Book Review

L. E. J. Brouwer—Topologist, Intuitionist, Philosopher: How Mathematics Is Rooted in Life

Reviewed by Dale M. Johnson

L. E. J. Brouwer—Topologist, Intuitionist, Philosopher: How Mathematics Is Rooted in Life
Dirk van Dalen
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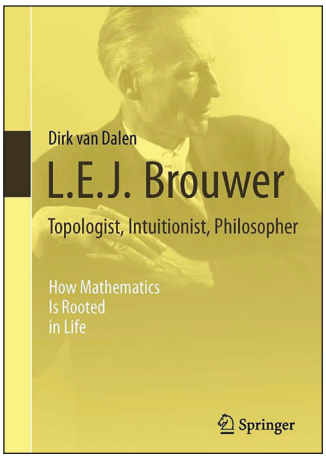
L. E. J. Brouwer (1881–1966) is well known to mathematicians primarily as a topologist, one of the two masters who around the beginning of the twentieth century shaped much of later topology; the other is Henri Poincaré (1854–1912). Every mathematician knows the Brouwer fixed point theorem, and most likely know a few other facts about him: that he was the first to prove the topological invariance of dimension and of domain; that he developed further the degree of a mapping as a topological tool; and that he offered a beautiful proof of the Jordan Curve Theorem and stated and proved its generalization, the Jordan-Brouwer Separation Theorem.

Brouwer is better known among mathematical logicians as a founder of a special brand of constructivist thinking in mathematics known as intuitionism. Philosophers may know the philosophical aspects of his mathematical foundational thinking associated with intuitionism. Hence, the subtitle of the book under review, *Topologist, Intuitionist, Philosopher,* is very apt. However, in terms of the *development* of Brouwer's intellectual life as revealed in this full-length biographical study, the order should probably be reversed: philosopher (and mystic to an extent), intuitionist, topologist.



Dirk van Dalen's book is a massive 875 pages; yet it is a revised and somewhat shortened version of his two-volume version published by Oxford University Press [10](xv + x + 946 pp.). Fortunately, the new version, though very large, is very reasonably priced. Van Dalen also published a collection of Brouwer's letters, *The Selected Correspondence of L. E. J. Brouwer* [11], which additionally offers a huge amount of archival material online (3,000+ pages) if one purchases the book. The Brouwer *Collected Works* was published in two volumes: one largely on intuitionism (1975) [2] and the other on mathematics and topology (1976) [3]. Thus in several extensive publications we have a very full picture of a great mathematician and seminal modern thinker.

Van Dalen's excellent biography is a very detailed examination of Brouwer's life; his connections with many other people, including the most famous mathematicians of his time; and his intellectual development and activity from his

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earliest years to his final days. Brouwer himself was a "saver"—he knew that he had created some important work and sought to preserve it. He saved almost every paper and letter he ever touched, and not just mathematical papers and technical notes. He had the good fortune of an unusual companion, Cor Jongejan, as well as his wife, Lize, to look after his letters and intellectual property. Some fires in his archive may have destroyed certain material, but much has been preserved. Van Dalen, I believe, has studied virtually every piece of material related to Brouwer and all the extant letters to put together his very detailed biography. He took on a massive task to chronicle the life of a giant intellect.

Luitzen Egbertus Jan (Bertus, as he was called) Brouwer was born on February 27, 1881, in Overschie, The Netherlands. At the early age of sixteen Bertus enrolled in the University of Amsterdam (also called the Municipal University). It was the academic institution with which he was associated and at which he taught and did research for the rest of his life, though he had several offers at various times to become a professor elsewhere.

As revealed in the first chapters of the book, Brouwer had an early interest in philosophy and a kind of romantic mysticism. His first book, which was published in 1905, is entitled *Leven, Kunst en Mystiek* (*Life, Art, and Mysticism*) [1]. His philosophy tended toward solipsism, and many aspects of his character tended toward the introspective and the interior life. However, in many other ways he was outgoing. Brouwer had many friendships with important mathematicians and other intellectuals. In his youth Brouwer had a close friendship with Adama van Scheltiema (1877–1924), who was a poet and more concerned with the artistic side of life. Throughout his life Brouwer maintained an interest in philosophy and certain philosophical movements, such as the Signific Circle (Signifische Kring), a group of Dutch thinkers interested in philosophy of language and social issues, of which Brouwer was a founding member.

