coprod B \to F(A \coprod B)$ is the composition

$$A \coprod B \xrightarrow{\xi \coprod \rho} F(A) \coprod F(B) \xrightarrow{\gamma} F(A \coprod B)$$

and γ is defined by $\gamma \circ i_{F(A)} = F(i_A)$, $\gamma \circ i_{F(B)} = F(i_B)$. If X is any set, we shall denote by $X \cdot A$ the coproduct of $|X|$ copies of A.

Note also that $\mathcal{A}$ has a terminal object T. Using 1 to denote a fixed one-element set, we have $T = F(1)$ equipped with the map $\delta_1: T \to F(T)$. Now we are ready to state the

THEOREM. *The functor $U: \mathcal{S} \to \mathcal{A}$ defined by $U(X) = X \cdot T$ is monadic.*

The left adjoint V to U is called the *component functor* associated with $\mathcal{A}$.

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2. **Proof of the theorem.** First we shall prove that U has a left adjoint V, by appealing to the adjoint-functor theorem of Freyd [4]. Observe that a map $f: (A, \xi)\to X\cdot T$ is a homomorphism iff the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "\xi"'] & & X\cdot T \arrow[d, "X\cdot\delta_1"] \\ & & X\cdot F(T) \arrow[d, "\gamma"] \\ F(A) \arrow[rr, "F(f)"] & & F(X\cdot T) \end{tikzcd}$$

commutes. There is a unique homomorphism $e_A: A\to T$ which is equal to $F(\bar{e}) \circ \xi: A\to F(A)\to F(1)=T$ where $\bar{e}: A\to 1$ is the unique set map. Then for each $a\in A, f(a)$ must be the element $e_A(a)$ in some summand of $X\cdot T$ labeled by $\chi(a)\in X$. In this way we obtain a function $\chi: A \to X$ and by chasing elements in the diagram it is easily seen that the diagram commutes iff for every $a\in A$, we have the additional property that

$$F(f)\xi(a) = F(i_{\chi(a)})\delta_1 e_A(a).$$

Using this characterization of homomorphisms $A\to X\cdot T$ we shall show that the functor U preserves limits.

To show that U preserves equalizers, let $f, g: X\rightrightarrows Y$ and let $h: E \to X$ be $eq(f, g)$. We claim that $h\cdot T: E\cdot T\to X\cdot T$ is the equalizer in $\mathfrak{A}$ of $f\cdot T$ and $g\cdot T$. For, suppose that $k: (A, \xi)\to X\cdot T$ is such that $(f\cdot T)\circ k=(g\cdot T)\circ k$. Then there is a unique set map $j: A\to E\cdot T$ such that $k=(h\cdot T)\circ j$, and we need only show that j is a homomorphism. Let $k(a)$ be the element $e_A(a)$ in the summand of $X\cdot T$ labeled by $\chi(a)$ and let $j(a)$ be the element $e_A(a)$ in the summand of $E\cdot T$ labeled by $\eta(a)$. Then

$$\begin{aligned} F(h\cdot T)F(j)\xi(a) &= F(k)\xi(a) \\ &= F(i_{\chi(a)})\delta_1 e_A(a) \\ &= F(h\cdot T)F(i_{\eta(a)})\delta_1 e_A(a). \end{aligned}$$

Since $F(h\cdot T)$ is monic, $F(j)\xi(a)=F(i_{\eta(a)})\delta_1 e_A(a)$ and thus j is indeed a homomorphism.

Next, it is rather more difficult to show that U preserves products; for simplicity we write the proof for binary products but the proof for infinitary products is similar. Thus we must prove that the product in $\mathfrak{A}$ of $X\cdot T$ and $Y\cdot T$ is $(X\times Y)\cdot T$ where $X\times Y$ is the product in $\mathcal{S}$. Suppose $f: (A, \xi)\to X\cdot T$ and $g: (A, \xi)\to Y\cdot T$, where $f(a)$ is the $e_A(a)$ in the $\chi(a)$-summand and $g(a)$ is the $e_A(a)$ in the $\eta(a)$-summand. Define $h: A\to(X\times Y)\cdot T$ by requiring $h(a)$ to be $e_A(a)$ in the summand labeled by $(\chi(a), \eta(a))$. Then clearly we need only show that h is a homomorphism.

