ON EQUATIONS OF THE WIENER-HOPF TYPE IN SEVERAL COMPLEX VARIABLES

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1. **Introduction.** In a previous communication [1] we established conditions under which the additive decomposition of a function of several complex variables is unique. In the analysis presented here we use the considerations of [1] to formulate a Wiener-Hopf problem in several complex variables which will be solved for a class of physically important kernels [2] by the method of successive approximations.

Statement of the Wiener-Hopf problem. Let $K(z_1, \cdots, z_n)\times g_1(z_1, \cdots, z_n)$, $z_j=x_j+iy_j$, *be analytic in a tube* T: $\{-\gamma_i<y_i<\delta_i, x_i\in(-\infty, \infty)\}$, *and let* K *have the form*

$$K(z_1, \cdots, z_n) = 1 - \lambda H(z_1, \cdots, z_n), \tag{1}$$

where λ *is a complex parameter which may be made small and* $H(z_1, \cdots, z_n)$ *is a uniformly bounded analytic function in the tube* T. *Suppose that the* L_2 *norm*

$$||g_1||_2 = \left\{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} | g_1(z_1, \cdots, z_n) |^2 dx_1 \cdots dx_n\right\}^{1/2}$$

converges in T, *then* $||Kg_1||_2$ *also converges in* T. *Consequently, there exists in* T *a unique additive decomposition* $Kg_1=\sum_{i=1}^{2^n} f_i$, *where each* f_i *is analytic and bounded in an octant shaped tube* T_i *containing the interior of* T, *and* $f_i\to 0$ *as any one of the* $y_j\to\pm\infty$ *in* T_i.

Given $f_1(z_1, \cdots, z_n)$ *and the particular kernel* $K(z_1, \cdots, z_n)$ *in* (1) *determine the remaining* 2^n *unknown analytic functions* $g_1, f_2, \cdots, f_{2^n}$ *appearing in the decomposition of* Kg_1.

2. **Solution of the Wiener-Hopf problem.** Let the functions $g_1(z_1, \cdots, z_n)$ and $f_1(z_1, \cdots, z_n)$ be assumed to be analytic in the octant shaped tube T_1: $\mathrm{Im}(z_i)>0$, $i=1, 2, \cdots$, and to have bounded L_2 norms in T. The decomposition of Kg_1 has the form

$$g_1 - \lambda H g_1 = \sum_{i=1}^{2^n} f_i, \tag{2}$$

and because this decomposition is unique, Cauchy integration yields

Received by the editors January 27, 1969.

$$g_1(z_1, \cdots, z_n) = f_1(z_1, \cdots, z_n) \tag{3}$$
$$+ \frac{\lambda}{(2\pi i)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{H(\zeta_1, \cdots, \zeta_n) g_1(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n,$$

where $\mathrm{Im}(z_1) > 0$, $\mathrm{Im}(z_2) > 0, \cdots, \mathrm{Im}(z_n) > 0$.

Equation (3) may be reduced to a singular integral equation for $g_1(z_1, \cdots, z_n)$ by allowing $\mathrm{Im}(z_1) \to 0^+, \cdots, \mathrm{Im}(z_n) \to 0^+$ and introducing the generalized Plemelj formulas [**3**]. The result is

$$g_1 = f_1 + \frac{\lambda}{2^n} \prod_{i=1}^{2^n} (I + S_i)(Hg_1), \tag{4}$$

where I is the identity operator and where

$$S_i = \frac{P}{\pi i} \int_{-\infty}^{\infty} \frac{d\zeta_i}{(\zeta_i - z_i)}, \tag{5}$$

are principal value integral operators. We state our main result for equation (4) as a

THEOREM. *Let the maximum modulus of* $H(z_1, \cdots, z_n)$ *on* $\mathrm{Im}(z_i) = 0$, $i = 1, 2, \cdots, n$ *be denoted by* $\max |H|$. *Then for* $0 < |\lambda| < (\max |H|)^{-1}$, *and* g_1 *in the complete normed linear space* L_2,

$$T(g_1) = f_1 + \frac{\lambda}{2^n} \prod_{i=1}^{2^n} (I + S_i)(Hg_1)$$

is a contraction mapping with respect to the L_2 *norm. Hence the integral equation* $T(g_1) = g_1$ *has one and only one fixed point belonging to* L_2. *This fixed point is the limit of a sequence of successive approximations converging in* L_2 *norm to* g_1.

PROOF. Let g_1 and h_1 be members of the L_2 function space and analytic in T_1, we have

$$\| T(g_1) - T(h_1) \|_2 = \frac{|\lambda|}{2^n} \left\| \prod_{i=1}^{2^n} (I + S_i) H(g_1 - h_1) \right\|_2 .$$

Expanding the operator $\prod_{i=1}^{2^n} (I + S_i)$ and applying Minkowski's inequality to the result yields

$$\| T(g_1) - T(h_1) \|_2 \leqq |\lambda| \max |H| \, \| g_1 - h_1 \|_2$$

where we have made use of the fact that the principal value operator S_i gives the Hilbert transform, with respect to the ith variable, of the function on which it operates and that the Hilbert transform is a bounded linear operator in L_2 satisfying [4] $||S_i H g_1||_2 = ||H g_1||_2$ and

$$\left\| \prod_{i=1}^{2^n} S_i H g_1 \right\|_2 = ||H g_1||_2.$$

We have established that $T(g_1)$ is a contraction mapping with respect to the L_2 norm provided $|\lambda| \max |H| < 1$. The remainder of the theorem is then a consequence of Banach's fixed point theorem [5].

Since we have now constructed a g_1 such that the conditions of Bochner's theorem [1] apply to the left-hand side of (2), the unknowns $f_2, f_3, \cdots, f_{2^n}$ in (2) are now uniquely determined by the Cauchy integral decomposition of $g_1 - \lambda H g_1$. This solves the Wiener-Hopf problem for kernels of the form indicated in (1).

References

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