PRODUCT SOLUTIONS FOR SIMPLE GAMES. III

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1. **Introduction.** For simple games solution theory, the main object of investigation now is to determine wider classes of solutions for wider classes of games and to develop the structural relationships between solutions of structurally related games. In this paper we obtain partial families of solutions to certain compound games.

2. **Definitions and notations.** We shall denote a simple game by the symbol $\Gamma(P, W)$ where P is a finite set (players) and W is a collection of subsets of P (the winning coalitions). We demand that $P \in W$ and the empty set is not an element of W.

Let $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ be two simple games with $P_1 \cap P_2 = \varnothing$ and let $P = P_1 \cup P_2$. Then the product $\Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2)$ [for simplicity we will write $P_1 \otimes P_2$] is defined as the game $\Gamma(P, W)$ where W consists of all $S \subseteq P$ such that $S \cap P_i \in W_i$ for $i = 1, 2$. We will use the same notation P for the number of elements in P. By an imputation we mean a nonnegative vector x such that $\sum_{i=1}^{P} x_i = 1$. A_P will stand for the collection of all imputations. We recall that a solution of the game $\Gamma(P, W)$ is a set X of imputations such that $X = A_P - \text{dom } X$ where dom X denotes the set of all $y \in A_P$ such that for some $x \in X$, the set $[i : x_i > y_i]$ is an element of W. The notations dom_1 and dom_2 will be used for domination with respect to special classes W_1 and W_2.

DEFINITION. A parametrized family of sets of imputations

$$[Y(\alpha): 0 \leq \alpha \leq 1]$$

will be called semimonotone if for every α, β, x such that $0 \leq \alpha \leq \beta \leq 1$ and $x \in Y(\beta)$ there exists $y \in Y(\alpha)$ with $\alpha y \leq \beta x$. [Here α, β are scalars, y, x are vectors and $\alpha y \leq \beta x$ means each one of the coordinates in αy is less than or equal to the corresponding coordinates in βx.]

DEFINITION. A semimonotonic family is called δ-monotonic $(0 \leq \delta \leq 1)$ if for every α, β, y such that $\delta \leq \alpha \leq \beta \leq 1$ and $y \in Y(\alpha)$ there exists $x \in Y(\beta)$ with $\alpha y \leq \beta x$.

We call a 0-monotonic family fully-monotonic. In general δ will stand for any positive number with $0 < \delta < 1$. Let $P = P_1 \cup P_2$ and let

$$A_{P_i} = \left\{ x : x \in A_P, \sum_{j \in P_i} x_j = 1 \right\} \qquad \text{for } i = 1, 2.$$

DEFINITION. Let Y be a solution to the product of simple games $P_1 \otimes P_2$. Call Y a product solution if the following conditions are met.

(i) There exists a semimonotonic family $\{Y_i(\alpha): 0 \leq \alpha \leq 1\}$ such that $Y_i(\alpha)$ are solutions to P_i for all α except $\alpha = 1$ where $i = 1, 2$ and

(ii) $Y = \bigcup_{0 \leq \alpha \leq 1} Y_1(\alpha) \times_\alpha Y_2(1-\alpha) \cup Y_1^* \cup Y_2^*$ where $Y_i^* = A_{P_i} - Y_i(1) - \mathrm{dom}_i Y_i(1)$ and $Y_1(\alpha) \times_\alpha Y_2(1-\alpha)$ defined as the set $= \{z: z = \alpha x_1 + (1-\alpha)x_2 \text{ for some } x_1 \in Y_1(\alpha) \text{ and } x_2 \in Y_2(1-\alpha)\}$.

DEFINITION. Let $\Gamma(P, W)$ be a simple game. Let X be any subset of A_P. Call X an internally stable set if $X \cap \mathrm{dom}\ X = \emptyset$. Call X an externally stable set if $X \cup \mathrm{dom}\ X = A_P$. Call X a solution if X is both internally stable and externally stable.

DEFINITION. Let $\Gamma(P, W)$ be a simple game. Let $x, y \in A_P$. Let $S \in W$. We say x dominates y via S (in notation $x > y$ via S or $x >_S y$) if $x_i > y_i$ for all $i \in S$.

