ON EULER TRANSFORMS

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Let E_0 denote the class of non-constant functions satisfying

(1) $$d\phi(x) \geqq 0 \qquad \text{where } 0 \leqq x < \infty.$$

Then the Euler transform

(2) $$\phi_\lambda(x) = \int_0^\infty (x+t)^{-\lambda} d\phi(t), \qquad \text{where } 0 < x < \infty,$$

is defined for some ϕ of class E_0. Let E_λ denote the class of all functions ϕ_λ belonging to a fixed $\lambda > 0$ and to some ϕ of class E_0. It will be shown that

(3) $$E_\lambda \text{ is a (proper) subset of } E_\mu \text{ if } \lambda < \mu.$$

This implies that

(4) $$E_\infty = \lim_{\lambda\to\infty} E_\lambda \text{ if } E_\infty = \sum_{0<\lambda<\infty} E_\lambda.$$

The class E_∞ is closely related to the Hausdorff-Bernstein class, consisting of all functions which are completely monotone for $0<x<\infty$. Let E^∞ denote the latter class. It will be shown that

(5) $$E_\infty \text{ is a (proper) subset of } E^\infty$$

and that, with reference to the "natural" topology on E^∞,

(6) $$E_\infty \text{ is dense on } E^\infty.$$

It should be noted that E^∞ consists of all functions representable in the form

(7) $$\phi^\infty(x) = \int_0^\infty e^{-xt} d\phi(t), \qquad \text{where } 0 < x < \infty,$$

provided that ϕ, instead of being subject to both restrictions (1), is subject only to the second of those restrictions and to the assumption that the integral (7) is convergent at every $x>0$ (but not necessarily at $x=0$). By the "natural" topology on E^∞ is meant that defined by the Helly convergence of monotone functions.

PROOF OF (3). It is readily verified from (1) and (2) that, as $x\to\infty$, no $\phi_\lambda(x)$ can tend to 0 as strongly as $x^{-\mu}$, if $\mu>\lambda$. On the

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other hand, (2) shows that $\phi_\mu(x)=x^{-\mu}$ if $\phi(t)=\operatorname{sgn} t$. Hence, the parenthetical assertion of (3) will need no further proof.

The main assertion of (3) is that, if $0<\lambda<\mu$, there belongs to every $\phi_\lambda(t)$ of class E_λ some $\phi^*(t)$ of class E_0 satisfying

$$(8) \qquad \phi_\lambda(x) = \overset{*}{\phi}_\mu(x), \qquad \text{where } 0 < x < \infty.$$

It will be shown that such a $\phi^*(t)$ is supplied by the absolutely continuous function having the derivative

$$(9) \quad d\phi^*(t)/dt = \mathrm{A}\int_0^t (t-s)^{\mu-\lambda-1}d\phi(s) \ (0 < t < \infty, \phi^*(0) = \phi^*(+0)),$$

where $\mathrm{A}=\mathrm{A}(\lambda, \mu)$ is a positive constant.

In view of (2), the assertion of (8) and (9) means that

$$\int_0^\infty (x+t)^{-\lambda}d\phi(t) = \mathrm{A}\int_0^\infty (x+t)^{-\mu}\left\{\int_0^t (t-s)^{\mu-\lambda-1}d\phi(s)\right\}dt,$$

where $0<x<\infty$. It follows therefore from (1), and from (the Stieltjes form of) Fubini's theorem, that it is sufficient to verify the identity

$$\int_0^\infty (x+t)^{-\lambda}d\phi(t) = \mathrm{A}\int_0^\infty\left\{\int_s^\infty (x+t)^{-\mu}(t-s)^{\mu-\lambda-1}dt\right\}d\phi(s).$$

But the latter holds for every ϕ of class E_0 if

$$(x+t)^{-\lambda} = \mathrm{A}\int_t^\infty (x+s)^{-\mu}(s-t)^{\mu-\lambda-1}ds$$

is an identity in (x, t), where $x>0$, $t>0$. Hence, if the integration variable s is replaced by $s-t$, and if $x+t$ is then called x, it follows that it is sufficient to verify the identity

$$x^{-\lambda} = \mathrm{A}\int_0^\infty (x+s)^{-\mu}s^{\mu-\lambda-1}ds,$$

where A is independent of x. Finally, the truth of this identity follows by changing s to xs (at a fixed $x>0$).

This proves (3). It also follows that the value of the constant $\mathrm{A}=\mathrm{A}(\lambda, \mu)$ which occurs in (9) is given by

$$1 = \mathrm{A}\int_0^\infty (1+s)^{-\mu}s^{\mu-\lambda-1}ds.$$

Incidentally, the last integral can readily be transformed into the integral defining $B(\lambda, \mu) = \Gamma(\lambda)\Gamma(\mu-\lambda)/\Gamma(\mu)$; so that, in (9),

(9 bis) $$A = A(\lambda, \mu) = 1/B(\lambda, \mu).$$

PROOF OF (5). It is seen from (1) that $(-1)^n$ times the nth derivative of the function (2) is non-negative for $n=0, 1, 2, \cdots$. This means that every class E_λ is contained of the Hausdorff-Bernstein class, E^∞. Hence, by (4), E_∞ is contained in E^∞. The parenthetical part of (5) follows from the fact that e^{-x} is in E^∞, but e^{-x} is not in E_∞. For otherwise e^{-x} would be in E_λ some λ, which would imply, for $x>0$ and $t_0>0$, that

$$e^{-x} \geqq (x + t_0)^{-\lambda}(\phi(t_0) - \phi(0)),$$

by (1) and (2). If t_0 is chosen so that $\phi(t_0) - \phi(0) > 0$, the last formula line leads to a contradiction for large x. This contradiction shows that e^{-x} cannot be in E_∞ and completes the proof of (5).

PROOF OF (6). Let $b>0$ and $\lambda>0$. Then it is readily verified that

$$b^\lambda \int_0^\infty (x + t)^{-\lambda} d \operatorname{sgn} (t - b) = (1 + x/b)^{-\lambda},$$

where $0 \leqq x < \infty$. Clearly, the expression on the left of this identity represents a function of class E_λ. On the other hand, the expression on the right tends to e^{-ax} if $\lambda \to \infty$ and $b = \lambda/a$, where a is any positive constant. Accordingly, every function of the form e^{-ax} is a limit, as $\lambda \to \infty$, of functions contained in E_λ. Hence, the same is true of every function of the form

$$\sum_{k=1}^{m} c_k e^{-a_k x},$$

where c_k, a_k are arbitrary non-negative constants. Since the latter sum is identical with the case

$$\phi(x) = \sum_{a_k \leqq x} c_k$$

of the transform (7), the assertion of (6) now follows by a standard application of Helly's theorems on monotone functions.

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