ON EUCLIDEAN LOCAL GROUPS SATISFYING CERTAIN CONDITIONS

MASATAKE KURANISHI

Let G be a euclidean local group, that is, a topological local group the space of which is homeomorphic to a euclidean space. The purpose of this note is to prove the following theorem:

THEOREM. *Let G have a neighborhood U of identity e, in which a metric $\rho(x, y)$ can be introduced satisfying the following conditions:*

(A) *If $x, y, xy \in U$,*[1] *then $K_2\rho(y, e) \leqq \rho(xy, x) \leqq K_1\rho(y, e)$.*

(B) *If $x, x^2, \cdots, x^{2^n-1}, x^{2^n} \in U$, $y, y^2, \cdots, y^{2^n-1}, y^{2^n} \in U$, then $K_4 2^n\rho(x, y) \leqq \rho(x^{2^n}, y^{2^n}) \leqq K_3 2^n\rho(x, y)$, with positive constants K_i, $i=1, 2, 3, 4$.*

Then G is a local Lie group, and vice versa.

P. A. Smith [2][2] has obtained a necessary and sufficient condition for G to be a local Lie group. He says that if we can introduce into a neighborhood of e a coordinate system $a^1, \cdots, a^r$, with respect to which the product function ab is expressible in the form, written vectorially,

$$ab = a + b + |a|F(a, b)$$

where $|a| = (\sum(a^i)^2)^{1/2}$ and where F satisfies the sole condition that $F \to 0$ as $a \to e$, $b \to e$, then G is a local Lie group and vice versa. A coordinate system satisfying this condition is called by him (right) regular. It is shown [1; 2] that a coordinate system in which the product function ab is of class C^1 with respect to b fixing a is regular and that the euclidean metric of a regular coordinate system satisfies our conditions (A) and (B).

Our assumptions (A) and (B) are mainly used to select a uniformly convergent subsequence from the function family $P_n(x, y) = (x^{1/2^n} \cdot y^{1/2^n})^{2^n}$, $n = 1, 2, \cdots$, which, in some sense, corresponds to differentiating the product function at the identity e.

The author is indebted to Prof. T. Nakayama for his kind encouragement and advice.

LEMMA 1. *Let U be a neighborhood of e in G which contains no sub-*

Received by the editors March 4, 1949.

[1] In details, we must write—instead of $xy \in U$—xy exists and is contained in U. But we shall often omit "exists" in such a case in the following for simplicity.

[2] Numbers in brackets refer to the references cited at the end of the paper.

group in the large except $\{e\}$. *Then there exists a neighborhood* V *of* e *such that* $x, y, x^2=y^2\in V$ *implies* $x=y$.

PROOF.[3] We can assume without loss of generality that $\overline{U}$ is compact and for any arbitrary three elements x, y, z of U the products $(xy)z$ and $x(yz)$ exist. Take W such that $U\supset W^4$, $W=W^{-1}$. As $\overline{U}$ is compact, there exists a neighborhood V of e such that for any $g\in\overline{U}$, we have $g^{-1}Vg\subset W$. Now if $x, y, x^2=y^2\in V$, put $x^{-1}y=a$. Then $axa=x$, $a\in V^2\subset W\subset U$, and it follows that if $a, a^2, \cdots, a^m\in U$, then $a^{2m}=x^{-1}a^{-m}xa^m\in x^{-1}W\subset W^2\subset U$, $a^{2m+1}=a^{2m}\cdot a\in W^2W^2\subset U$. This means that for any integer n, $a^n\in U$; consequently from the assumption $a=e$, $x=y$.

REMARK. In Lemma 1 the assumption that G is euclidean is not necessary. Let σ be a mapping $y\rightarrow y^2$. If G is euclidean and if U is sufficiently small, then $\sigma(U)$ is an open subset of G and σ is a homeomorphism between U and $\sigma(U)$. The proof is clear from the Brouwer's theorem on the invariance of domain and from the fact that $\overline{U}$ is compact and that σ is one-to-one.