3. A theorem on the simple game J.
The four-person game J is defined by

$$J = \Gamma((1, 2, 3, 4), \{124, 134, 234, 1234\})$$

where members in the curly brackets denote the winning coalitions of the game J.

$$J = M_{123} \otimes B_4.$$

M_{123} denotes the 3-person simple majority game and B_4 the 1-person pure bargaining game.

In [2] we have proved the following theorem.

THEOREM 1. *Let $0 \leq \delta < 1$ and let $[Y_1(\alpha): 0 \leq \alpha \leq 1]$ be any δ-monotonic family of product solutions except that $Y_1(1)$ need not be externally stable to the game $J = M_{123} \otimes B_4$. Then*

$$Y = \bigcup_{0 \leq \alpha \leq 1} Y_1(\alpha) \times_\alpha Y_2(1 - \alpha) \cup Y_1^*$$

is a solution for $J \otimes K$ where K is any arbitrary simple game and $Y_1^* = A_4 - Y_1(1) - \mathrm{dom}_1 Y_1(1)$ and $Y_2(1-\alpha) \equiv Y_2$ being any solution to K.

REMARK 1. Theorem 1 has its own limitations, as every solution to J need not be a product solution. For example, consider the following solution.

Let $s = (0, 1/4, 1/4, 1/2)$ and let $L = L_1 \cup L_2 \cup L_3$ where

$$L_1 = \bigcup_{0 \leq t \leq 1/2} [(1/4, x, y, t): x, y \geq 0 \quad \text{and} \quad x + y = 3/4 - t];$$

$$L_2 = \bigcup_{1/2 < t \leq 3/4} [(3/4 - t, x, y, t): x, y \geq 0 \quad \text{and} \quad x + y = 1/4];$$

$$L_3 = \bigcup_{3/4 < t \leqq 1} [(0, x, y, t): x, y \geqq 0 \quad \text{and} \quad x + y = 1 - t].$$

Define $H = L \cap \text{dom } s$.

Then it can be shown that $\{s\} \cup (L - H)$ is a solution for J. For proof of this assertion see Shapley [4]. Furthermore, this is *not* a product solution to J. In this section we are going to show that Theorem 1 remains true if one includes solutions of the type constructed by Shapley [4] containing an arbitrary closed component for the game J.

Let C be an arbitrary closed subset of $[0, 1)$ and let S be the image of C under the one-one map

$$u_4 \in C \rightarrow \left\langle 0, \frac{1 - u_4}{2}, \frac{1 - u_4}{2}, u_4 \right\rangle.$$

Consider the set L which consists of all imputations of the form

$$\left\{ \left\langle \frac{1 - u_4 - \rho(u_4, C^0)}{2}, x, y, u_4 \right\rangle : u_4 \in [0, 1] \right\}$$

where $C^0 = C \cup \{1\}$, $\rho(u_4, C^0) = \text{Inf}_{u \in C^0} |u_4 - u|$, x, $y \geqq 0$ and $x + y = [1 - u_4 + \rho(u_4, C^0)]/2$.

Let $H = L \cap \text{dom } S$.

The $S \cup (L - H)$ is a solution for J. For a proof refer [4]. We will call such solutions C-solutions.

THEOREM 2. *Let $0 \leqq \delta < 1$ and let $Y_1(\alpha)$ be C-solutions to J (except that $Y_1(1)$ need not be externally stable) and $[Y_1(\alpha): 0 \leqq \alpha \leqq 1]$ be δ-monotonic. Then*

$$Y = \bigcup_{0 \leqq \alpha \leqq 1} Y_1(\alpha) \times_\alpha Y_2(1 - \alpha) \cup Y_1^*$$

is a solution to $J \otimes K$ where $Y_2(1 - \alpha) \equiv Y_2$ is any solution to K and $Y_1^* = A_4 - Y_1(1) - \text{dom}_1 Y_1(1)$.

REMARK 2. Since $\{Y_1(\alpha)\}$ is semimonotonic it follows that $Y_1(1)$ is internally stable and hence $Y_1(1) \cup Y_1^*$ is externally stable. Hence external stability of Theorem 2 can be proved as in the case of Theorem 5 of Shapley (see [3, pp. 282–283]) or as in [1].

REMARK 3. As the family $\{Y_1(\alpha): 0 < \alpha < 1\}$ with $\overline{Y}_1(1)$ [closure of $Y_1(1)$] is also semimonotonic, we will assume $Y_1(1)$ itself is closed.

