# SIMPLY CONNECTED HOMOGENEOUS SPACES

DEANE MONTGOMERY

**Introduction.** Let $G$ be a topological group which acts transitively, as a transformation group, on a space $M$. If $x$ is any point of $M$, let $G_x$ be the closed subgroup made up of all elements in $G$ which leave $x$ fixed. There is a natural one-to-one correspondence between points of $M$ and points of the coset space or homogeneous space $G/G_x$. This one-to-one correspondence is not a homeomorphism in the most general cases, but it is clearly a homeomorphism if $G$ is a compact Lie group, and it is also a homeomorphism if $G$ is a Lie group and $M$ is a manifold [1].[1] In the present note the following theorem will be proved:

THEOREM A. *If $G$ is a connected Lie group which acts transitively on a compact manifold $M$, and if $G_x$ is connected, then $G$ contains a compact subgroup which acts transitively on $M$.*

Because of the facts stated in the introduction this theorem could be given an equivalent formulation in the slightly different language of homogeneous spaces, but this will be omitted. When $M$ is simply connected, $G_x$ must be connected so that the theorem implies the following:

COROLLARY 1. *If $G$ is a connected Lie group which acts transitively on a compact simply connected manifold $M$, then $G$ contains a compact subgroup which also acts transitively on $M$.*

PROOF OF THE THEOREM. The proof depends on a theorem about Lie groups which was proved in essence by Cartan, Chevalley, and Malcev which will now be stated.

*If $G$ is a connected Lie group, then $G$ is the direct product, as a space, of a compact subgroup $K$ and a euclidean space $E$.*

For a proof and references to original sources see [2]. It is true that maximal compact subgroups of $G$ exist which contain any given compact subgroup, that any two such are conjugate, and that $K$ may be chosen as one of these maximal compact groups.

In view of this theorem let the group $G$ of Theorem A be the direct product, as a space, of $K$ and $E$

$$G = K \cdot E.$$

Received by the editors May 21, 1949.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

Since $G_x$ is connected, $G_x$ is the direct product of a compact subgroup $L$ and a euclidean space $F$

$$G_x = L \cdot F.$$

It may be assumed that $L \subset K$.

Let $f_1$ be the map from $G/L$ to $G/K$ under which each coset of $L$ is carried to the coset of $K$ which contains it.

$$f_1: \quad G/L \to G/K.$$

This is a fiber mapping in the sense, for example, that each point of $G/K$ is in a neighborhood whose inverse is a product of a fiber and a cross section. Each fiber is homeomorphic to $K/L$. But $G/K$ is homeomorphic to $E$, a euclidean space of some dimension, and when a fibering has such a base space, it is a product. Therefore $G/L$ is homeomorphic to the topological product of $K/L$ and $E$, which is written as

$$G/L = K/L \times E. \tag{1}$$

Let $f_2$ be the map from $G/L$ to $G/G_x = M$ under which each coset of $L$ is carried to the coset of $G_x$ in which it is contained.

$$f_2: \quad G/L \to G/G_x.$$

This is also a fiber mapping and each of the fibers is homeomorphic to $G_x/L$, that is, to $F$, a euclidean space. But when all the fibers are homeomorphic to euclidean space it is known that a cross section $A$ exists. The set $A$ is a closed set in $G/L$, touching each fiber of $f_2$ precisely once, and $A$ is homeomorphic to the base space $G/G_x = M$.

There exists a mapping $P$ which retracts $G/L$ onto $A$

$$P: \quad G/L \to A,$$

that is, $P$ is defined on all of $G/L$ to $A$ and each point $A$ is fixed under $P$. It follows that if a cycle in $A$ bounds in $G/L$, it also bounds in $A$. The set $A$ is a manifold and hence contains a cycle $z$, at least mod 2, such that

$$\dim z = \dim A$$

and $z$ does not bound in $A$. Therefore $z$ does not bound in $G/L$.

From (1) it follows that the homology properties of $G/L$ are the same as those of $K/L$, and therefore $\dim M = \dim A = \dim z \leq \dim K/L$.

Since $L$ is a maximal compact group in $G_x$, it follows that

$$L = K \cap G_x.$$

Then $K(x)$, the orbit of $x$ under $K$, is homeomorphic to $K/L$ so that

$$\dim M \geqq \dim K(x) = \dim K/L \geqq \dim M.$$

Since $K(x)$ is a manifold,

$$K(x) = M$$

and this completes the proof of the theorem.

COROLLARY 2. *If $G$ is a connected Lie group which acts transitively on a compact manifold $M$, and if $G_x$ has a finite number of components, then $G$ contains a compact subgroup which also acts transitively on $M$.*

COROLLARY 3. *If $G$ is a connected Lie group which acts transitively on a compact manifold $M$, where $M$ has a finite fundamental group, then $G$ has a compact subgroup which also acts transitively on $M$.*

Under the hypothesis of Corollaries 2 and 3, $G_x$ has at most a finite number of components. Let $G_x^*$ be the identity component of $G_x$. Since $G/G_x^*$ is a finite covering of $G/G_x$ and $G/G_x$ is compact, it follows that $G/G_x^*$ is compact. By Theorem A, $G$ contains a compact group $K$ such that

$$KG_x^* = G.$$

But then

$$KG_x = G,$$

which proves Corollary 2.

As a matter of fact it is not necessary to assume in Theorem A that $G$ is connected, for if $G$ has a countable number of components and is transitive on $M$, then the identity component of $G$ is also transitive on $M$. The same remark applies to the corollaries.

The Klein bottle is a homogeneous space of a connected Lie group (Mostow) but not of a compact Lie group, which shows that some such extra hypothesis as the one used here is necessary.

## BIBLIOGRAPHY

1. R. Arens, *Topologies for homeomorphism groups*, Amer. J. Math. vol. 68 (1946) pp. 593–610.

2. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949) pp. 507–558.

3. G. D. Mostow, *Extendability of local transformation groups and groups on surfaces*, Ann. of Math. vol. 52 (1950).

INSTITUTE FOR ADVANCED STUDY