LEMMA 2. *Let* G *have a neighborhood* U *of* e *such that for any element* y *of* $U-\{e\}$ *there exists an integer* n *satisfying* $y^{2^n}\notin\overline{U}$. *Then, if* V *is a sufficiently small neighborhood of* e, *we can construct a real-valued continuous function* $f(y)$ *defined on* V *satisfying the following conditions*:

(i) *If* $y, y^2\in V$, *then* $f(y^2)\geqq f(y)$.

(ii) $f(y)=0$ *if and only if* $y=e$ *provided* $y\in V$.

PROOF. *If* $y, y^2, y^{2^2}, \cdots, y^{2^n}\in\overline{V}$, $y^{2^{n+1}}\bar{\in}\overline{V}$, we put $n=\delta_V(y)$. Take an element p of V, we note that if $k\leqq\delta_V(p)$, then $p\neq p^{2^k}$. Consequently if we take a sufficiently small neighborhood W of e, then for any $i<j\leqq\delta_V(p)$, we have $\sigma^i(Wp)\cap\sigma^j(Wp)=\Theta$, $\sigma^{\delta_V(p)+1}(Wp)\cap\overline{V}=\Theta$, where $\sigma^0(y)=y$, $\sigma^2(y)=y^2$, $\sigma^{i+1}(y)=\sigma^1(\sigma^i(y))$.

Construct a continuous function $f_p^0(y)$ defined on V such that

$$0\leqq f_p^0(y)\leqq 1;\ \text{if } y\notin Wp,\ \text{then } f_p^0(y)=0;\ \text{and } f_p^0(p)=1.$$

For any integer $k\leqq\delta_V(p)$ put

$$\text{if } y\notin\sigma^k(Wp),\qquad f_p^k(y)=0,$$
$$\text{if } y\in\sigma^k(Wp),\qquad f_p^k(y)=f_p^0(y^{1/2^k}),\quad\text{where}\quad (y^{1/2^k})^{2^k}=y.$$

If V is sufficiently small, by the remark to Lemma 1, $f_p^k(y)$ is a continuous function defined on V. Put

[3] This simple proof was suggested to the author by T. Hayashida. The author's original proof is more complicated.

$$f_p(y) = \sum_{k=0}^{\delta_V(p)} f_p^k(y).$$

It is easy to see that $f_p(y)$ is a continuous function defined on V and $0 \leqq f_p(y) \leqq 1$, $f_p(p) = 1$, $f_p(y^2) \geqq f_p(y)$. As $\bigcup_{p \in -V\{e\}} V_p = V - \{e\}$, where $V_p = \{y;\ f_p(y) > 0\}$, we can select a countable set $\{p_n\}$ such that $\bigcap_{n=1}^{\infty} V_{p_n} = V - \{e\}$. Put

$$f(y) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{p_n}(y).$$

It is clear that $f(y)$ satisfies our conditions (i) and (ii).

LEMMA 3. *Under the same assumption as in Lemma 2, there exists a neighborhood W of e such that for an arbitrary element x of W, W contains a unique element $x^{1/2}$ satisfying $(x^{1/2})^2 = x$.*

PROOF. Take a sufficiently small neighborhood V of e, where we can construct the function $f(y)$ as stated in Lemma 2. Put $V_\epsilon = \{y; f(y) < \epsilon\}$. If V is sufficiently small, by the Remark to Lemma 1, $\sigma(V)$ is an open set. From the condition (ii) which $f(y)$ satisfies, there exists $\epsilon > 0$, $V_\epsilon \subset \sigma(V) \cap V$. This means that for any element y of V_ϵ there exists $y^{1/2} \in V$ such that $(y^{1/2})^2 = y$. But since $f(y^{1/2}) \leqq f(y) < \epsilon$, $y^{1/2} \in V_\epsilon$. The uniqueness of $y^{1/2}$ has already been proved in Lemma 1.