First, we note that

$$\begin{aligned} F(f)\xi(a) &= F(i_{\chi(a)})\delta_1 e_A(a) \\ &= F(i_{\chi(a)})\delta_1 F(\bar{e})\xi(a) \\ &= F(i_{\chi(a)})F^2(\bar{e})\delta_A\xi(a) \\ &= F(i_{\chi(a)} \circ F(e) \circ \xi)\xi(a) \\ &= F(i_{\chi(a)} \circ e_A)\xi(a), \end{aligned}$$

and similarly $F(g)\xi(a)=F(i_{\eta(a)} \circ e_A)\xi(a)$. To show that h is a homomorphism, i.e., to show that the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dd, "\xi"'] & (X\times Y)\cdot T \arrow[d, "(X\times Y)\cdot\delta_1"] \\ & (X\times Y)\cdot F(T) \arrow[d, "\gamma"] \\ F(A) \arrow[r, "F(h)"] & F((X\times Y)\cdot T) \end{tikzcd}$$

commutes, we must show that for all $a\in A$,

$$F(h)\xi(a) = F(i_{(\chi(a),\eta(a))} \circ e_A)\xi(a).$$

The result would be immediate if we knew that the map $(F(p_1), F(p_2))$: $F((X\times Y)\cdot T)\to F(X\cdot T)\times F(Y\cdot T)$ is one-to-one. Unfortunately, there are many examples of comonads for which it is *not* one-to-one. However, it will suffice to show that $(F(p_1), F(p_2))$ is one-to-one *on the image of* $\gamma \circ (X\times Y)\cdot\delta_1$. In fact, we shall show that the composition

$$k = (F(p_1), F(p_2)) \circ \gamma \circ (X \times Y)\cdot\delta_1 : (X \times Y)\cdot T \to F(X\cdot T) \times F(Y\cdot T)$$

is actually a one-to-one function.

A little computation using the definition of γ shows that $k =(\gamma_X, \gamma_Y) \circ (X\times Y)\cdot\delta_1$, where γ_X: $(X\times Y)\cdot F(T)\to F(X\cdot T)$ is defined by $\gamma_X \circ i_{(x,y)} = F(i_x)$. $F(T)\to F(X\cdot T)$ and likewise $\gamma_Y \circ i_{(x,y)} = F(i_y)$. Then we have $k \circ i_{(x,y)} =(F(i_x), F(i_y)) \circ \delta_1$: $T\to F(T)\to F(X\cdot T) \times F(Y\cdot T)$. But δ_1, $F(i_x)$, and $F(i_y)$ are one-to-one and hence $k \circ i_{(x,y)}$ is one-to-one for every $(x, y)\in X\times Y$.

To show that k is one-to-one, it now suffices to show that the images of distinct summands under k are *disjoint*; that is, if $(x, y) \neq (\bar{x}, \bar{y})$ then the maps $(F(i_x), F(i_y)) \circ \delta_1$ and $(F(i_{\bar{x}}), F(i_{\bar{y}})) \circ \delta_1$: $T \to F(X \cdot T) \times F(Y \cdot T)$ have disjoint images. In fact, more is true: if $x \neq \bar{x}$ then the images of $F(i_x) \circ \delta_1$ and $F(i_{\bar{x}}) \circ \delta_1$ are disjoint (and similarly if $y \neq \bar{y}$). For, if they were not disjoint, they would continue to fail to be disjoint upon application of the map $\epsilon_{X \cdot T}$. But the diagram

$$\begin{tikzcd} T \arrow[r, "\delta_1"] \arrow[dr, "1_T"'] & F(T) \arrow[r, shift left, "F(i_x)"] \arrow[r, shift right, "F(i_{\bar{x}})"'] \arrow[d, "\epsilon_T"] & F(X \cdot T) \arrow[d, "\epsilon_{X \cdot T}"] \\ & T \arrow[r, shift left, "i_x"] \arrow[r, shift right, "i_{\bar{x}}"'] & X \cdot T \end{tikzcd}$$

shows that this would be impossible. Hence k is one-to-one, and we have completed the proof that U preserves limits.

Now, in order to show that U has a left adjoint V, it suffices to show that for each (A, ξ) there is a family $\{B_\alpha\}$ of sets such that whenever f: $(A, \xi) \to X \cdot T$, there is a homomorphism η: $(A, \xi) \to B_\alpha \cdot T$ and a set map g: $B_\alpha \to X$ such that $f = (g \cdot T) \circ \eta$. But given f, its image contains at most $|A|$ elements and thus is contained in $B \cdot T$ where $B \subseteq X$ and $|B| \leqq |A|$. Thus, for a solution set for (A, ξ) we can take the set of all subsets of A. This establishes the existence of the component functor V: $\mathcal{A} \to \mathcal{S}$, by Freyd's adjoint functor theorem.