PROOF OF THEOREM 2. We will only indicate the proof of internal stability. Suppose there exist $x, y \in X$ such that $x > y$ via $S \in W$. Let $S_1 = S \cap P_1$, $S_2 = S \cap P_2$ and $x = \alpha x_1 + (1 - \alpha) x_2$, $y = \beta y_1 + (1 - \beta) y_2$ where $x_1, y_1 \in A_4$ and $x_2, y_2 \in A_K$. Since $x > y$ on $S_1 \cup S_2 = S$ it follows that $0 < \alpha < 1$.

Case 1a. $1 > \beta \geq \alpha$. Since $\alpha > 0$, $\beta > 0$. Also $\alpha x_1 > \beta y_1$ on S_1. Since $Y_1(\alpha)$ is semimonotonic there exists $x_1' \in Y_1(\alpha)$ such that $\beta y_1 \geq \alpha x_1'$. Hence $\alpha x_1 > \beta y_1 \geq \alpha x_1'$, that is $x_1 > x_1'$ on S_1 contradicting the internal stability of $Y_1(\alpha)$.

Case 1b. $\alpha \geq \beta > 0$. This means $1 > 1 - \beta \geq 1 - \alpha$. Since $\alpha \neq 1$ this case is similar to the previous case except that we have to utilize the semimonotonic property of $Y_2(\alpha)$.

Case 2a. $\beta_1 = 1$, that is, $\alpha x_1 > y_1$ on S_1. If $y_1 \in Y_1(1)$ then using the semimonotonic property of $\{Y_1(\alpha) : 0 \leq \alpha \leq 1\}$ and $Y_1(1)$ [see Remark 3], we will arrive at a contradiction. Let if possible $\alpha x_1 > y_1$ on S_1 with $y_1 \in Y_1^*$

$$S_1 = \{1, 2, 4\}, \quad x_1 = (a_1, a_2, a_3, a_4), \quad y_1 = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4).$$

[For simplicity we are not writing $x_1 = (a_1 a_2 a_3 a_4 0000)$ etc.] Using the δ-monotonic property and the fact that $Y_1(\alpha)$ except $\alpha = 1$ are C-solutions it can be shown that the following two vectors belong to $Y_1(1)$:

$$w_1 = \left(\frac{1 - u_4 - \rho^0}{2}, \frac{1 - u_4 + \rho^0}{2}, 0, u_4 \right) \in Y_1(1),$$

$$w_2 = \left(\frac{1 - u_4 - \rho^0}{2}, 0, \frac{1 - u_4 + \rho^0}{2}, u_4 \right) \in Y_1(1),$$

where $\alpha a_4 > u_4 > \epsilon_4$ and $\rho^0 \geq 0$. If

$$(1 - u_4 - \rho^0)/2 \leq \epsilon_1$$

then $\alpha x_1 > w_2$ via S_1. This means there exists a vector x' such that

$$\alpha x_1 > w_2 \geq \alpha x' \quad \text{via } S_1 \quad \text{which is impossible.}$$

We will assume $(1 - u_4 - \rho^0)/2 > \epsilon_1$ and hence it follows that $(1 - u_4 + \rho^0)/2 \leq \epsilon_2$. Otherwise, there will be a contradiction to the fact that $y_1 \in Y_1^*$. If $a_3 > 0$, then $\alpha x_1 > w_1$ via 234 which will in turn contradict the internal stability of $Y_1(\alpha)$. So we will assume $a_3 = 0$.

Let C_α be the closed subset of $[0, 1]$ corresponding to $Y_1(\alpha)$. If $a_4 \in C_\alpha$ then

$$\left(0, \frac{1 - a_4}{2}, \frac{1 - a_4}{2}, a_4 \right) \in X_1(\alpha).$$

Since $a_3 = 0$ it follows that $a_1 = a_2 = (1 - a_4)/2$. Hence $\alpha x > w_1$ via 124 which is impossible. Let $a_4 \notin C_\alpha$. This means

$$a_1 = 1 - a_4 - \rho(a_4, C_\alpha^0)/2 \quad \text{and} \quad a_2 = 1 - a_4 + \rho(a_4, C_\alpha^0)/2$$

where $C_\alpha^0 \cup C_\alpha = \{1\}$, $\rho(a_4, C_\alpha^0) > 0$ for $a_4 \notin C_\alpha^0$ and C_α^0 is compact. Let