By Lemma 3, there exists $x^{1/2^n}$ such that $(x^{1/2^n})^{2^n} = x$ and $x^{1/2^n} \in W$ for any element x of W. We can assume without loss of generality $W \subset U$, where U is a neighborhood of e satisfying the assumption of our theorem. We shall write $|x|$ instead of $\rho(x, e)$. Then by (B), $|x^{1/2^n}| \leqq (1/K_4)(1/2^n)|x|$. We can prove, by induction, the existence of $(x^{1/2^n})^m$ and

$$(*) \qquad |(x^{1/2^n})^m| \leqq \frac{K_1}{K_4} \frac{m}{2^n} |x|$$

for any arbitrary integer $m \leqq 2^n$. Assume, in fact, $|(x^{1/2^n})^m| \leqq (K_1/K_4)(m/2^n)|x|$ for some $m < 2^n$, then

$$\begin{aligned} |(x^{1/2^n})^{m+1}| &= |(x^{1/2^n})^m x^{1/2^n}| \leqq K_1 |x^{1/2^n}| + |(x^{1/2^n})^m| \\ &\leqq \frac{K_1}{K_4} \frac{1}{2^n} |x| + \frac{K_1}{K_4} \frac{m}{2^n} |x| \\ &= \frac{K_1}{K_4} \frac{m+1}{2^n} |x|, \end{aligned}$$

as we can see easily from (A) that

$$\text{(C)} \qquad |xy| \leqq K_1|y| + |x|.$$

By using the uniqueness of $x^{1/2}$, it is easy to see that $m/2^n = m'/2^{n'}$ implies $(x^{1/2^n})^m = (x^{1/2^{n'}})^{m'}$. Now we put $f(m/2^n) = (x^{1/2^n})^m$; then $f(m/2^n)f(m'/2^{n'}) = f(m/2^n + m'/2^{n'})$, and from these and from the inequality (*) follow the uniform continuity of the function f on the set $\{m/2^n; m \leqq 2^n\}$, which is everywhere dense in the interval $[0, 1]$. Therefore f can be extended to a continuous function $\bar{f}$ defined on $[0, 1]$. If we put $x^\lambda = \bar{f}(\lambda)$, $x^{-\lambda} = (x^{-1})^\lambda$, for any $0 \leqq \lambda \leqq 1$, then $|x^\lambda| \leqq (K_1/K_4)|x|$ and $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$ if both sides have meaning. This proves the following lemma.

LEMMA 4. *Under the same assumption as in the theorem, there exist neighborhoods W_1 and W_2 of e such that for any element x of W_1, there exists a unique one-parameter subgroup x^λ contained in W_2.*

From (A),

$$\text{(D}'\text{)} \quad \rho(ax, ay) \leqq K_1\rho((ay)^{-1}\cdot ax, e) = K_1\rho(y^{-1}\cdot x, e) \leqq \frac{K_1}{K_2}\rho(x, y).$$

Take a sufficiently small neighborhood V of e, and put

$$\Sigma = \{x; x \in V, X^2 \notin V\}, \qquad C = \min_{x \in \Sigma} |x|,$$

$$\Omega = \{w; w = a^{-1}xa, x \in \Sigma, a \in V\}, \quad D = \max_{x \in \Omega} |x|.$$

Then $\bar{\Sigma} \not\ni e$, $C > 0$, $D > 0$, and

$$(**) \qquad |a^{-1}xa| \leqq \frac{D}{C}|x| \qquad \text{for } x \in \Sigma,\ a \in V.$$

Take $y \in V$, then for some integer n, $y^{2^n} \in \Sigma$ and for every integer $0 \leqq m \leqq n$, $y^{2^m} \in V$. From (B) and (**)

$$|a^{-1}ya| \leqq \frac{1}{2^n}\frac{1}{K_4}|a^{-1}y^{2^n}a| \leqq \frac{1}{2^n}\frac{1}{K_4}\frac{D}{C}|y^{2^n}| \leqq \frac{K_3D}{K_4C}|y|,$$

$$\text{(D}''\text{)} \quad \rho(xa, ya) \leqq K_1\rho(a^{-1}y^{-1}xa, e) \leqq \frac{K_1K_3D}{K_4C}\rho(y^{-1}x, e)$$

$$\leqq \frac{K_1K_3D}{K_2K_4C}\rho(x, y),$$

if x, y, a is sufficiently near to e.

