THE GREEN'S FUNCTIONS FOR THE RECTANGLE OBTAINED BY THE FINITE FOURIER TRANSFORMATIONS

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1. **Introduction.** When the finite Fourier transformation method is employed, the solutions of very general boundary value problems (in contrast to initial value and mixed problems) can be expressed in terms of special functions.[1] In this paper we propose to derive, in terms of a special function to be introduced, a new formula for the Green's function for the Laplace equation $\Delta u=0$ in a rectangle such that u vanishes on the boundary.

2. **The function** $B_{12}(x, u)$. In this section we introduce a key function and obtain its inverse Fourier cosine transform. If $|r|<1$, then $\log (1+re^{i\theta})=-\sum_{\nu=1}^{\infty}(-r)^\nu e^{i\nu\theta}/\nu$. Also if $R^2=(1+r\cos\theta)^2+r^2\sin^2\theta$ and $\tan\Phi=r\sin\theta/(1+r\cos\theta)$, then

$$\begin{aligned}\log Re^{i\Phi} &= \log (1+re^{i\theta}) \\ &= -\sum_{\nu=1}^{\infty}\frac{(-r)^\nu e^{i\nu\theta}}{\nu}.\end{aligned} \tag{1}$$

Equating the real part of (1), we get

$$\begin{aligned}\log R &= \log\left[1+2r\cos\theta+r^2\right]^{1/2} \\ &= -\sum_{\nu=1}^{\infty}\frac{(-1)^\nu r^\nu\cos\nu\theta}{\nu}.\end{aligned} \tag{2}$$

When $|r|\leqq 1$, the series (2) is a Fourier cosine series. The Fourier cosine transforms, defined by the operation $C\{F(x)\}=\int_0^\pi F(x)\cos nx\,dx=f_c(n)$ $(n=0, 1, 2, \cdots)$, are except for the factor $\pi/2$ the coefficients of a Fourier cosine series. Hence, in view of (2), we get

$$\begin{aligned}C\left\{\log\frac{1}{1+2r\cos\theta+r^2}\right\} &= \frac{\pi(-1)^n r^n}{n} && (n=1, 2, \cdots) \\ &= 0 && (n=0).\end{aligned}$$

It can be shown that $(-1)^n f_c(n)=C\{F(\pi-x)\}$. Using this fact and

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[1] A. W. Jacobson, *A generalized convolution for the finite Fourier transformations*, Thesis (Microfilmed), University of Michigan, 1948. Also see *Solution to steady state temperature problems with the aid of a generalized fourier convolution*, Quarterly of Applied Mathematics, vol. 7 (1949).

putting $r=e^{-z}$, $\theta=x$, we obtain when $z\geqq 0$ the following formula:

$$C\left\{\log\frac{e^{z}}{2(\cosh z-\cos x)}\right\}=\frac{\pi e^{-nz}}{n}\quad(n=1,2,\cdots)\tag{3}$$
$$=0\qquad(n=0).$$

Let $b_1(n, u)$ denote the function

$$\frac{\cosh nu}{n\sinh ny_0}=\sum_{\nu=0}^{\infty}\frac{1}{n}\left\{e^{-[(2\nu+1)y_0-u]n}+e^{-[(2\nu+1)y_0+u]n}\right\}.\tag{4}$$

Also let $B_{12}(x, u)$ denote the inverse cosine transform $C^{-1}\{b_1(n, u)\}$, of the function $b_1(n, u)$. When $\pi-u\geqq 0$, we have according to formula (3)

$$B_{12}(x,u)=\frac{1}{\pi}\log\prod_{\nu=0}^{\infty}\frac{e^{2(2\nu+1)y_0}}{4[\cosh(2\nu y_0+y_0-u)-\cos x][\cosh(2\nu y_0+y_0+u)-\cos x]}.\tag{5}$$

