# Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World

*Reviewed by Slava Gerovitch*

**Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World**
*Amir Alexander*
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From today's perspective, a mathematical technique that lacks rigor or leads to paradoxes is a contradiction in terms. It must be expelled from mathematics, lest it discredit the profession, sow chaos, and put a large stain on the shining surface of eternal truth. This is precisely what the leading mathematicians of the most learned Catholic order, the Jesuits, said about the "method of indivisibles", a dubious procedure of calculating areas and volumes by representing plane figures or solids as a composition of indivisible lines or planes, "infinitesimals". While the method often produced correct results, in some cases it led to spectacular failures generating glaring contradictions.

This kind of imprecise reasoning seems to undermine the very ideals of rationality and certainty often associated with mathematics. A rejection of infinitesimals might look like a natural step in the progress of mathematical thinking, from the chaos of imprecise analogies to the order of disciplined reasoning. Yet, as Amir Alexander argues in his fascinating book *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World*, it was precisely the champions of offensive infinitesimals who propelled mathematics forward, while the rational critics slowed the development of mathematical thought. Moreover, the debate over infinitesimals reflected a larger clash in European culture between religious dogma and intellectual pluralism and between the proponents of traditional order and the defenders of new liberties.

Known since antiquity, the concept of indivisibles gave rise to Zeno's paradoxes, including the famous "Achilles and the Tortoise" conundrum, and was subjected to scathing philosophical critique by Plato and Aristotle. Archimedes used the method of indivisibles with considerable success, but even he, once a desired volume was calculated, preferred proving the result with a respectable geometrical method of exhaustion. Infinitesimals were revived in the works of the Flemish mathematician Simon Stevin, the Englishman Thomas Harriot, and the Italians Bonaventura Cavalieri and Evangelista Torricelli in the late sixteenth to the early seventeenth century. The method of indivisibles was appealing not only because it helped solve difficult problems but also because it gave an insight into the structure of geometrical figures. Cavalieri showed, for example, that the area enclosed within an Archimedean spiral was equal to one-third of its enclosing circle because the indivisible lines comprising this area could be rearranged into a parabola. Torricelli, in order to demonstrate the power and flexibility of the new method, published a remark-

*Slava Gerovitch is a lecturer in history of mathematics at the Massachusetts Institute of Technology. His email address is* `slava@math.mit.edu`.



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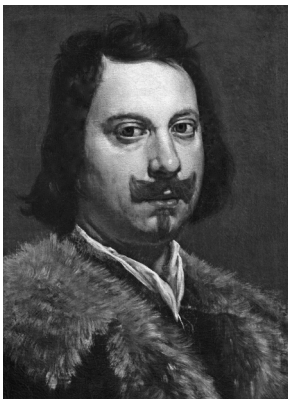
**Bonaventura Cavalieri (1598–1647).**



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**Evangelista Torricelli (1608–1647), by Lorenzo Lippi (circa 1647, Galleria Silvano Lodi & Due).**

able treatise with twenty-one different proofs of an already-known result (the area inside a parabola). In ten different proofs, Torricelli used indivisibles, producing effective explanatory arguments instead of cumbersome Euclidean constructions. He called the method of indivisibles "the Royal Road through the mathematical thicket," while the traditional Euclidean approach deserved "only pity" (p. 110). Due to their calculating power and explanatory appeal, infinitesimals quickly gained popularity until they faced stern opposition from the Jesuits.

The method of indivisibles had glaring flaws. Comparing the infinitesimals composing one figure to the infinitesimals composing another could produce different results, depending on the procedure used. For example, if one drew a diagonal in a rectangle with a greater horizontal side, it would split into two equal triangles, as in Figure 1 (below). Using the method of indivisibles, however, one could argue that each horizontal segment in the upper triangle was greater than the vertical segment drawn through the same point on the diagonal in the lower triangle, and therefore the two triangles differed in size. Torricelli found a way out of this conundrum by claiming that the indivisible lines of the lower triangle were "wider" than the lines of the upper and built a whole mathematical apparatus around the concept of indivisibles of different width. However ingenious, this explanation did not fly with the Jesuits.

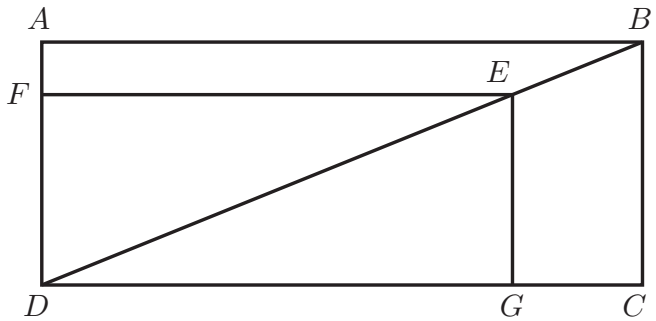


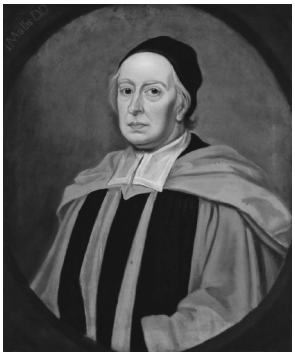

**Figure 1. The method of indivisibles can lead to contradictions. Here each horizontal segment in the upper triangle is longer than the corresponding vertical segment in the lower triangle, wrongly implying that the upper triangle has greater area.**

