TWO-ENDED TOPOLOGICAL GROUPS

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Introduction. Let G be a locally compact, connected topological group (satisfying the second countability axiom). Let G^* be a compact space which contains a dense subset G' homeomorphic to the space G and is such that G^*-G' is totally disconnected. Then, Freudenthal has proved [1, Satz 1X, p. 277][1] that the set G^*-G' consists of at most two distinct points. Actually, Freudenthal's theorem even for topological groups is more general than here stated, and this theorem is an application to group spaces of a wider theory of "ends" of topological spaces. However, we shall quote only so much of Freudenthal's results as are necessary to this paper.

It will be convenient to regard G' as identical with G so that G is topologically imbedded in G^*. We shall call a locally compact, connected group G *two-ended* if a G^* exists such that G^*-G consists of two distinct points. The simplest example of such a group is the additive group of reals. Other examples are afforded by the direct product of this group and any compact connected topological group; it is likely that these are the only examples.

The principal objective of this note is the following theorem.

THEOREM A. *If a locally compact, connected topological group G is two-ended, then G contains a closed subgroup T isomorphic to the group of reals such that the coset-space G/T is compact; moreover, the space G is the topological product of the axis of reals by a compact connected set homeomorphic to the space G/T.*

1. **Definitions.** Now let G be two-ended and G^* compact, and necessarily connected, such that G^*-G consists of a pair of points. We shall denote one of these by e_L and the other by e_R. The space G^* is not a group, but we may continue to speak of the group product fg when $f, g \in G \subset G^*$. Moreover [1] each $f \in G$ may be regarded as a homeomorphism $f(G^*)=G^*$ by the definitions: $f(e_L)=e_L$, $f(e_R)=e_R, f(g)=fg\in G$ when $g\in G$. To the product of homeomorphisms f and g there corresponds the homeomorphism associated with $fg\in G$. We shall denote by gK (resp. Kg), $g\in G$, $K\subset G$, the set of points gk (resp. kg), where $k\in K$.

The two following properties [1] of G^* are of great importance in this work.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

(1.1) If $g_n \in G$, $n=1, 2, \cdots$, and $g_n \to e_R$, then $g_n^{-1} \to e_L$; and if $g_n \to e_L$ then $g_n^{-1} \to e_R$.

(1.2) If K is a compact subset of G and $g_n \to e_R$ (resp. e_L), then $g_n K \to e_R$ (resp. e_L).

In virtue of the known structure of locally compact abelian groups we may, in any locally compact group, distinguish between two classes of elements [2, Lemma 2, p. 96]. One of these is:

(1.3) The class which we shall denote by C consisting of those element of G which are contained in *compact* subgroups of G. For any elements $c \in C$, and any neighborhood V of the identity of G, there exists an integer n such that $c^n \in V$.

Each element of G not in C generates a subgroup of G isomorphic to the group of integers. For such elements no sequence of distinct powers has any limit in G. Now in a two-ended G we may distinguish further:

(1.4) The class R of elements of G such that if $r \in R$ then $r^n \to e_R$ (in G^*), and

(1.5) The class L such that $l \in L$ if $l^n \in e_L$ (in G^*).

It is clear from (1.1) that R and L are each the set of inverses (in G) of the other, are mutually exclusive, and are homeomorphic.

It is another matter, of course, to assert that none of these classes L, C, and R is empty, or that they comprise all elements of G. This will be shown in the succeeding paragraphs. The assumption that G is two-ended is to be understood throughout; many of the lemmas are not valid in the more general case.

2. The key lemma.

PRINCIPAL LEMMA. *If $g_n \to e_R$, $g_n \in G$, then for all but a finite number, $g_n \in R$.*

PROOF. From the definition of ends [1] it follows at once that there exists a closed compact set $K \subset G$, and open sets A and B of G^* such that

$$G^* = A \cup K \cup B, \tag{2.1}$$
$$e_L \in A,\ e_R \in B,\ A \cap K = K \cap B = B \cap \overline{A} = A \cap \overline{B} = 0.$$

By (1.1) for almost all n, $g_n K \subset B$. Let g denote any g_n for which $gK \subset B$. Then $\overline{A} \cap gK = 0$.

Now let A_0 denote the component of $G^* - K$ which contains e_L. Then A_0 is a maximally connected subset of A, A_0 is closed in A, and $\overline{A}_0 \cap K$ is not empty since G^* is connected [3, p. 16]. Let k denote some element of the set $\overline{A}_0 \cap K$.