$0 < \epsilon < \rho(a_4, C_\alpha^0)$, then

$$\left\langle \frac{1 - a_4 - \rho(a_4, C_\alpha^0)}{2}, \frac{1 - a_4 + \rho(a_4, C_\alpha^0) - \epsilon}{2}, \epsilon/2, a_4 \right\rangle \in Y_1(\alpha).$$

That is

$$\alpha \left\{ \frac{1 - a_4 - \rho(a_4, C_\alpha^0)}{2}, \frac{1 - a_4 + \rho(a_4, C_\alpha^0) - \epsilon}{2}, \epsilon/2, a_4 \right\} > w_1 \text{ via 234}$$

which will once again contradict the internal stability of $Y_1(\alpha)$. Consider the case where

$$a_3 = (1 - a_4 - \rho(a_4, C_\alpha^0))/2 = 0,$$

that is, $\rho(a_4, C_\alpha^0) = 1 - a_4$. From the way in which every C-solution is constructed it follows that the first two coordinates can run between 0 to $1 - a_4$. Hence we have

$$\alpha(a_2, a_1, 0, a_4) > w_2 \quad \text{via 124}$$

which leads to a contradiction. Other cases can be disposed of similarly. Thus the proof is complete.

In fact, one can prove the following theorem, which includes both Theorems 1 and 2.

THEOREM 3. *Let $0 \leq \delta < 1$ and let $\{ Y_1(\alpha) : 0 \leq \alpha \leq 1 \}$ be δ-monotonic solutions to J except that $Y_1(1)$ need not be externally stable. Further, suppose each $Y_1(\alpha)$, except $\alpha = 1$, is either a product solution or a C-solution. Then*

$$Y = \bigcup_{0 \leq \alpha \leq 1} Y_1(\alpha) \underset{\alpha}{\times} Y_2(1 - \alpha) \cup Y_1^*$$

is a solution for $J \otimes K$ where $Y_2(1 - \alpha) \equiv Y_2$ is any solution for K and $Y_1^* = A_4 - Y_1(1) - \mathrm{dom}_1 Y_1(1)$.

REMARK 4. We are unable to settle the following question: Suppose $Y_1(\alpha)$ is any δ-monotonic family of solutions (nor necessarily product solutions or C-solutions) to J except that $Y_1(1)$ need not be externally stable. Then will this family yield product solutions to games of the form $J \otimes K$? However, one can prove the following partial result in this direction.

THEOREM 4. *Let $Y_1(\alpha)$ be $1/2$-monotonic family of solutions to J*

except that $Y_1(1)$ need not be externally stable. Further suppose $[Y_1(\alpha): \alpha \geqq 1/2]$ are product solutions. Then

$$Y = \bigcup_{0 \leqq \alpha \leqq 1} Y_1(\alpha) \times Y_2(1-\alpha) \cup Y_1^*$$

is a solution for $J \otimes K$ where $Y_2(1-\alpha) \equiv Y_2$ is any solution for K.

PROOF. We will prove the internal stability of Y. For this it is enough if we prove that there does not exist a vector $x \in Y_1(\alpha)$ with $\alpha x > y$ via S for some $y \in Y_1^*$. Let, if possible,

$$\alpha(a_1, a_2, a_3, a_4) > y = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \text{ via } 124$$

where $y \in Y_1^*$. For every $\beta \in [0, 1]$, there exists a set N_β which belongs to $Y_1(1)$. Also N_β has the following representation.

$$N_\beta = [\langle \beta(C_\beta, t, 1 - t - C_\beta), 1 - \beta \rangle : 0 \leqq t \leqq 1 - C_\beta].$$

For details regarding this fact refer [2]. This is a consequence of the fact that $Y_1(\alpha)$ is a 1/2-monotonic family and that $\{Y_1(\alpha): \alpha \geqq 1/2\}$ are product solutions except $\alpha = 1$.