The Jesuits were largely responsible for raising the status of mathematics in Italy from a lowly discipline to a paragon of truth and a model for social and political order. The Gregorian reform of the calendar of 1582, widely accepted in Europe across the religious divide, had very favorable political ramifications for the Pope, and this project endeared mathematics to the hearts of Catholics. In an age of religious strife and political disputes, the Jesuits hailed mathematics in general, and Euclidean geometry in particular, as an exemplar of resolving arguments with unassailable certainty through clear definitions and careful logical reasoning. They lifted mathematics from its subservient role well below philosophy and theology in the medieval tree of knowledge and made it the centerpiece of their college curriculum as an indispensable tool for training the mind to think in an orderly and correct way.

*The Jesuits viewed the notion of infinitesimals as a dangerous idea.*

The new, enviable position of mathematics in the Jesuits' epistemological hierarchy came with a set of strings attached. Mathematics now had a new responsibility to publicly symbolize the ideals of certainty and order. Various dubious innovations, such as the method of indivisibles, with their inexplicable paradoxes, undermined this image. The Jesuits therefore viewed the notion of infinitesimals as a dangerous idea and wanted to expunge it from mathematics. In their view, infinitesimals not only tainted mathematics but also opened the door to subversive ideas in other areas, undermining the established social and political order. The Jesuits never aspired to mathematical originality. Their education was oriented toward an unquestioning study of established truths, and it discouraged open-ended intellectual explorations. In the first decades of the seventeenth century the Revisors General in Rome issued a series of injunctions against infinitesimals, forbidding their use in Jesuit colleges. Jesuit mathematicians called the indivisibles "hallucinations" and argued that "[t]hings that do not exist, nor could they exist, cannot be compared" (pp. 154, 159).

The champions of infinitesimals chose different strategies to deal with the Jesuit onslaught. In 1635 Cavalieri expounded the method of indivisibles in a heavy volume, filled with impenetrable prose, which even the best mathematicians of the day found hard to get through. He dismissed the paradoxes generated by his method with long and convoluted explanations aimed to intimidate more than persuade. Later on, when asked about the paradoxes, the defenders of infinitesimals often simply gestured toward Cavalieri's volumes, claiming that he had already resolved all of them. Neither the critics nor the support-



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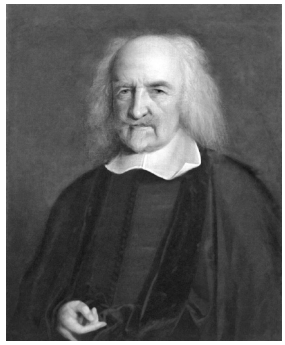
**John Wallis (1616–1703), after Sir Godfrey Kneller, Bt oil on canvas, feigned oval, (1701), NPG 578.**

ers of the indivisibles dared to penetrate Cavalieri's obscure fortress. Evangelista Torricelli, by contrast, found the paradoxes the most fascinating part of the topic and published several detailed lists of them, believing that a study of such paradoxes was the best way to understand the structure of the continuum. For him, the study of paradoxes was akin to an experiment, for it pushed a phenomenon to its extreme in order to reveal its true nature.

The dispute over infinitesimals was at the same time a dispute over the nature of mathematics. Do mathematical proofs have to show only the correctness of a theorem (as in Euclidean geometry) or to explain why it is true (as in the method of indivisibles)? Should one pursue a top-down approach, starting with universal first principles, put mathematical objects in order, and then impose this order on the world, or should one build mathematics from the bottom up, beginning with one's intuition about the world and moving up to higher and higher abstractions? The latter question pitted the Jesuits against Galileo, which led to his eventual condemnation and lifetime house arrest. Harsh administrative measures were also taken against the remaining stalwarts of infinitesimals, who lost their jobs and were forbidden to teach or publish. The Jesuits even went so far as to engineer the dissolution of a small monastic order, the Jesuats, which had sheltered Cavalieri and Stefano degli Angeli, the leading promoters of the method of indivisibles. The Jesuit mathematicians saw their mission in preserving the eternal truths of Euclidean geometry and in suppressing any threat of potentially disruptive innovation. This led, Alexander argues, to "the slow suffocation and ultimate death of a brilliant Italian mathematical tradition" (p. 165).

*...a clash between two visions of modernity.*

The battle over the method of indivisibles played out differently in England, where the Royal Society proved capable of sustaining an open intellectual debate. One of the most prominent critics of infinitesimals in England was philosopher and amateur mathematician Thomas Hobbes. A sworn enemy of the Catholic Church, he nevertheless shared with the Jesuits a fundamental commitment to hierarchical order in society. He believed that only a single-purpose organic unity of a nation, symbolized by the image of Leviathan, could save England from the chaos and strife sowed by the civil war. In the 1650s–70s his famously acrimonious dispute with John Wallis, the Savil-ian Professor of Geometry at Oxford and a leading propo-nent of the method of indivisibles, again pitted a champion of social order against an advocate of intellectual freedom.