(2.2) $$G^* = g(G^*) = gA \cup gK \cup gB,$$

and it is easily seen that gA_0 is the component of G^*-gK which contains $e_L=g(e_L)$. Then, since $gK\subset B$, $\bar{A}_0\cap gK=0$ and $\bar{A}_0\subset gA_0$. Moreover, since $gk\in B$, $gk\notin \bar{A}_0$ and there is some neighborhood V of k such that $gV\cap\bar{A}_0=0$. Since $k\in\bar{A}_0$, $V\cap A_0$ is not empty. Choose some $a\in V\cap A_0$. There is a neighborhood W of the identity such that $Wa\subset V$. Then $gWa\subset gV$, and $(gWa)\cap\bar{A}_0=0$. Thus we have shown that

(2.3) $$\bar{A}_0 \subset gA_0 - g(Wa \cap \bar{A}_0),$$

and if we denote g^{-1} by f, we may write this as

(2.4) $$f\bar{A}_0 \subset A - Wa \cap \bar{A}_0.$$

Now, for every integer n,

(2.5) $$f^n\bar{A}_0 = f^{n-1}f\bar{A}_0 \subset f^{n-1}\bar{A}_0 \subset \cdots \subset f\bar{A}_0 \subset A - Wa \cap \bar{A}_0.$$

In consequence of this, for every n, $f^na\notin Wa$, and therefore $f^n\notin W$.

But now it follows from (1.3) that no sequence of powers of f can converge to any element of G. On the other hand, since $f^na\subset A$, no subsequence of the points f^na, $n=1, 2, \cdots$, can converge to $e_R\in B$. Because of (1.2), no subsequence of the set f^n, $n=1, 2, \cdots$, can converge to e_R. Finally, then, $f^n\to E_L$, and in consequence of this $g^n\to e_R$, since $g=f^{-1}$. This proves the lemma.

Corollary. *There exists an open set $O^*\subset G^*$, $e_R\in O^*$, such that if $g\in O^*\cap G$, then $g\in R$.*

This follows from the lemma by the observation that if no such O^* existed one could construct a sequence $g_n\to e_R$ such that $g_n\notin R$, and this would contradict the lemma.

Lemma. *If $g\in G$ and for some positive integer k, $g^k\in R$, then $g\in R$*

Let Q denote an arbitrary open subset of G^* containing e_R. Since $g^k\in R$, the sequence $g^{nk}\to e_R$, k fixed. Therefore for each $m=0, 1, 2, \cdots, k-1$, the sequences $g^{nk}g^m=g^{nk-m}\to e_R$ (k and m fixed) and there exists an integer N such that for all $n\geqq N$ and all $m=0, 1, \cdots, k-1$, $g^{nk+m}\in Q$. This means, of course, that in the sequence $g, g^2, g^3, \cdots$, all but a finite number belong to Q. Since Q is an arbitrary neighborhood of e_R, this implies that $g^n\to e_R$ and, consequently, $g\in R$.

Corollary. *R is open, and $R\cup e_R$ is open.*

Let $g\in R\subset G$. Then for some integer k, $g^k\in O^*\cap G$, for the O^* of the preceding corollary. The set $O^*\cap G$ is open and contained in R. Then there is a neighborhood V of g such that $V^k\subset O^*\cap G$. Now for any $f\in V$, $f^k\in O^*\cap G\subset R$ and consequently $f\in R$. This establishes that R is open and, since $O^*\subset R\cup e_R$, it is clear that $R\cup e_R$ is open.

3. **Another lemma.** We have proved now that R is open and not empty. By symmetrical arguments, or by an application of (1.1), the set L is not empty and open. Moreover, in view of the preceding corollary, $e_L\cup L\cup R\cup e_R$ is an open set whose complement clearly is a closed and *compact* subset of G. Further, if c is an element of this complement, then $c^n\notin L\cup R$ for every integer n, by a preceding corollary. Then $c\in C$ as defined in (1.3). Since $R\cap L=0$, and since G is connected, it follows that

$$G = L \cup C \cup R, \tag{3.1}$$

where C is closed, compact, not empty. We observe, from the form of (3.1), that C separates G^* between e_L and e_R; that is, there is no connected subset of G^*-C which contains both e_L and e_R.

LEMMA. *The identity e of G is a limit of element of R.*

We know that $C\cap\overline{R}$ is not empty, since G is connected. Let $c\in C\cap\overline{R}$, and let V be an arbitrary neighborhood of the identity. There exists an integer n such that $c^n\in V$, by (1.3), and there must exist a neighborhood W of c such that $W^n\subset V$. This implies that there are in V elements of the form r^n, $r\in W\cap R$. For such r, $r^n\in R$. Therefore $R\cap V$ is not empty, and since V is an arbitrary neighborhood of the identity we have established the lemma.

4. **One-parameter subgroups.** It is our next task to construct in G a one-parameter group T which is *closed* in G. Now if T denotes any one-parameter subgroup of any locally compact group G, then by a result of which we have already taken partial advantage [**2**, p. 96] *either* T belongs to some compact subgroup of G, and in that case $T\subset C$, *or* T *is* a closed subgroup of G. Therefore, to construct in G a subgroup isomorphic to the additive group of reals it suffices to construct a one-parameter group which contains an element not in C.