3. **A boundary value problem.** Let it be required to determine the temperature $V(x, y)$ in a rectangle R when the source function $H(x, y)$ is a prescribed, sectionally continuous function of x and y and such that V vanishes on the boundary. That is, let $V(x, y)$ be the solution of the following problem:

$$\begin{aligned}&V_{xx}+V_{yy}=H(x,y)\quad\text{in}\quad R,\\&V(x,+0)=V(x,y_0-0)=0\qquad(0<x<\pi),\\&V(+0,y)=V(\pi-0,y)=0\qquad(0<y<y_0).\end{aligned}\tag{A}$$

The Fourier sine transformation, defined by the operation $S\{F(x)\}=\int_0^\pi F(x)\sin nx\,dx=f_s(n)$ $(n=1, 2, \cdots)$, in particular, yields when applied to the second derivative of $F(x)$ the formula

$$S\{F''(x)\}=-n^2f_s(n)+n[F(0)-(-1)^nF(\pi)].$$

Applying the sine transformation with respect to x to the equations in (A), we obtain in view of the last condition the following transformed problem:

$$\begin{aligned}&\frac{d^2v}{dy^2}-n^2v(n,y)=h(n,y),\\&v(n,0)=v(n,y_0)=0,\end{aligned}\tag{A$'$}$$

where

$$h(n,y)=S\{H(x,y)\}.$$

The solution of the transformed problem, in terms of the Green's function $g(n, y, \mu)$, is[2]

$$v(n, y) = \int_0^{y_0} g(n, y, \mu)h(n, \mu)d\mu, \tag{6}$$

where

$$g(n, y, \mu) = -\frac{\sinh n\mu \sinh n(y_0 - y)}{n \sinh ny_0} \qquad (\mu \leqq y)$$

$$= -\frac{\sinh ny \sinh n(y_0 - \mu)}{n \sinh ny_0} \qquad (\mu \geqq y)$$

or

$$g(n, y, \mu) = \frac{\cosh n(y_0 - y - \mu)}{2n \sinh ny_0} - \frac{\cosh n(y_0 - y + \mu)}{2n \sinh ny_0} \qquad (\mu \leqq y),$$

and the same expression with y and μ interchanged when $\mu \geqq y$. According to formula (4) this can be written

$$g(n, y, \mu) = 2^{-1}b_1(n, y_0 - y - \mu) - 2^{-1}b_1(n, y_0 - y + \mu), \qquad (\mu \leqq y).$$

The inverse cosine transform of $b_1(n, u)$ is the function $B_{12}(x, u)$. Hence, if $C^{-1}\{g(n, y, \mu)\} = G(x, y, \mu)$, we have

$$G(x, y, \mu) = 2^{-1}B_{12}(x, y_0 - y - \mu) - 2^{-1}B_{12}(x, y_0 - y + \mu). \tag{7}$$

The solution (6) of the transformed problem can then be written

$$S\{V(x, y)\} = \int_0^{y_0} C\{G(x, y, \mu)\}S\{H(x, \mu)\}d\mu. \tag{8}$$

The product of the two transforms in (8) can be expressed in terms of the sine transform of the convolution of the two functions. For if $P(x)$ is even and $Q(x)$ is an odd sectionally continuous function, and if $Q(x+2\pi) = Q(x)$, then

$$C\{P(x)\}S\{Q(x)\} = 2^{-1}S\{P(x) * Q(x)\},$$

where the convolution is defined thus:[3]

$$P(x) * Q(x) = \int_{-\pi}^{\pi} P(x - \lambda)Q(\lambda)d\lambda.$$

Hence the solution (8) becomes, upon inverse transformation,

[2] E. L. Ince, *Ordinary differential equations*, Dover, 1944, p. 258.