From (D′) and (D″) we can see that for a sufficiently small neighborhood V of e, there exists $A>0$ such that

(D)[4] if $x, y, x', y' \in \bar{V}$, $\rho(xy, x'y') \leqq A\{\rho(x, x') + \rho(y, y')\}$.

If we take a sufficiently small neighborhood V of e, from (C), (B), and (D) we can easily prove the existence of $P_n(x, y) = (x^{1/2^n}y^{1/2^n})^{2^n}$ and

$$\text{(E)} \quad |P_n(x, y)| = |(x^{1/2^n}y^{1/2^n})^{2^n}| \leqq \frac{K_1K_3}{K_4}(|x| + |y|),$$

$$\text{(F)} \quad \rho(P_n(x, y), P_n(x', y')) \leqq \frac{K_3A}{K_4}\{\rho(x, x') + \rho(y, y')\}$$

for arbitrary $x, y, x', y' \in \bar{V} \subset U$.

Therefore $P_n(x, y)$, $n = 0, 1, 2, \cdots$, is an equi-continuous and uniformly bounded family of functions defined on $\bar{V}\times\bar{V}$, and consequently a uniformly convergent subsequence $\{P_{n'}(x, y)\}$ can be selected from $\{P_n(x, y)\}$. Put $P(x, y) = \lim_{n'\to\infty} P_{n'}(x, y)$.

Now we shall prove that if we define the product $x \circ y = P(x, y)$, V becomes an abelian local Lie group. We denote this local group by H.

(1) Commutativity is clear from $(x^{1/2^n}y^{1/2^n})^{2^n} = x^{1/2^n}(y^{1/2^n}x^{1/2^n})^{2^n}(x^{1/2^n})^{-1}$.

(2) Associative law: Put $P(x, y) = \epsilon_{n'}P_{n'}(x, y)$, then $\lim_{n'\to\infty} \epsilon_{n'} = e$. Using (D), (E), and (F),

$$\rho(P_{n'}(P(x, y), w), P_{n'}(P_{n'}(x, y), w)) \leqq \frac{K_3A^2}{K_4}\rho(\epsilon_{n'}, e),$$

from the definition $P_{n'}(P_{n'}(x, y), w) = P_{n'}(x, P_{n'}(y, w))$. From these we can see $(x \circ y) \circ w = x \circ (y \circ w)$.

(3) $x^\lambda \circ x^\mu = x^{\lambda+\mu}$, which shows the existence of the inverse, and that every one-parameter subgroup of G is also that of H.

(4) As H is abelian, we can find easily $x_1, x_2, \cdots, x_r$ of H (where r is the dimension of G) such that $W = \{y;\ y = x_1^{\lambda_1} \circ x_2^{\lambda_2} \circ \cdots \circ x_r^{\lambda_r},\ |\lambda_i| \leqq 1,\ i = 1, 2, \cdots, r\}$ is a neighborhood of e and mapping $(\lambda_1, \lambda_2, \cdots, \lambda_r) \to x_1^{\lambda_1} \circ x_2^{\lambda_2} \circ \cdots \circ x_r^{\lambda_r}$ is topological. Moreover

$$(x_1^{\lambda_1} \circ \cdots \circ x_r^{\lambda_r}) \circ (x_1^{\lambda_1'} \circ \cdots \circ x_r^{\lambda_r'}) = (x_1^{\lambda_1+\lambda_1'} \circ \cdots \circ x_r^{\lambda_r+\lambda_r'}).$$

Thus we can see that H is a Lie group.

LEMMA 5. *Under the same assumptions as in the theorem the local group consisting of inner automorphisms of G is a linear group.*

PROOF. From the uniqueness of $x^{1/2}$, it is easy to see $px^{1/2}p^{-1}$

[4] The proof of this inequality is essentially due to P. A. Smith [2].