LEMMA. *G contains a connected abelian group A which contains an element of R.*

The group A will be constructed with the aid of the preceding lemma. Let $r_n\to e$, $r_n\in R$. Denote by Q an open compact subset of G such that $C\subset Q$, and denote by A_n the group generated by r_n. This group is isomorphic to the additive group of integers. Then A_n will have elements in Q and also in the complement of $\bar{Q}$, and it is easy to see that there is a subsequence A_{n_i} of the sequence A_n such that some point of $\bar{Q}-Q$ is in the sequential limit set of the sequence of sets A_{n_i}. This sequential limit set is the desired group A. We have merely sketched the argument because it occurs in substantially the form needed here in [4, Lemma 5, p. 112].

COROLLARY. *G contains a subgroup T isomorphic to the additive group of reals.*

The closure of the group A of the preceding lemma is a locally compact connected abelian group and is therefore the direct product of a compact subgroup by an n-dimensional vector group [5]. Since $\bar{A}$ cannot be compact, containing an element of R, it follows that the vector group is not the identity and must contain a subgroup T of the desired property. This is a closed subgroup of G, as well as of $\bar{A}$.

5. **Conclusion of the proof of Theorem A.** We are nearing the end of our task. We know that G contains at least one subgroup isomorphic to the reals. Suppose that T is any such closed one-parameter subgroup. Then the set $e_L \cup T \cup e_R$ is a closed subset of G^* homeomorphic to a simple arc with end points e_R and e_L. Moreover, this is also true of the set $e_L \cup gT \cup e_R$ for every $g \in G$. Then it follows from the remark following (3.1) that every coset gT contains at least one point of C. Let us now consider the coset-space, for definiteness the right coset space G/T. Since T is closed in G, G/T may be topologized and is a locally compact topological space [5]. It may assist the reader to remark that an essential feature of this topology is the fact that if $g_n \to g$, where $g, g_n \in G$, and if the sequence $g_n t_n$, $t_n \in T$, has any limit point in G then this limit point is of the form gt for some $t \in T$.

LEMMA. *The coset-space G/T is compact and homeomorphic to a space C^* which is a continuous image of C.*

Let us denote by C_g the set $C \cap gT$, for $g \in C$. This is a closed set. Suppose now that $g_n \to g$, $g_n \in C$, and consider the associated sets $C_n = C \cap g_n T$. Let $k_n \in C_n$. Then $k_n = g_n t_n$, $t_n \in T$, $g_n \in C$. Since C is compact, and $k_n \in C$, it is an easy consequence that the set of t_n, $n=1, 2, \cdots$, is compact. From this it is clear that any limit point of the set k_n, $n=1, 2, \cdots$, is of the form $gt \in C_g$, for some $t \in T$. Consequently the sets C_g, $g \in C$, give rise to an upper semi-continuous decomposition [3] of C, and there is a compact space C^* which is a continuous image of C, $\phi(C)=C^*$. For each $c^*\in C^*$, the set $\phi^{-1}(c^*)$ is some decomposition set $C_g=C\cap gT$.

Now it is clear that each point of C^* is associated uniquely with a coset gT and therefore with a point of the coset-space G/T. Since each coset intersects C, this correspondence extends over all of G/T and all of C^*. From the definition, above, of C^* and the topology of G/T it follows easily that this correspondence is a homeomorphism. This establishes the major part of Theorem A.

The fact that the coset-space G/T is homeomorphic to a compact set C^* is not alone sufficient to insure that G may be expressed as the topological product of C^* and T. This would imply, and is implied by, the existence in G of a closed set $\tilde{C}$ meeting each coset gT in one and only one point. But the fact that T is a closed one-parameter group is enough to insure the existence of such a "cross-section." This was proved by Montgomery and the author [6] in a somewhat more general context, and depends on certain local "sections" of regular families of curves constructed by Whitney. We may conclude, then, that G is representable as the direct product of two closed subsets, the set T and a cross-sectioning set $\tilde{C}$ which is of necessity homeomorphic to C^*. Since G is connected, it is clear that $\tilde{C}$ and C^* are connected as well as compact. This concludes the proof of Theorem A.

6. **An application.** As an application of Theorem A, let us suppose that a two-ended group G possesses a connected subgroup H which is not compact. Then $\overline{H}$ is locally compact and connected and also two-ended. Therefore, $\overline{H}$ contains a subgroup T isomorphic to the group of reals. Since T is in G, it is evident from the preceding section that G/T is compact. All the more, then, $G/\overline{H}$ is compact.

It is an immediate consequence of this remark that if a locally compact connected group G contains a non-compact connected subgroup H such that the coset space $G/\overline{H}$ is *not* compact, then G cannot be a two-ended group. Therefore, by Freudenthal's theorem which forms the point of departure for this note, G can have only a single "end." This sufficient condition for "one-endedness" generalizes a result due to Freudenthal, appearing in an appendix to his work [7] on the "ends" of discrete-spaces.

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