Brouwer's doctoral dissertation represents a highly significant stage in his intellectual development. The dissertation, "Over de grondslagen der wiskunde" (On the foundations of mathematics), completed and defended in 1907, covers several topics, such as the foundational issues of set theory and logic, intuitionism, and mathematical domains, as well as philosophy (in [2], pp. 11–104). Brouwer's thesis advisor was D. J. Korteweg (1848–1941). Korteweg mentored the independent-minded Brouwer and had a significant impact on the thesis. He had Brouwer remove some of the more philosophical parts (see van Stigt [13]). Korteweg was highly influential in Brouwer's career. He campaigned for Brouwer's appointment at Amsterdam and later proposed a plan to enable Brouwer to obtain a full (ordinarius) professorship at Amsterdam. In effect, Korteweg exchanged his ordinary professorship with Brouwer's extraordinary professorship. Korteweg was very discerning in his judgment of Brouwer's mathematical talents. At that time (as now) it was difficult to obtain a good academic position. Many young mathematicians taught in high schools before becoming university lecturers and professors. Brouwer did not take that route.

Brouwer thought of the primary activity of mathematics as an internal mental construction of mathematical objects. The presentation of mathematics in language was very much secondary; logic (one might say) was a distant third. In the dissertation of 1907, as a critique of Hilbert's early work on the foundations of mathematics, Brouwer distinguished eight levels of mathematical treatment, the first three of which are ([2], pp. 94–95) (as translated in the current volume under review, pp. 106–7):

1. The pure construction of intuitive mathematical systems, which, if applied, we externalize by viewing the world mathematically.
2. The language parallel of mathematics: mathematical speaking or writing.
3. The mathematical consideration of the language: logical language construction….

There is a meta-level on top of a meta-level in the eight levels, so to speak.

In his earliest mathematical work Brouwer took up problems such as developing Lie group theory independent of analytical machinery (Hilbert's Fifth Problem in the celebrated list of challenging problems of 1900) and Cantor-style point-set theoretic topology. He studied closely the work of Arthur Schoenflies (1853–1928) in point-set topology and found the work defective; hence, he wrote a highly important critique, "Zur Analysis Situs" (published in 1910) ([3], pp. 352–370), filled with counterexamples to Schoenflies's results. In that paper Brouwer introduced indecomposable continua. At first he and Schoenflies argued about the results, but subsequently they also became lifelong friends. In the mathematical realm Schoenflies was no match for Brouwer, whose incisive topological thinking yielded refutations of many of Schoenflies's results.

On the eve of his greatest discoveries and results in topology, Brouwer presented on October 12, 1909, his inaugural address as a new *privaat docent* at the university, "Het wezen der meetkunde" (The essence of geometry) ([2], pp. 112–120). In this short talk he enumerated some of the hard problems of topology, many of which he went on to solve by proving new theorems. I think that it is important to see Brouwer as a problem solver in mathematics, not as a formal

developer of mathematics. He went after the difficult problems and in many cases solved them. He was not interested so much in the cleanup afterwards into neat theories. He was rigorous but not encyclopedic.

Van Dalen does a very able job of telling the story of Brouwer's topological development in chapters 4 and 5. Brouwer published a veritable flood of fundamental topological results in the years 1909–1913. The First World War, 1914–1918, was a break point in his mathematical career.

From the beginning of his career Brouwer interacted with the greatest mathematicians at conferences, in letters, and in other encounters. David Hilbert (1862–1943) noticed his work very early on, and as a result they corresponded. Other mathematicians with whom he became acquainted early in his career included Jacques Hadamard (1865–1963), Henri Lebesgue (1875–1941), Henri Poincaré, Paul Koebe (1882–1945), Otto Blumenthal (1876–1944), Hermann Weyl (1885–1955), and many others. With some of these mathematicians he had vigorous disputes both on mathematical and on cultural and political grounds. By analyzing the many related letters and other documents, van Dalen carefully lays out these disputes in great detail. Some disputes relate to the aftermath of the First World War, when German mathematicians were largely excluded from the international community because of the widely held perception that the Germans were the main cause and offenders in the war.

Brouwer's intuitionism is famous or perhaps infamous. His rejection of the universal validity of the law of excluded middle is startling to many mathematicians, since it fundamentally calls into question many important proofs in mathematics. His constructivism does not permit commonly accepted existence proofs. Even Brouwer's own proofs in topology do not remain valid under these strictures. He recognized that fact and in the 1920s sought to modify some of his earlier proofs to make them intuitionistically acceptable.

In the years just after the First World War, Brouwer found an ally in intuitionism, Hermann Weyl, who developed his own approach to foundations and then strongly supported Brouwer's foundational position: "… and Brouwer—that is the revolution." In developing his own foundational and constructivist thinking, Weyl wrote his classic monograph *Das Kontinuum* (The Continuum) (1918) [15], as well as other papers (in [16]).

Brouwer's brand of intuitionistic mathematics developed over the years. There are several formulations. Van Dalen does an excellent job in telling this story of mathematical development, including also the human interactions (contentious at times) with other mathematicians (on constructivism see also [9], [12]). Brouwer's introduction of choice sequences in his constructivism was an important step.

