WIENER PROCESS DISTRIBUTIONS OF THE "ARCSINE LAW" TYPE

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1. Introduction and summary of the results. By the Wiener process we shall mean a Gaussian stochastic process $\{x(t), 0 \leqq t \leqq T\}$ with mean function $E\{x(t)\}=0$ and covariance function $E\{x(s)x(t)\}=\min\{s, t\}$ for which the sample points $x(t)$ are the continuous functions defined over the interval $[0, T]$ and vanishing at $t=0$. Lebesgue measure on the interval $[0, T]$ is denoted by $m(\cdots)$.

The famous arcsine law of Erdös and Kac [2] states that

$$P\{m(s \in [0, T]: x(s) > 0) \leqq t\} = \frac{2}{\pi} \arcsin (t/T)^{1/2}.$$

Below, a generalization of this result is obtained by evaluating the distribution function of $m(s \in [0, T]: x(s) > as)$ for any non-negative number a. Also there is evaluated the probability density in T representing the first time T that the functional $m(s \in [0, T]: x(s) > as)$ has a given value t. Finally, the case in which T is infinite is considered. It is shown that the strong law of large numbers for the Wiener process is actually implied by a *weak* probability law.

2. Preliminary result. The calculations below are based on the following equality between conditional probabilities on the Wiener process: Let a, b be any two real numbers and define $S(a, T)$ to be the last zero of $x(s)-as$ before $s=T$. Then

$$\begin{aligned} &P\{m(s \in [0, T]: x(s) > 0) \leqq t, S(0, T) \leqq u \mid x(T) = b\} \\ &\quad = P\{m(s \in [0, T]: x(s) > as) \leqq t, S(a, T) \leqq u \mid x(T) = b + aT\}. \end{aligned} \tag{1}$$

To demonstrate the equality we point out that the finite dimensional distribution functions of the process $\{x(t), 0 \leqq t \leqq T\}$ conditioned by the event $x(T)=b$ are exactly equal to the finite dimensional distribution functions of the process $\{y(t)=x(t)-at, 0 \leqq t \leqq T\}$ conditioned by the event that $y(T)=x(T)-aT=b$. In fact, both conditional processes are Gaussian with mean function $u(t)=bt/T$ and covariance function

$$r(s, t) = \begin{cases} s(T-t)/T & s \leqq t, \\ t(T-s)/T & s \geqq t. \end{cases}$$

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Since the mean and covariance functions of a Gaussian stochastic process completely determine the distributions of functionals like $m(s\in[0, T]: x(s)>as)$ and $S(0, T)$ and since $S(a, T)$ is the last zero of $y(s)$ before $s=T$, it follows that

$$\begin{aligned}
&P\{m(s \in [0, T]: x(s) > 0) \leqq t, S(0, T) \leqq u \mid x(T) = b\} \\
&\quad = P\{m(s \in [0, T]: y(s) > 0) \leqq t, S(a, T) \leqq u \mid y(T) = b\} \\
&\quad = P\{m(s \in [0, T]: x(s) > as) \leqq t, S(a, T) \leqq u \mid x(T) = b + aT\}.
\end{aligned}$$

This proves (1).

3. **The calculations.** In the following $P\{X=\alpha\}$ will denote the density function for the distribution function $P\{X\leqq\alpha\}$ of the random variable X.

We first evaluate the joint density function of $S(0, T)$ and $m(s\in[0, T]: x(s)>0)$ under the condition that $x(T)=-\xi$ for $\xi>0$. Note that the density function $P\{S(0, T)=u|x(T)=-\xi\}$ for $0\leqq u\leqq T$ is

$$\left(\frac{T}{2\pi u(T-u)^3}\right)^{1/2} \xi \exp\left\{-\frac{u\xi^2}{2T(T-u)}\right\}. \tag{2}$$

For the class of functions $x(s)$ with $S(0, T)=u$ and $x(T)=-\xi$, clearly $m(s\in[0, T]: x(s)>0)=m(s\in[0, u]: x(s)>0)$ and it is well known [1] that the functional $m(s\in[0, u]: x(s)>0)$ has a uniform distribution between 0 and u under the condition that $x(u)=0$. Thus for $0\leqq u\leqq T$, by (1) and (2)

$$\begin{aligned}
&P\{m(s \in [0, T]: x(s) > as) = t, S(a, T) = u \mid x(T) = -\xi + aT\} \\
&\quad = P\{m(s \in [0, T]: x(s) > 0) = t, S(0, T) = u \mid x(T) = -\xi\} \\
&\quad = \begin{cases} \left(\dfrac{T}{2\pi u^3(T-u)^3}\right)^{1/2} \xi \exp\left\{-\dfrac{u\xi^2}{2T(T-u)}\right\} & u > t, \\ 0 & u \leqq t. \end{cases}
\end{aligned} \tag{3}$$

It follows from an integration and a subsequent simplification of terms that for $0<t<T$ and non-negative a

$$\begin{aligned}
&P\{m(s \in [0, T]: x(s) > as) = t, x(T) < aT\} \\
&\quad = \frac{1}{\pi T}\left(\frac{T-t}{t}\right)^{1/2} e^{-a^2T/2} + \frac{a}{2\pi}\int_t^T du \frac{e^{-a^2u/2}}{(u^3)^{1/2}} \int_{-a(T-u)^{1/2}}^{\infty} e^{-\xi^2/2} d\xi.
\end{aligned} \tag{4}$$

