ON BERMAN'S VERSION OF THE LÉVY-BAXTER THEOREM

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In this note we derive the numerical value of the constant B_k in Berman's version of the Lévy-Baxter theorem [2]. Let $X(t)$, $t=(t_1, t_2, \cdots, t_k), -\infty<t_1, \cdots, t_k<\infty$, $\|t\|=(t_1^2+t_2^2+\cdots+t_k^2)^{1/2}$, be Lévy's Brownian process of k parameters: it is a Gaussian process with mean 0 and covariance function

$$\rho(s, t) = E\{X(s)X(t)\} = (1/2)\{\|s\| + \|t\| - \|t - s\|\}. \tag{1}$$

For each integer $n\geqq 1$, let the unit cube $\{t: 0\leqq t_1\leqq 1, \cdots, 0\leqq t_k\leqq 1\}$ be broken up into 2^{nk} cubes whose edges have the common length 2^{-n} and whose corner-points are of the form $(i_1 2^{-n}, \cdots, i_k 2^{-n})$, where the i's are integers between 0 and 2^n. Let $Y_{i,n}$ denote the kth-order difference of the sample function X over the cube $C(i, n) = \{t: (i_1-1)2^{-n}\leqq t_1\leqq i_1 2^{-n}, \cdots, (i_k-1)2^{-n}\leqq t_k\leqq i_k 2^{-n}\}$:

$$\begin{aligned} Y_{i,n} = \Delta_1 \cdots \Delta_k X = X(i_1 2^{-n}, \cdots, i_k 2^{-n}) - \sum_{r=1}^{k} p_r + \sum_{r<s} p_{rs} \\ - \cdots + (-1)^k X((i_1 - 1)2^{-n}, \cdots, (i_k - 1)2^{-n}) \end{aligned} \tag{2}$$

where $p_{rs\cdots t}$ denotes $X(c_1, \cdots, c_k)$ for $c_r=(i_r-1)2^{-n}$, $c_s=(i_s-1)2^{-n}$, $\cdots, c_t=(i_t-1)2^{-n}$ and the remaining c_j equal $i_j 2^{-n}$. S. Berman [2] proved: For $n\geqq 1$, let $\sum|Y_{i,n}|^{2k}$ be the sum of the $2k$th powers of the $Y_{i,n}$ over all cubes $C(i, n)$. Its limit, for $n\to\infty$, exists with probability 1 and is equal to a numerical constant B_k.

The theorem below gives the numerical value of B_k.

THEOREM.

$$B_k = \frac{(2k)!}{k!2^k}\left[2^{k-1}\sum_{r=1}^{k}(-1)^{r-1}\binom{k}{r}\sqrt{r}\right]^k \tag{3}$$

where

$$\binom{k}{r} = \frac{k!}{r!(k - r)!}\,.$$

PROOF. Berman showed in [2] that

$$B_k = ((2k)!/k!2^k)D_k^k \tag{4}$$

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where D_k is the variance of the kth-order difference of $X(\cdot)$ over the corner-points of the unit cube. We shall show that

$$D_k = 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}. \tag{5}$$

Represent the 2^k different corner points by $t_1, \cdots, t_{2^k}$. Let e_j, $j=1, \cdots, 2^k$ denote ± 1 according to the rule: $e_j = +1$ if there are an even number of zeros in the k coordinates of t_j, whereas, $e_j = -1$ if there are an odd number of zeros in the k coordinates of t_j. We can then write D_k as

$$\begin{aligned} D_k &= \operatorname{Var}\left[\sum_{j=1}^{2^k} e_j X(t_j) \right] = \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \rho(t_i, t_j) \\ &= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j (1/2) \left\{ \|t_i\| + \|t_j\| - \|t_i - t_j\| \right\} \\ &= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \|t_i\| - (1/2) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \|t_i - t_j\|. \end{aligned} \tag{6}$$

There are $C_{k,j}$ distinct ways of forming k-tuples consisting of exactly j zeros and $k-j$ ones, thus

$$\sum_{j=1}^{2^k} e_j = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = 0 \tag{7}$$

(For the last equality see [**3**, p. 63].) We therefore have

$$\sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \|t_i\| = \sum_{j=1}^{2^k} e_j \sum_{i=1}^{2^k} e_i \|t_i\| = 0. \tag{8}$$

Now, since the t_j's are corner points of the unit cube in k-dimensional Euclidean space, we have $\|t_i - t_j\| = \sqrt{r}$ where r is the number of coordinates in t_i that differ from the corresponding coordinates in t_j. Also, note that if $\|t_i - t_j\| = \sqrt{r}$, then $e_i e_j = (-1)^r$. Thus,

$$\begin{aligned} -(1/2) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \|t_i - t_j\| &= -(1/2) \sum_{r=1}^{k} \binom{k}{r} 2^k \sqrt{r} (-1)^r \\ &= 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}. \end{aligned} \tag{9}$$

Equations (4), (6), (8) and (9) yield the desired result of the theorem.

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