CONVERGENCE OF A SEQUENCE OF POWERS

R. E. DEMARR[1]

A well-known theorem states that if a stochastic matrix (definition below) of finite order has all positive entries in it, then the sequence of its powers (or iterates) converges to a limit; see [**3**, p. 173]. In this paper we will give a new proof of this result using elementary ideas from the theory of partially ordered linear algebras. Our proof does not use the internal structure of the given matrix; therefore, it can be applied to nonnegative operators.

The basic definition of a partially ordered linear algebra (pola) is as follows. A pola A is first of all a linear algebra with real numbers as scalars. Real numbers will usually be denoted by small Greek letters. Multiplication of elements of A is assumed to be associative, but not necessarily commutative. Next, the linear algebra A is a partially ordered set subject to the following conditions (x, y, z denote arbitrary elements of A and α denotes an arbitrary real number under the specified restrictions in each condition):

(a) if $x \leqq y$, then $x+z \leqq y+z$;

(b) if $0 \leqq x$ and $0 \leqq y$, then $0 \leqq xy$;

(c) if $0 \leqq \alpha$ and $0 \leqq x$, then $0 \leqq \alpha x$;

(d) for any $x \in A$ there exists $y \geqq 0$ and $z \geqq 0$ such that $x = y - z$.

We may also introduce a form of order completeness described as follows. The pola A is said to be Dedekind σ-complete if it satisfies the following condition: if $\{x_n\}$ is a sequence of elements from A such that $x_1 \geqq x_2 \geqq \cdots \geqq 0$, then $\inf\{x_n\}$ exists. See [**4**, pp. 9–11]. Of course, $\inf\{x_n\}$ denotes the infinum (greatest lower bound) of the sequence $\{x_n\}$. It is defined as follows: $\inf\{x_n\} = x$ means that

(1) $x \leqq x_n$ for all n;

(2) if $y \leqq x_n$ for all n, then $y \leqq x$.

We now introduce a concept of order convergence: a sequence $\{y_n\}$ of elements from A is said to order converge to $y \in A$ if and only if there exists a sequence $\{z_n\}$ of elements from A such that $z_1 \geqq z_2 \geqq \cdots \geqq 0$, $\inf\{z_n\} = 0$, and $-z_n \leqq y_n - y \leqq z_n$ for all n. In this case we write o-lim $y_n = y$.

In general, multiplication is *not* continuous with respect to order convergence; see [**2**]. We say that multiplication is continuous if the following holds: for every sequence $\{x_n\}$ such that $x_1 \geqq x_2 \geqq \cdots \geqq 0$

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and $\inf\{x_n\}=0$ and for every $y\geqq 0$ we have $\inf\{x_ny\}=\inf\{yx_n\}=0$.

The reader may find more basic information on partially ordered sets, etc., in [1] and [5].

If m is a fixed positive integer and if A denotes the real linear algebra of all matrices of order m with real entries, then A can be regarded as a pola as follows. If $x\in A$ and $y\in A$, where $x=[\alpha_{ij}]$ and $y=[\beta_{ij}]$, than $x\leqq y$ means that $\alpha_{ij}\leqq\beta_{ij}$ for all i, j. It is easy to show that in A multiplication is continuous. A stochastic matrix $[\alpha_{ij}]$ is one such that $\alpha_{ij}\geqq 0$ for all $i, j, =1, \cdots, m$ and $\sum_{i=1}^m \alpha_{ij}=1$ for all $i=1, \cdots, m$. Now suppose $x=[\alpha_{ij}]$ is a stochastic matrix with $\alpha_{ij}\geqq\delta>0$ for all i, j. It is easily seen that $0\leqq x^n\leqq\delta^{-1}x$ for all n. It turns out that this is all that is needed to prove that o-lim x^n exists. One can easily construct other kinds of nonnegative matrices which satisfy this condition. By referring to [2] the reader will see these ideas can be applied to bounded operators on a real Banach space. We now prove the main theorem.