Using an argument similar to the one just given it is possible to show that for $0<t<T$ and non-negative a

$$P\{m(s \in [0, T]: x(s) > as) = t,\ x(T) > aT\}$$

$$= \frac{1}{\pi T}\left(\frac{t}{T-t}\right)^{1/2} e^{-a^2T/2} - \frac{a}{2\pi}\int_{T-t}^{T} du \frac{e^{-a^2u/2}}{(u^3)^{1/2}} \int_{a(T-u)^{1/2}}^{\infty} e^{-\xi^2/2} d\xi. \tag{5}$$

The results in (4) and (5) combine together to give an evaluation of $P\{m(s \in [0, T]: x(s) > as) = t\}$ from which the distribution function of $m(s \in [0, T]: x(s) > as)$ comes immediately. It is easy to check that the distribution function of $m(s \in [0, T]: x(s) > as)$ has the proper value when $a = 0$ (see [2]).

4. **Implications of the results.** Consider for now the set of functions $x(t)$ whose probability is given in (5). Since $x(T) > aT$ for functions in this set, $m(s \in [0, u]: x(s) > as)$ must be increasing in u for u in the neighborhood of T. Thus for the functions $x(t)$ in this set, $m(s \in [0, u]: x(s) > as) < t$ for all $u \in [0, T)$ and we may describe the set as those functions for which $m(s \in [0, u]: x(s) > as) = t$ for the first time at $u = T$. Relation (5), therefore, gives the first passage time probability density in T required for the functional $m(s \in [0, T]: x(s) > as)$ to attain the value t. The case in which $a = 0$ is of particular interest because this first passage time density is quite simple, i.e.

$$\begin{cases} \dfrac{1}{\pi T}\left(\dfrac{t}{T-t}\right)^{1/2} & T > t, \\ 0 & T \leqq t. \end{cases}$$

Thus, for $T > t$ we may write

$$\begin{aligned} P\{m(s \in [0, T]: x(s) > 0) \geqq t\} &= \int_t^T \frac{1}{\pi u}\left(\frac{t}{u-t}\right)^{1/2} du \\ &= \frac{2}{\pi}\arctan\left(\frac{T-t}{t}\right)^{1/2} = 1 - \frac{2}{\pi}\arcsin\left(\frac{t}{T}\right)^{1/2} \end{aligned}$$

which agrees with the value dictated by the arcsine law.

By means of (3) we deduce that for $T > u > t > 0$

$$\begin{aligned} P\{m(s \in [0, T]&: x(s) > as) = t,\ S(a, T) = u,\ x(T) < aT\} \\ &= \int_0^\infty \exp\{-(\xi - aT)^2/2T\} \frac{1}{((2\pi)^2 u^3 (T-u)^3)^{1/2}} \\ &\quad \cdot \xi \exp\{-u\xi^2/2T(T-u)\} d\xi \\ &= \frac{1}{2\pi}\frac{e^{-a^2T/2}}{(u^3(T-u))^{1/2}} + \frac{a}{2\pi}\frac{e^{-a^2u/2}}{(u^3)^{1/2}}\int_{-a(T-u)^{1/2}}^{\infty} e^{-\xi^2/2} d\xi. \end{aligned} \tag{6}$$

Now, if we let T become infinite in (6) we obtain for $t>0$

$$P\{m(s\in[0,\infty):x(s)>as)=t,\,S(a,\infty)=u\}$$
$$=\begin{cases}(a/(2\pi u^3)^{1/2})\exp\{-a^2u/2\} & u>t,\\ 0 & u\leqq t,\end{cases}\tag{7}$$

and in particular we find that for $u>0$

$$\begin{aligned}P\{S(a,\infty)\leqq u\}&=\int_0^u\frac{a}{(2\pi v)^{1/2}}\exp\left\{-\frac{a^2v}{2}\right\}dv\\&=\left(\frac{2}{\pi}\right)^{1/2}\int_0^{au^{1/2}}e^{-\xi^2/2}d\xi.\end{aligned}\tag{8}$$

The interpretation of (8) is very interesting if it is borne in mind that for almost all functions of the Wiener process $\{x(t),\ 0\leqq t\leqq T\}$ the random variable $S(a,T)$ defined in §2 is equal to the minimum of values $u\in[0,T]$ for which $m(s\in[0,u]:x(s)>as)=m(s\in[0,T]:x(s)>as)$. The strong law of large numbers for the Wiener process states that for any $a>0$ with probability one the s-set for which $x(s)>as$ is bounded. In other words with probability one the functional $m(s\in[0,u]:x(s)>as)$ remains constant for all u sufficiently large. This so-called strong law result is clearly implied by the weak probability law in (8). Moreover, (8) gives an exact estimate of the probability that $x(s)>as$ for no s greater than a fixed value of u. It seems the weak law is stronger than the strong law.

From (7) it is also possible to calculate the density function of $m(s\in[0,\infty):x(s)>as)$. In fact for $t>0$

$$P\{m(s\in[0,\infty):x(s)>as)=t\}=a\int_t^\infty\frac{1}{(2\pi u^3)^{1/2}}e^{-a^2u/2}du.$$

References

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