$=(pxp^{-1})^{1/2}$, for arbitrary $x, y, p \in V$. From this and from the sufficient smallness of $|P_n(x, y)|$, we can deduce immediately that for arbitrary $p, x, y \in V$, $P_n(p^{-1}xp, p^{-1}yp) = p^{-1} \cdot P_n(x, y)p$, which implies $(p^{-1}xp) \circ (p^{-1}yp) = p^{-1}(x \circ y)p$. Thus $x \to p^{-1}xp$ is an automorphism of the abelian Lie group H, which proves the lemma.

LEMMA 6.[5] *Let N be a closed local normal subgroup of G, satisfying the following conditions:*

(1) *N is an abelian local Lie group,*

(2) *G/N is a local Lie group,*

(3) *there exists a set M such that for any a sufficiently near to e, Na contains one and only one element a' of M depending continuously on G/N.*

If G contains N satisfying the above conditions, G is a local Lie group.

PROOF. As N is an abelian Lie group, we can introduce a coordinate system $u(\xi^1, \cdots, \xi^n)$ in N satisfying $u(\xi^1, \cdots, \xi^n) \cdot u(\xi'^1, \cdots, \xi'^n) = u(\xi^1+\xi'^1, \cdots, \xi^n+\xi'^n)$. The transformation by an element a of G induces an automorphism in N, given by

$$au(\xi^1, \cdots, \xi^n)a^{-1} = u(\xi'^1, \cdots, \xi'^n),$$
$$(\xi'^1, \cdots, \xi'^n) = (\xi^1, \cdots, \xi^n)A_a,$$

where $(A^{a}{}_{i}^{j})$ is a real matrix of degree n. As $G = NM$, we can put for arbitrary elements $p, q \in M$ sufficiently near to e,

$$pq = \Psi(p, q)(p \circ q),$$
$$\Psi(p, q) = u(\Psi^1(p, q), \cdots, \Psi^n(p, q)) \in N, \qquad p \circ q \in M,$$

where, by (3), both $p \circ q$ and $\Psi(p, q)$ are continuous functions on $M \times M$. Then by the product $p \circ q$, M is a local Lie group isomorphic with G/N, and $M \ni p \to A_p$ is a representation of the local Lie group M. Consequently, introducing the canonical coordinate $v(\eta^1, \cdots, \eta^m)$ in M, $A_{v}(\eta^1, \cdots, \eta^m)_j^i$ are analytic functions of $(\eta^1, \cdots, \eta^m)$.

From the associative law of the product of G, if p, q, r are sufficiently near to e, we can see easily that

$$(1) \quad \overset{i}{\Psi}(p, q) + \overset{i}{\Psi}(p \circ q, r) = \sum_{j=1}^{n} \overset{i}{A}_{pj} \overset{j}{\Psi}(q, r) + \overset{i}{\Psi}(p, q \circ r), \; i = 1, \cdots, n.$$

Let $\phi(p)$ be an arbitrary continuous function defined on M and

[5] In the proof of this lemma, the fact that the space of G is euclidean is not used. To prove our theorem we need only the case when N is the center of G. But we state the lemma in this general form, because this lemma is applicable to some theorem on locally compact groups (cf. [3]).

taking values in N. $M_\phi=\{p_\phi;\ p_\phi=\phi(p)p,\ p\in M\}$ also satisfies the condition (3), if M is replaced by M_ϕ. We can put

$$p_\phi\cdot q_\phi = \Psi_\phi(p, q)(p_\phi \circ q_\phi),$$
$$\Psi_\phi(p, q) \in N, \qquad p_\phi \circ q_\phi \in M_\phi.$$

As $p_\phi \circ q_\phi=(p \circ q)_\phi=\phi(p \circ q)(p \circ q)$, the relation between Ψ^i and Ψ^i_ϕ is given by

$$(2)\ \Psi^i_\phi(p, q) = \phi^i(p) + \sum_{j=1}^{n} A^i_{pj}\phi^j(q) + \Psi^i(p,q) - \phi^i(p\circ q), i = 1,2, \cdots, n.$$