Let us choose a β' such that $\alpha a_4 > 1 - \beta' > \epsilon_4$ and consider the set $N_{\beta'}$. $\beta' C_{\beta'}$ cannot be less than or equal to ϵ_1 as this will contradict the internal stability of $Y_1(\alpha)$. Hence $\beta' C_{\beta'} > \epsilon_1$. Since $y \in Y_1^*$, it follows, $\beta'(1 - C_{\beta'}) \leqq \epsilon_2, \epsilon_3$. So we have the following inequalities

$$\alpha a_4 > 1 - \beta'$$
$$\alpha a_2 > \epsilon_2 \geqq \beta'(1 - C_{\beta'}) \geqq \beta'/2 \quad [\text{for } C_{\beta'} \leqq 1/2].$$

Therefore $\alpha > 1 - \beta'/2 \geqq 1/2$. Since $\alpha > 1/2$ and $Y_1(\alpha)$ is 1/2-monotonic, we arrive at a contradiction with regard to the assumption that $y \in Y_1^*$. Let if possible

$$N_{\beta'} = (\beta'/2, \beta'/2, 0, 1 - \beta')$$
$$\cup (\beta'/2, 0, \beta'/2, 1 - \beta')$$
$$\cup (0, \beta'/2, \beta'/2, 1 - \beta').$$

As before we have,

$$\beta'/2 > \epsilon_1, \quad \beta'/2 \leqq \epsilon_2, \quad \alpha x_4 > 1 - \beta', \quad \alpha x_2 > \beta'/2 \quad \text{or } \alpha > 1/2.$$

This once again contradicts the hypothesis that $y \in Y_1^*$. Similarly other cases can be disposed of and thus the proof is complete.

REMARK 5. We suspect Theorem 4 might be true if we replace 1/2 by any positive δ near 1.

4. A theorem on simple majority game. Let $\Gamma(N, W)$ be an n-person simple game. $N = \{1, 2, \cdots, n\}$. We are going to assume that the only winning coalitions are those which contain $(n-1)$-elements. That is $S \in W$ means $S = N$ or S contains $(n-1)$-elements from N.

Let $Y_1(1) \subset A_N$ be an internally stable set not necessarily an externally stable set. Assume $Y_1(1)$ to be a closed set. Let

$$Y_1^* = A_N - Y_1(1) - \text{dom}_1 \, Y_1(1).$$

Let $y \in E$. Define $y(S) = \sum_{i \in S} y_i$ for $S \in W$ where $y = (y_1, y_2, \cdots, y_N)$. Let $a_S = \text{Inf} \, [y(S) : y \in Y_1^*]$ for $S \in W$ and $\delta_0 = \text{Min} \, [a_S : a_S > 0]$. If $a_S \equiv 0$, $\forall S \in W$, we take any $\delta \in [0, 1)$.

THEOREM 5. *Let $\{Y_1(\alpha) : 0 \leq \alpha \leq 1\}$ be δ_0-monotonic (where δ_0 is defined as above) family of solutions except that $Y_1(1)$ need not be externally stable to $\Gamma(N, W)$. Then*

$$Y = \bigcup_{0 \leq \alpha \leq 1} Y_1(\alpha) \times_{\alpha} Y_2(1 - \alpha) \cup Y_1^*$$

is a solution to $N \otimes K$ where $Y_2(1 - \alpha) \equiv Y_2$ any solution to the simple game K.

PROOF. We need only establish that there does not exist a vector $x \in Y_1(\alpha)$ with $\alpha x > y$ via S for some $y \in Y_1^*$. Let if possible $\alpha x > y$ via S. We will consider two cases according as $a_S = 0$ or $a_S > 0$.

Case 1. $a_S = 0$. We will take without loss of generality $S = \{1, 2, \cdots, n-1\}$. Let $u = (0, 0, 0, \cdots, 0, 1)$. Since $a_S = 0$ and $Y_1(1) \cup Y_1^*$ is a closed set, it follows that $u \in Y_1(1) \cup Y_1^*$. Also we have

$$\alpha x > u \text{ via } S \quad (\text{for } \alpha x > y \text{ via } S).$$

Now, $u \notin Y_1(\beta)$ for any $\beta \geq \alpha$ otherwise internal stability of $Y_1(\alpha)$ will be contradicted. This simply means $u \in Y_1^*$. Let $\beta > \delta$. There exists a vector $w \in Y_1(\beta)$ with $w > u$ via S. That is, $\beta w > u$ via S. Since the family $Y_1(\alpha)$ is δ-monotonic, one can find a vector $v \in Y_1(1)$ with

$$v \geq \beta w > u \quad \text{via } S, \qquad v > u \quad \text{via } S.$$

This means $u \notin Y_1(1) \cup Y_1^*$ contradicting the assumption that $u \in Y_1(1) \cup Y_1^*$. Hence we are through in this case.

