# The Shape of Inner Space

*Reviewed by Nigel Hitchin*

**The Shape of Inner Space**
*Shing-Tung Yau and Steve Nadis*
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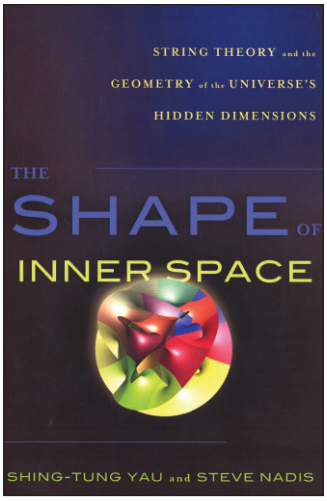
"The End of Geometry": a curious title for the final chapter of a book whose first author is perhaps the best-known differential geometer in the world. And this from someone who confesses that geometry is "the field closest to nature and therefore closest to answering the kinds of questions I care most about.'' Yau's concluding theme concerns the challenge to find the mathematical language to describe both general relativity (which differential geometry has done very successfully) and quantum physics (whose language in recent years has penetrated algebraic geometry). It is string theory and its practitioners that have brought these new ideas into geometry, it is they who use and christened *Calabi-Yau* manifolds, and it is they who are demanding from us a new geometry to suit their needs. The book aims to tell us how this came about via two themes: one is a personal story about Yau himself and his Fields Medal-winning proof of the Calabi conjecture, the other a description for the lay reader of string theory and the role of Calabi-Yau manifolds in its attempt to describe the universe.

First the Calabi conjecture. The version most relevant here concerns objects that had been of interest to algebraic geometers for a long time—varieties with trivial canonical bundle. These are higher-dimensional analogues of elliptic curves. In two dimensions they are the so-called K3 surfaces. Yau's theorem asserts the existence of a special kind of Riemannian metric on these—a Kähler metric with zero Ricci tensor, equivalently a Riemannian solution to Einstein's vacuum equations. Until the theorem in 1977, there were no




nontrivial compact examples, but the proof, together with algebraic geometric constructions, generated many. Not only that, but the metric, in particular for K3 surfaces, provided a tool for understanding the algebraic geometry of moduli problems. It gave an analytical object that could somehow interpolate between algebraic ones—just as it helps to see rational numbers inside the reals, one can use the Calabi-Yau metric to pass from one algebraic surface to another.

The path to proving the theorem was not easy, and Yau's description of his successes and setbacks on the way makes for interesting reading and is instructive for young mathematicians to see. His first attempt at a counterexample, presented to an eminent audience, failed. By examining how it failed, he convinced himself the conjecture must be true and then worked hard on it, talking to people, learning new techniques, until finally in a Christmas Day meeting in 1976 with Calabi and Nirenberg, his proof seemed to hold up. "I may have my shortcomings," he confesses, "but no one has ever accused me of being lazy!" The second author, a seasoned science writer, gives a smooth, literate account here, but on occasion one can detect the authentic Yau voice breaking through.

Then comes string theory, historically described in a fashion that is recognizable to readers of, for example, Brian Greene's books. Around 1984 the physicists Michael Green and John Schwarz found that a consistent supersymmetric string theory required six extra real dimensions that had a Kähler metric with

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zero Ricci tensor. It seems that they were erroneously told that no higher-dimensional analogues of K3 surfaces existed, but soon they found out about Yau's theorem and before long the objects acquired the name of Calabi-Yau manifolds. The theorem is an existence theorem and doesn't provide any detailed information about the metric that the physicists might need, so what they actually had from it was a number of examples of the underlying spaces. In fact, too many, because quite soon, in cooperation with algebraic geometers, they were able to generate thousands and thousands of examples. Somewhat difficult to choose from if you want a model of the universe.

But a little later Calabi-Yau manifolds took on a different role, by means of a link with conformal field theories. The symmetries of those theories provided a new viewpoint on Calabi-Yau geometry, and out of it came the phenomenon of mirror symmetry—it seemed as if these manifolds all came in pairs. The thousands of examples provided a good testing ground for this, and in the end, as the authors remark, this symmetry would "revitalize Calabi-Yau manifolds and rejuvenate a somnolent branch of geometry". Those mathematicians working in enumerative algebraic geometry might resent the use of this adjective, but it is true that mirror symmetry awakened them to a new dawn in this area, presenting new problems and new methods. Enumerative geometry concerns itself with counting objects within a bigger one—curves, sheaves, etc.—and the physicist's partition function turns out to be a generating function for some of these. The symmetries of the conformal field theory suggested properties of the manifolds that were to a large extent invisible in the more traditional approach. One steps back and looks at things in a much wider context and in particular finds a systematic way of counting degenerate objects. Indeed, the extra structure that is observed emanates from these degenerations—in fact, it is still not clear whether mirror symmetry applies to a single Calabi-Yau or to a degenerating family. In any case, this symmetry and its ramifications is one of the most profound ways in which physical ideas have been absorbed into pure mathematics.

These, and related issues, are described in the book quite effectively, and it becomes a rather different story from Yau's personal achievement, in his dogged pursuit of the big theorem. Throughout the book attempts are made to describe in layperson's terms the mathematical and physical concepts, but I suspect that many of these will only make sense to a practicing mathematician despite the multiple diagrams and two-dimensional analogies. On the other hand, the lay reader may well enjoy being presented with some views on what geometry is, how it links up with other areas of mathematics, and how it provides a language for expressing some of the key concepts in physics. The reader may also find intriguing the authors' view on the motivation of mathematicians when they attack a problem and how they feel on solving it. Drawing on the analogy of mountain climbing, we are told, for example, that "at the end of a long proof, the scholar does not plant a flag." (I wonder if the reader would be so naive as to believe that.) What is slightly irritating, and seems to be a common method in science writing, is the multiplicity of sound bites from eminent scholars: "… as X of Harvard explains …" In an area in which either the mathematics is very abstract or the physics untested, falling back on the comments of the great and good for justification is a poor substitute for real evidence.

So what really was the impact of Yau's theorem in string theory? It established the existence of Ricci-flat Kähler metrics, but, as we read in the book, this may not actually be what the physicists wanted. Making the beta function vanish involves an infinite number of adjustments to the metric, moving away from the Ricci-flat condition; and non-Kähler Calabi-Yaus seem to be needed to complete the moduli space picture. It seems that the main concrete outcome of the theorem was to open the door for the physicists to invade algebraic geometry, to provide new questions, new tools, and new ways of organizing information.

In his earlier years Einstein commented that "since the mathematicians have invaded the theory of relativity, I do not understand it myself." It may be that some of us geometers have the same feeling about the incursion of ideas from physics today, but it is a fact that is not going to change. So how do we now take that final chapter—is the death of geometry real or simply greatly exaggerated? "Geometry as we know it will undoubtedly come to an end," say the authors, but it seems more likely a statement about string theory. For, despite Yau's energetic pursuit of string-related mathematical problems, despite his important facilitation of interactions between physicists and mathematicians, and despite his encouragement of students and postdocs on these problems, his *oeuvre* contains highly influential results in many other branches of geometry. In the book he steps aside ("one of the luxuries of being a mathematician") from the controversial discussions of the landscape or of multi-universes. Another such luxury is to accept the mathematical challenges of problems from whatever source, so long as they intrigue us, and I suspect the first author will continue to do this.