THEOREM. *Let A be a partially ordered linear algebra which is Dedekind σ-complete. If $x\in A$ and if for some $\beta\geqq 1$ we have $0\leqq x^n\leqq\beta x$ for all $n=1, 2, \cdots$, then* o-lim $x^n=u$ *exists. Also, $0\leqq u^2\leqq u$. If, in addition, we assume that multiplication is continuous, then $u=u^2=xu=ux$.*

PROOF. We begin by defining $\lambda_1=\beta$ and then by induction $\lambda_{n+1}=\lambda_n(1+\lambda_1)(\lambda_1+\lambda_n)^{-1}$ for all $n=1, 2, \cdots$. This latter expression can be rewritten $\lambda_{n+1}=1+\lambda_1(\lambda_n-1)(\lambda_1+\lambda_n)^{-1}$ which means that since $\lambda_1=\beta\geqq 1$, we have $\lambda_n\geqq 1$ for all n. Consequently, we see that $0\leqq\lambda_{n+1}-1\leqq\lambda_1(1+\lambda_1)^{-1}(\lambda_n-1)$ for all n. If we put $\alpha=\beta(1+\beta)^{-1}<1$, then we can show by induction that $\lambda_n-1\leqq\alpha^{n-1}(\beta-1)$ for all n.

We now show by induction that for each n we have $x^k\leqq\lambda_nx^n$ for all $k\geqq n$. The assumption in our theorem states that this is true if $n=1$. Now suppose that for some $n=p\geqq 1$ we have $x^k\leqq\lambda_px^p$ for all $k\geqq p$. Take any $q\geqq p$ and define $r=q+1-p\geqq 1$. Now note that $0\leqq(\lambda_1x-x^r)(\lambda_px^p-x^q)$ or $(\lambda_1+\lambda_p)x^{q+1}\leqq\lambda_1\lambda_px^{p+1}+x^{q+r}$, which is obtained from the previous inequality after multiplying and using the fact that $r+p=q+1$. Now since $q+r-1\geqq p$, we see that $x^{q+r-1}\leqq\lambda_px^p$ which means that $x^{q+r}\leqq\lambda_px^{p+1}$. Therefore,

$$(\lambda_1+\lambda_p)x^{q+1}\leqq(\lambda_1\lambda_p+\lambda_p)x^{p+1},$$

which means that $x^{q+1}\leqq\lambda_{p+1}x^{p+1}$ for all $q+1\geqq p+1$. This completes the proof by induction.

Now let us define $z_n=\mu\alpha^nx$ and $y_n=x^n+z_n$, where $\mu=(\beta-1)(1+\beta)^2$. It is clear that $z_1\geqq z_2\geqq\cdots\geqq 0$ and $\inf\{z_n\}=0$. We note that

$z_n - z_{n+1} = \alpha^{n-1}\beta(\beta-1)x$, which can easily be computed by recalling that $\alpha = \beta(1+\beta)^{-1}$. Now $0 \leqq \lambda_n x^n - x^{n+1} = x^n - x^{n+1} + (\lambda_n - 1)x^n \leqq x^n - x^{n+1} + \alpha^{n-1}(\beta-1)\beta x = x^n - x^{n+1} + z_n - z_{n+1} = y_n - y_{n+1}$. Consequently, $y_1 \geqq y_2 \geqq \cdots \geqq 0$. Since A is Dedekind σ-complete, we know that $u = \inf\{y_n\}$ exists. It is easy to show that $-z_n \leqq x^n - u \leqq y_n - u$ for all n. Since $\inf\{y_n - u + z_n\} = 0$, we have that o-lim $x^n = u$.

It is easily seen that $xu \leqq xy_n = x^{n+1} + xz_n \leqq x^{n+1} + (1+\beta)$. $z_{n+1} = y_{n+1} + \beta z_{n+1}$ for all n. Thus, $xu \leqq u$. From this it follows that $x^n u \leqq u$ for all n. Hence it follows that $u^2 \leqq y_n u = x^n u + z_n u \leqq u + \mu\alpha^n u$ for all n. Hence $u^2 \leqq u$.

Now let us assume that multiplication is continuous. Since $0 \leqq xy_n - xu = x(y_n - u)$ and since $\inf\{x(y_n - u)\} = 0$, we see that $\inf\{xy_n\} = xu$. It is clear that $xy_n \geqq x^{n+1}$, which means that $xy_n + z_n \geqq y_{n+1} \geqq u$ for all n. Since $\inf\{xy_n + z_n\} = xu$, we see that $xu \geqq u$. We have already shown that $xu \leqq u$. Hence, $xu = u$. Similarly, we can show that $ux = u$.

Now $0 \leqq uy_n - u^2 = u(y_n - u)$. Since $\inf\{u(y_n - u)\} = 0$, we see that $\inf\{uy_n\} = u^2$. From what was just proved above we see that $uy_n = u + \mu\alpha^n u$ for all n. Hence, $u^2 = \inf\{uy_n\} = u$.

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University of New Mexico