A GEOMETRIC PROOF OF THE LEBESGUE DIFFERENTIATION THEOREM

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We prove the Lebesgue differentiation theorem using only the most elementary concepts of measure theory. The proof is an instance of a general method of attack which has yielded a number of probabilistic limit theorems.

Preliminaries. We consider a real-valued function $f(x)$ on an interval $[a, b]$; Lebesgue measure on $[a, b]$ will be denoted by μ. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are collections of intervals then $\mu\mathcal{G}_1$ is the measure of the point set covered by $\mathcal{G}_1$, while $\mathcal{G}_1 - \mathcal{G}_2$ is the collection of intervals in $\mathcal{G}_1$ which are not in $\mathcal{G}_2$. The notation $f(c, d)$ will mean the slope of the chord connecting $(c, f(c))$ with $(d, f(d))$. If p is a partition, that is, a finite set of points containing a and b, then $\pi(x)$ is the polygonal approximation to $f(x)$ on p. We assume that f is rectifiable or, equivalently, of bounded variation (i.e., $lf = \text{lub}_\pi\, l\pi < \infty$, where $l\pi$ is the length of the graph of π); thus f is continuous except on a countable set. The upper (lower) right (left) derivates are measurable; in fact, at points of continuity of f, we have for the upper right derivate:

$$f^+(x) = \lim_{n\to\infty} \ \operatorname*{lub}_{0<r_i<1/n} f(x, r_i),$$

where r_i is an ordering of the rationals.

THEOREM. *For $f(x)$ we have* (i) *the derivative exists a.e. and* (ii) *the derivative is finite a.e.*

We note two elementary lemmas:

LEMMA 1. *Any finite collection of intervals $\mathcal{G}$ contains a disjoint subcollection $\mathcal{G}_1$ such that $\mu\mathcal{G}_1 \geqq (1/3)\mu\mathcal{G}$.*

PROOF. Let I_1 be an interval of $\mathcal{G}$ of maximal length and $\{I_1\}$ the intervals of $\mathcal{G}$ which intersect I_1. Inductively, I_j is an interval of $\mathcal{G} - \{\{I_1\}, \cdots, \{I_{j-1}\}\}$ of maximal length and $\{I_j\}$ the intervals intersecting I_j. There is a k such that $\mathcal{G} = \{\{I_1\}, \cdots, \{I_k\}\}$ and then $\mathcal{G}_1 = \{I_1, \cdots, I_k\}$ has the desired properties.

LEMMA 2. *If $\pi(x)$ is linear on $[a, b]$ with $\pi(a) \leqq \pi(b)$ and if $q(x)$ is a polygon coinciding with π at a and b and such that $q(c_i, d_i) < -\alpha$*

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($\alpha > 0$), *where* $[c_i, d_i]$ *is a finite disjoint collection of intervals the sum of whose lengths is* d, *then* $lq > l\pi + d[\sqrt{(1+a^2)} - 1]$.

PROOF. By translations of the sides of q parallel to the coordinate axes we obtain an auxiliary polygon q_1 coinciding with q at a and b and with $lq_1 = lq$ and $q_1(a, a+d) < -\alpha$. Let q_2 and q_3 be the two-sided polygons determined by the endpoints of q and $(a+d, q(a+d))$ and $(a+d, q(a))$, respectively. Then the following relation is obvious and establishes the lemma:

$$l\pi \leqq lq_3 < lq_2 - d[\sqrt{(1+\alpha^2)} - 1] < lq_1 - d[\sqrt{(1+\alpha^2)} - 1] \leqq lq - d[\sqrt{(1+\alpha^2)} - 1].$$

PROOF OF THEOREM. Suppose (i) is not true, then there must be numbers β and $\alpha > 0$ such that if E is the set where f is continuous and $f^+ > \beta + \alpha$, $f_- < \beta - \alpha$ then $\mu E > 0$. Since we may add a linear function, it is no restriction to assume that $\beta = 0$. Let $\pi(x)$ be any polygonal approximation to $f(x)$. We may cover each point $x \in E - p$ with an open interval (a_x, b_x) such that p is linear on $[a_x, b_x]$ and $f(a_x, b_x) < -\alpha$, $> \alpha$, according as $\pi(a_x, b_x) \geqq 0$, $\leqq 0$. Using the Lindelöf theorem we may pick a finite collection $\mathcal{G}$ of the (a_x, b_x) such that $\mu\mathcal{G} \geqq (1/2)\mu E$. Thus by Lemma 1 we may pick a disjoint subcollection $\mathcal{G}_1$ such that $\mu\mathcal{G}_1 \geqq (1/6)\mu E$. Finally by Lemma 2 we see that the polygon q, determined by the partition consisting of p and the endpoints of the intervals of $\mathcal{G}_1$, satisfies $lq > l\pi + (1/6)\mu E[\sqrt{(1+a^2)} - 1]$ and hence $lf = \infty$, contradicting that f is of bounded variation.

Thus $f^+ \leqq f_-$ and (i) follows at once; to verify (ii) we again use an indirect argument and assume that on a set E as above, $f^+ = +\infty$, then for any $M > 0$ we may pick the covering (a_x, b_x) of E so that $f(a_x, b_x) > M$. We pick $\mathcal{G}_1$ as before and let q be the partition determined by the endpoints of $\mathcal{G}_1$; then clearly $lq > d\sqrt{(1+M^2)}$ and since M is arbitrary, we again get a contradiction. This completes the proof.

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