Case 2. $a_S > 0$. If $a_S = 1$, then $\alpha x > y$ via S is impossible. So we assume $0 < a_S < 1$. This means $0 < \delta < 1$. It is not hard to check that $\alpha > y(S) \geq a_S \geq \delta$. Since the family is δ-monotonic, we can find a vector v in $Y_1(1)$ with $v \geq \alpha x$. That is $v > y$ via S which once again contradicts the assumption regarding y. This completes the proof of Theorem 5.

We will close this section with one more theorem on product solutions for simple games.

THEOREM 6. *Let* $\{Y_i(\alpha): 0 \leq \alpha \leq 1\}$ *for* $i = 1, 2$ *be semimonotonic families of solutions to simple games* $\Gamma(P_i, W_i)$ *except that* $Y_i(1)$ *need not be externally stable. Suppose there exist two vectors u, v where* $u \in Y_1(1)$, $v \in Y_2(1)$ *with* $u \geq \delta$ *and* $v \geq \delta$. [*Here each one of the coordinates of u and v are greater than or equal to* δ.] *Further, suppose the families to be* δ-*monotonic. Then*

$$Y = \bigcup_{0 \leq \alpha \leq 1} Y_1(\alpha) \times Y_2(1 - \alpha) \cup Y_1^* \cup Y_2^*$$

is a solution to $P_1 \otimes P_2$ *where* $Y_i^* = A_{P_i} - Y_i(1) - \mathrm{dom}_i Y_i(1)$.

REMARK 6. This theorem is a slight variation of Theorem 4, of [1]. The limitation of this theorem is that δ tends to be small as the number of players in the component games increase.

5. **An example of a product solution.** The five-person game F is defined by $F = \Gamma(12345, \{1235, 1245, 1345, 2345, 12345\})$ where members in the curly brackets denote the winning coalitions of the game F.

$$F = M_{1234} \otimes B_5$$

where M_{1234} denotes the 4-person simple majority game and B_5 the 1-person pure bargaining game. Define, for $0 \leq \alpha \leq 1/3$

$$Y_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \{(\beta^2/3, x, y, z, 1 - \beta) : x, y, z \geq 0 \quad \text{and} \quad x + y + z = \beta - \beta^2/3\}$$
$$\cup \{0, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0\}.$$

Define, for $1/3 < \alpha \leq \alpha_0$ where α_0 is so chosen that

$$\alpha_0(1 - 4/9(1 + \alpha_0)) = 2/3,$$
$$Y_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \left\{\left(\frac{4}{9} \frac{\beta^2}{1 + \alpha}, x, y, z, 1 - \beta\right) : x, y, z \geq 0 \text{ and } x + y + z \right.$$

$$\left. = \beta - \frac{4}{9} \cdot \frac{\beta^2}{1 + \alpha}\right\}.$$

For $\alpha_0 < \alpha < 1$ define

$$Y_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \big\{ [(1 - 2/3\alpha)\beta^2, x, v, z, 1 - \beta] : x, y, z \geq 0 \text{ and } x + y + z$$
$$= \beta - (1 - 2/3\alpha)\beta^2 \big\}.$$

For $\alpha = 1$, define

$$Y_1(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \big\{ (\beta^2/3, x, y, z, 1 - \beta) : x, y, z \geq 0 \text{ and } x + y + z = \beta - \beta^2/3 \big\}.$$

Now it is not hard to check that the family $[Y_1(\alpha) : 0 \leq \alpha \leq 1]$ is semi monotonic and that each $Y_1(\alpha)$ is a solution to the game F except that $Y_1(1)$ is not externally stable because $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ is not an element of $Y_1(1) \cup \mathrm{dom}\ Y_1(1)$. If $Y_1'(1) = Y_1(1) \cup \big\{ (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) \big\}$, then $Y_1'(1)$ is a solution to F. But $Y_1(\alpha)$ for $0 \leq \alpha < 1$ together with $Y_1'(1)$ is not semimonotonic. Further it is *not* hard to check that the family $[Y_1(\alpha) : 0 \leq \alpha \leq 1]$ is 2/3-monotonic but not fully monotonic. Moreover, all the conditions of Theorem 5 are satisfied. Hence, this family can be used to produce solutions to games of the form $F \otimes K$.

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