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**Thomas Hobbes (1588–1679), by John Michael Wright, oil on canvas, circa 1669–1670, NPG 225.**

The Royal Society was initially suspicious of mathematics. Society fellows prized experimental science, public demonstrations, and open intellectual debate as a model for peaceful resolution of societal tensions. Mathematics, with its reputation as a solitary, private pursuit, its claims for incontrovertible truth, its reliance on obscure professional language, and its inaccessibility to laymen, seemed like a poor match for the liberal ideals of the society. Wallis, the only mathematician among the founders, took upon himself the task of reconciling mathematics with the spirit of the society's ideals. Claiming that "[m]athematical entities exist not in the imagination but in reality" (p. 263), he put forward a new, "experimental" mathematics. In contrast to the Euclidean approach of constructing geometrical objects from the first principles, Wallis assumed that geometrical figures already existed in the world. Modifying the method of indivisibles, he viewed a triangle as actually composed not of lines but of infinitely thin trapezoids, two-dimensional objects making up the original triangle, just like mountains formed by geological strata. Studying geometrical objects for him was akin to the work of a scientist probing geological formations. His method relied on induction, was open to discussion, and aimed to persuade the reader by examining a series of particular cases, much like the laboratory experiments that became the hallmark of the Royal Society's approach to studying nature. In the eyes of its fellows, this kind of mathematics was aligned with the society's epistemological ideals, and its legitimation paved the way for the later transformation of the method of infinitesimals into calculus in the hands of Isaac Newton.

Alexander persuasively argues that the fight over infinitesimals was a reflection of a more fundamental clash between what he calls two "visions of modernity." While the Jesuits and Hobbes embodied the desire to achieve a societal unity through the imposition of a single truth and suppression of debate, the champions of infinitesimals valued the freedom of discussion and investigation and a pluralism of opinions. Their opponents feared that intellectual pluralism might lead to political and religious pluralism and wanted to squash the seeds of instability before they produced full chaos. Following the Jesuits' purge of creative mathematicians, not only Italy's mathematical tradition declined but the country itself became unreceptive to innovation and began falling behind. In England, by contrast, the support of mathematical novelty by the Royal Society was part of greater openness in intellectual and social debates and resulted in rapid scientific and

technological development, leading up to the Industrial Revolution. The author implies that the different fate of infinitesimals in different countries shaped the fortunes of these nations in the long run.

Alexander clearly outlines a cultural split between political conservatives and "liberalizers" with respect to the method of indivisibles. His own discussion of Hobbes's early fascination with infinitesimals, however, somewhat challenges this overly neat separation. Despite his royalist and traditionalist convictions, Hobbes carefully read and absorbed Cavalieri's subversive mathematical treatises. Reinterpreting the indivisibles as material objects, he developed an unconventional geometry in which mathematical objects were generated by the motion of simpler objects—lines by motion of points, surfaces by motion of lines, and solids by motion of surfaces—before he turned against infinitesimals in his personal vendetta against Wallis. Well, good history of mathematics, like good mathematics, might occasionally benefit from a paradox or two.

*Innovation often grows out of outlandish ideas.*

In the 1960s, three hundred years after the Jesuits' ban, infinitesimals eventually earned a rightful place in mathematics by acquiring a rigorous foundation in Abraham Robinson's work on nonstandard analysis. They had played their most important role, however, back in the days when the method of indivisibles lacked rigor and was fraught with paradoxes. Perhaps it should not come as a surprise that today's mathematics also borrows extremely fruitful ideas from nonrigorous fields, such as supersymmetric quantum field theory and string theory.

Alexander's book meaningfully points to a fundamental tension between the popular image of mathematics as a collection of eternal truths which never changes and knows no debate and its actual practice, filled with uncertainty, frustration, failure, and rare glimpses of profound insight. If, as in the case of the Jesuits, maintaining the appearance of infallibility becomes more important than exploration of new ideas, mathematics loses its creative spirit and turns into a storage of theorems. Innovation often grows out of outlandish ideas, but to make them acceptable one needs a different cultural image of mathematics—not a perfectly polished pyramid of knowledge, but a freely growing tree with tangled branches.

### About the Author

Photo courtesy of Slava Gerovitch.

Slava Gerovitch teaches cultural history of mathematics at MIT. His research interests include history of twentieth-century mathematics, cybernetics, astronautics, and computing. His current project explores how the Soviet mathematics community creatively adapted to various political, institutional, and cultural pressures. He is the author of *From Newspeak to Cyberspeak: A History of Soviet Cybernetics, Voices of the Soviet Space Program,* and *Soviet Space Mythologies.* A poet and a translator, he has also published *Wordplay: A Book of Russian and English Poetry*.