Take two sufficiently small numbers α and β such that $U=\{v(\eta^1, \cdots, \eta^m);\ |\eta^i|<\alpha\}$ and $V=\{v(\eta^1, \cdots, \eta^m);\ |\eta^i|<\beta\}$ satisfy $U\supset V \circ V \circ V$ and such that if $p,\ q,\ r\in U$, equalities (1) and (2) have meanings. Take a real function $c(p)$ defined on U such that $c(v(\eta^1, \cdots, \eta^m))$ is of class C^3 with respect to $\eta^1, \cdots, \eta^m$ and which satisfies the following conditions:

If $p\in V$, then $c(p)=0$,

$$\int_V c(v(\eta^1, \cdots, \eta^m))d\eta^1 \cdots d\eta^m = \int_V c(v(\eta))d\eta = \int_U c(v(\eta))d\eta = 1.$$

Put

$$\phi^i_0(p) = -\int_V \Psi^i(p, v(\eta))c(v(\eta))d\eta = -\int_U \Psi^i(p, v(\eta))c(v(\eta))d\eta$$

and $\phi_0(p)=u(\phi_0^1(p), \cdots, \phi_0^n(p))\in N$. Then from (1) and (2), if $p,\ q\in V$

$$(3)\quad \begin{aligned} \Psi^i_{\phi_0}(p, q) &= \int_V \Psi^i(p, q\circ v(\eta))c(v(\eta))d\eta - \int_V \Psi^i(p, v(\eta))c(v(\eta))d\eta \\ &= F_1(p, q) - F_2(p), \qquad i = 1, 2, \cdots, n. \end{aligned}$$

Put $q \circ v(\eta^1, \cdots, \eta^m)=v(\eta'^1, \cdots, \eta'^m)$, and changing the integral variable η by η', we can see

$$\begin{aligned} F^i_1(p, q) &= \int_V \Psi^i(p, q\circ v(\eta))c(v(\eta))d\eta \\ &= \int_{q\circ V} \Psi^i(p, v(\eta'))c(q^{-1}\circ v(\eta'))J(\eta')d\eta' \\ &= \int_U \Psi^i(p, v(\eta'))c(q^{-1}\circ v(\eta'))J(\eta')d\eta', \end{aligned}$$

because $\eta \to \eta'$ is an analytic function of η and $q \circ V \subset U$. As $c(v(\eta'))$, $q \circ r$, and q^{-1} are functions of class C^3, the above equality shows that $F_1^i(p, v(\eta))$ are functions of class C^3 with respect to η when p is fixed. Consequently, by (3), $\Psi_{\phi_0}^i(p, v(\eta))$ are functions of class C^3 with respect to η when p is fixed. From (1), we can get

$$\sum_{j=1}^{n} \left\{ \int_U A_{p\,j}^{i} c(p) dp \right\} \Psi^j(q, r) \tag{4}$$
$$= \int_U \Psi^i(p, q) c(p) dp + \int_U \Psi^i(p \circ q, r) c(p) dp - \int_U \Psi^i(p, q \circ r) c(p) dp.$$

From the fact that if U is sufficiently small, A_p $(p \in U)$ is sufficiently near to the identity matrix and that $\int_U c(p) dp = 1$, it follows that the determinant $\left| \int_U A_{pj}^i c(p) dp \right|$ is not zero. Consequently, using the same argument as above on $\int_U \Psi^i(p \circ q, r)\, c(p) dp$, we can deduce from (4) that if $\Psi^i(p, q)$ is of class C^3 with respect to q when p is fixed, $\Psi^i(p, q)$ is of class C^3 with respect to p when q is fixed. Thus we have proved that $\Psi_{\phi_0}^i(p, q)$ is of class C^3 with respect to both p and q in the coordinate system $v(\eta^1, \cdots, \eta^m)$. As any element p of G is written uniquely as $p = nm_\phi$, $n \in N$, $m_\phi \in M$, we can introduce a coordinate system $w(\zeta^1, \cdots, \zeta^{n+m})$ in G by

$$w(\zeta^1, \cdots, \zeta^{n+m}) = u(\zeta^1, \cdots, \zeta^n)\phi_0(v(\zeta^{n+1}, \cdots, \zeta^{n+m}))v(\zeta^{n+1}, \cdots, \zeta^{n+m}).$$