Finally we proceed to show that U is monadic by appealing to Beck's Triplability Theorem [1], [3]. Let d_0, d_1. $X \rightrightarrows Y$ and let g: $Y \to W$ be coeq(d_0, d_1). Suppose that in $\mathcal{A}$ we have a diagram

$$\begin{tikzcd} X \cdot T \arrow[r, shift left, "d_0 T"] \arrow[r, shift right, "d_1 T"'] & Y \cdot T \arrow[l, bend right=60, "t"'] \arrow[r, shift left, "z"] & Z, \arrow[l, shift left, "s"] \end{tikzcd}$$

where $zs = \text{id.}$, $sz = (d_1 \cdot T) \circ t$, $(d_0 \cdot T) \circ t = \text{id.}$, and $z \circ (d_0 \cdot T) = z \circ (d_1 \cdot T)$. Then $z = \text{coeq}(d_0 \cdot T, d_1 \cdot T)$, and in this situation Beck's theorem requires that $Z \cong W \cdot T$, $z \cong q \cdot T$. These equations are certainly true set-wise, so what we have to show is that the $\boldsymbol{F}$-algebra structure on Z is that of $W \cdot T$. But this is easily seen by observing that $W \cdot T$ does function as the coequalizer of $d_0 \cdot T$ and $d_1 \cdot T$ in $\mathcal{A}$. This completes the proof of our theorem.

3. **Examples.** A. Let M be a monoid, let T be a set on which M acts, and let Mac(M, T) be the category whose objects are sets A on

which M acts, equipped with M-homomorphisms $e_A: A \to T$. A map $f: (A, e_A) \to (B, e_B)$ is to be an M-homomorphism such that $e_B \circ f = e_A$. $\mathrm{Mac}(M, T)$ is comonadic over $\mathcal{S}$ by [2]. Then $U(X) = (X \times T, e_{X\times T})$ where $m(x, t) = (x, mt)$ and $e_{X\times T} = p_2$. The component functor takes (A, e_A) to the set of connected components of A under the action of M. (This example was pointed out to the author by John Isbell in a personal communication.)

B. Let $\mathcal{A}$ be the category of sets equipped with equivalence relations. Using $[a]$ to denote the equivalence class of a, define a map $f: (A, \sim) \to (B, \sim)$ to be a function $f: A \to B$ such that $f[a] = [fa]$ for all $a \in A$. Then $\mathcal{A}$ is comonadic, $U(X) = (X, =)$, and $V(A, \sim) = A/\sim$.

C. For a more elaborate example, construct a comonad (F, ϵ, δ) as follows. Let $F(A) = A \times 2^A$, $\epsilon_A = p_1$, and define δ_A by $\delta_A(a, \alpha) = (a, \alpha, \{(b, \alpha) \mid b \in \alpha\})$. Then it is easy to compute that an algebra over this monad is essentially a set A equipped with a function $\phi_A: A \to 2^A$ with the property that $\phi_A(a) = \phi_A(a')$ whenever $a' \in \phi_A(a)$. A homomorphism $f: (A, \phi_A) \to (B, \phi_B)$ is a function such that $f(\phi_A(a)) = \phi_B(f(a))$ for all $a \in A$. The terminal object is $T = \{0, 1\}$ where $\phi_T(0) = \varnothing$ and $\phi_T(1) = \{1\}$. The component functor takes (A, ϕ_A) to the set

$$\{a \mid a \in A, \phi_A(a) = \varnothing\} \cup \{\phi_A(a) \mid a \in A, \phi_A(a) \neq \varnothing\}.$$

This case is particularly interesting since it admits another monadic functor $U: \mathcal{S} \to \mathcal{A}$ where $U(A) = (A \amalg \{1\}, \phi)$, $\phi(a) = \{a\}$, and $\phi(1) = \varnothing$. The left adjoint to this functor takes (A, ϕ_A) to the set $\{\phi_A(a) \mid a \in A, \phi_A(a) \neq \varnothing\}$. Thus, a category can admit reasonable "component functors" other than those for which we have provided a categorical framework.

References

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