[3] R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill, 1944, p. 274.

$$V(x, y) = \frac{1}{2}\int_0^{y_0} G(x, y, \mu) * H(x, \mu)d\mu,$$

or

$$V(x, y) = \frac{1}{2}\int_0^{y_0}\int_{-\pi}^{\pi} G(x - \lambda, y, \mu)H(\lambda, \mu)d\lambda d\mu.$$

Since $H(\lambda, \mu)$ is extended as an odd function of λ, we get

$$V(x, y) = \frac{1}{2}\int_0^{y_0}\int_0^{\pi} [G(x - \lambda, y, \mu) - G(x + \lambda, y, \mu)]H(\lambda, \mu)d\lambda d\mu.$$

Let $\Gamma(x, y, \lambda, \mu) = 2^{-1}[G(x-\lambda, y, \mu) - G(x+\lambda, y, \mu)]$. Then the function Γ is the Green's function for the boundary value problem (A). In terms of our function $B_{12}(x, u)$, formula (7), it can be written

$$\Gamma(x, y, \lambda, \mu) = 4^{-1}[B_{12}(x - \lambda, y_0 - y - \mu) - B_{12}(x - \lambda, y_0 - y + \mu) \tag{9}$$
$$- B_{12}(x + \lambda, y_0 - y - \mu) + B_{12}(x + \lambda, y_0 - y + \mu)].$$

4. **Relation between the function B_{12} and the Weierstrass's sigma function.** The Green's function K for $\Delta u = 0$ in the rectangle R: $0 < x < \pi$, $0 < y < y_0$, with $u = 0$ on the boundary is in terms of the Weierstrass's sigma function σ:

$$K(x, y, \lambda, \mu) = \frac{1}{2\pi}\Re\left(\log\frac{\sigma(z - \zeta, \omega_1, \omega_2)\sigma(z + \zeta, \omega_1, \omega_2)}{\sigma(z - \bar{\zeta}, \omega_1, \omega_2)\sigma(z + \bar{\zeta}, \omega_1, \omega_2)}\right), \tag{10}$$

where

$$z = x + iy, \qquad \zeta = \lambda + i\mu, \qquad \bar{\zeta} = \lambda - i\mu, \qquad \omega_1 = \pi, \qquad \omega_2 = iy.$$ [4]

The expression for the Green's function Γ for the same problem, obtained in the preceding section, is given in terms of the function B_{12} by formula (9). The two functions, K and Γ, are equivalent. Comparing the two expressions above, it can be shown[5] that the function B_{12} is related to the sigma function. For the various arguments of B_{12} the following relations hold:

$$B_{12}(x - \lambda, y_0 - y - \mu) = -\frac{2}{\pi}\log|\sigma(z - \bar{\zeta})|$$
$$+ \frac{\eta_1}{\pi}[(x - \lambda)^2 - (y + \mu)^2] + C,$$

[4] R. Courant and D. Hilbert, *Methoden der mathematischen Physik* I, Berlin, 1931, p. 335.

[5] A. W. Jacobson, *A generalized convolution for the finite Fourier transformations*, Thesis, University of Michigan, 1948, pp. 76–84.

$$B_{12}(x - \lambda, y_0 - y + \mu) = -\frac{2}{\pi} \log |\sigma(z + \zeta)|$$
$$+ \frac{\eta_1}{\pi}[(x - \lambda)^2 - (y - \mu)^2] + C,$$

$$B_{12}(x + \lambda, y_0 - y - \mu) = -\frac{2}{\pi} \log |\sigma(z + \zeta)|$$
$$+ \frac{\eta_1}{\pi}[(x + \lambda)^2 - (y + \mu)^2] + C,$$

and

$$B_{12}(x + \lambda, v_0 - y + \mu) = -\frac{2}{\pi} \log |\sigma(z - \zeta)|$$
$$+ \frac{\eta_1}{\pi}[(x + \lambda)^2 - (y - \mu)^2] + C,$$

where the constants η_1 and C are defined as follows:

$$\eta_1 = \frac{\pi}{12} - \frac{\pi}{2} \sum_{1}^{\infty} \frac{1}{\sinh^2 \nu y_0},$$

and

$$C = \frac{1}{\pi} \log 2 - \frac{4}{\pi} \log \prod_{\nu=1}^{\infty} (1 - e^{-2\nu y_0}).$$

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