As $\phi_0(v(\eta)) \in N$ and N is abelian, we can easily verify that the product function $f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m})$ $(i = 1, 2, \cdots, n+m)$ of G with respect to this coordinate system is given by

$$f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m}) = \zeta^i + A^i_{v(\zeta^{n+1}, \cdots, \zeta^{n+m})\,j}\zeta'^j + \Psi^i_{\phi_0}(v(\zeta^{n+1}, \cdots, \zeta^{n+m}), v(\zeta'^{n+1}, \cdots, \zeta'^{n+m})) \quad (i = 1, 2, \cdots, n),$$

$$f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m}) = g^i(\zeta^{n+1}, \cdots, \zeta^{n+m}; \zeta'^{n+1}, \cdots, \zeta'^{n+m}) \quad (i = n+1, \cdots, n+m),$$

where $g^{i-n}(\zeta^{n+1}, \cdots, \zeta^{n+m}; \zeta'^{n+1}, \cdots, \zeta'^{n+m})$ is the product function of $p \circ q$ with respect to $v(\zeta^{n+1}, \cdots, \zeta^{n+m})$.

Thus, with respect to the coordinate system $w(\zeta)$, the product function of G is of class C^3 and hence G is a local Lie group.

PROOF OF THE THEOREM. When G has the discrete center, Lemma 5 shows that G is a local Lie group.

When G has the non-discrete center N, by Lemma 4, N is an abelian Lie group, and by Lemma 5 again, G/N is a local Lie group. Then we can introduce a canonical coordinate of the second kind by $x_1^{*\lambda_1} \cdots, x_m^{*\lambda_m}$, where $x_i^{*\lambda_i}$ $(i=1, 2, \cdots, m)$ are one-parameter subgroups of G/N. Take x_i of G from the coset x_i^* for each i, we can easily show that the set $M=\{y;\ y=x_1^{\lambda_1} \cdots x_m^{\lambda_m},\ |\lambda_i| \leqq 1\}$ satisfies the condition (3) of Lemma 6. Consequently, by Lemma 6, G is a local Lie group. This completes our proof.

REFERENCES

1. Garrett Birkhoff, *Analytical groups*, Trans. Amer. Math. Soc. vol. 43 (1934) pp. 61–101.
2. P. A. Smith, *Foundation of Lie groups*, Ann. of Math. vol. 48 (1947).
3. K. Iwasawa, *On some types of topological groups*, Ann. of Math. vol. 50 (1949).
4. L. Pontrjagin, *Topological groups*, Princeton, 1939.

TOKYO INSTITUTE OF TECHNOLOGY

NOTE ON A THEOREM OF KOKSMA

WM. J. LEVEQUE[1]

In 1935 Koksma [2][2] showed, among other things, that the sequence $x, x^2, x^3, \cdots$ is uniformly distributed (mod 1) for almost all $x>1$; that is, that if $N(n, \alpha, \beta, x)$ denotes the number of elements x^i of the sequence $x, x^2, \cdots, x^n$ for which

$$0 \leqq \alpha \leqq x^i - [x^i] < \beta \leqq 1,$$

then

$$\lim_{n\to\infty} \frac{N(n, \alpha, \beta, x)}{n} = \beta - \alpha$$

Presented to the Society, September 10, 1948, under the title *A metric theorem on uniform distribution* (mod 1); received by the editors January 24, 1949 and, in revised form, February 10, 1949.

[1] The author is indebted to Professor Mark Kac for his help in connection with this paper.

[2] Numbers in brackets refer to the bibliography at the end of the paper.