In the 1920s the most celebrated dispute was between Brouwer and Hilbert over foundations of mathematics: Brouwer's intuitionist program versus Hilbert's formalism and metamathematical program. Some papers of the era, especially those by Hilbert, are highly polemical. A strange outcome of the dispute was the elimination of Brouwer from the editorial board of *Mathematische Annalen*, a premier journal of the day. Brouwer worked hard as an editor to bring papers for publication to the highest standard, and he did not impose his intuitionism on them. Yet Hilbert saw fit to have him removed. The reasons are not entirely clear. The story is complicated, so one needs to read carefully the analysis that van Dalen provides. His analysis seems deeper than that of Constance Reid [8], given all the documents that are now available.

Another highly significant dispute was over priority in dimension theory, not just proving the topological invariance of dimension. The dispute was complicated by the fact that Brouwer made a "slip of the pen" (*Schreibfehler*) in his great paper on the definition of dimension. Brouwer interacted with Paul (Pavel) Urysohn (1898–1924) on the matter, and they came to agree on a common position. However, Brouwer ended up with a considerable difference of opinion on priority with the then very young and new mathematician Karl Menger (1902–1985). Van Dalen relates the details in the volume under review (pp. 595–601); in the earlier version of his biography he gives greater space to this unpleasant dispute ([10], pp. 643–671).

Brouwer often was deeply involved with academic matters and pure politics at Amsterdam. One can see the pettiness of some of these things in van Dalen's careful treatment of them. No doubt such things still happen in academia.

Brouwer was often a loner. He conducted his life and research mostly at his hut (house) in Blaricum, outside Amsterdam. Many mathematicians went there to discuss mathematics, and the discussions were very fruitful. He seems to have had a condescending view of the world at times. One can imagine that he was just plain difficult to deal with on many issues. He preferred being right to being forgiving.

Regardless of his personality one cannot deny Brouwer's great achievements in topology, in intuitionism, and in foundational thinking. He was truly a giant of twentieth-century mathematics. Van Dalen really does a great service to the memory of such an intellect. The reader may become overwhelmed by all the details that van Dalen covers, but one is free to be selective in reading parts of the book according to one's interests.

By surveying recent general interest mathematical literature, one realizes that several full-length biographies have been written about important

About the cover

From L. E. J. Brouwer's "Zur Analysis Situs"

The cover image was suggested by the review in this issue of a biography of the Dutch mathematician L. E. J. Brouwer. The original diagram was one of two inserted as separate sheets into Brouwer's article "Zur Analysis Situs" in the 1910 volume of the *Mathematischen Annalen*. As far as we can tell, it was the first colored diagram to be published in any mathematical research journal, and is apparently the only colored diagram ever to have been published in the *Annalen*.

Brouwer is probably best known for his strong stance on constructive proofs, but of course he is also well known for his theorem on fixed-points. He was in fact one of the founders of modern topology. "Zur Analysis Situs" was his response to some prior work of the mathematician Schoenflies, who was at that time considered the world expert on the topology of point sets in the plane. But Brouwer had found Schoenflies's work to contain serious errors. Basically, what he pointed out was that Schoenflies had not been rigorous enough, and that the topological structure of point sets in the plane was often far from intuitive.

The figure on the cover served in his paper to illustrate several of his objections. It exhibited among other things Brouwer's construction of the first known *indecomposable continuum*—that is to say, a compact, connected subset of the plane that could not be expressed as the union of two proper closed connected subsets. This set is constructed as the intersection of a sequence of annuli. Each one is obtained from the previous one by removing two disjoint sets from the previous one. In the cover image, the construction is at the third stage. The annulus is the light region, the interior of the annulus is the region hatched in red, and its exterior is hatched in black. On the cover, Brouwer's diagram has been extended to include a larger background area hatched in black.

We find, as Schoenflies also found, Brouwer's explanation of his figure a bit hard to follow. His results were soon incorporated into the literature of topology, but rather different and simpler diagrams replaced his. The clearest explanation of Brouwer's diagram itself that we have found is the survey "A brief historical view of continuum theory" in the 2006 volume of *Topology and its Applications*, by W. T. Ingram. He interprets the intersection of the sequence of annuli as an

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mathematicians of the late nineteenth and early twentieth centuries, including ones for Georg Cantor [4]; Henri Poincaré [7], [14]; David Hilbert [8]; Ernst Zermelo [5]; and Constantin Carathéodory [6]. Van Dalen's biography is certainly one of the most extensive. The author must have spent countless hours collecting and analyzing the many extant documents related to the famous Dutchman. In this book we have a detailed and valuable picture of a great mathematician.

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