Book Review

Strange Curves, Counting Rabbits, and Other Mathematical Explorations

Reviewed by Harold R. Parks

Strange Curves, Counting Rabbits, and Other Mathematical Explorations
Keith Ball
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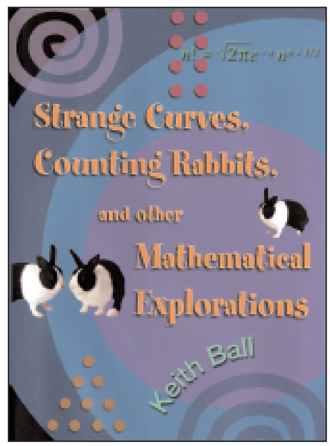
According to its preface, the genesis of this book was a lecture given to the math club at a school where a friend of the author was a teacher. That lecture and subsequent lectures on "recreational" or "popular" mathematics in schools and to school students, together with a few other topics, formed the basis for the book. The schools mentioned above were in the United Kingdom, not the United States, so the "math club" was a "maths club," and if you have formed an impression of the level of this book, you need raise it substantially: The author's lectures would probably be appropriate for the math club at a college or university in the United States and would be way over the top for any U.S. high school students other than the most advanced, talented, and enthusiastic. Even though the Library of Congress classification of this book is "QA93 Popular works," a sound knowledge of calculus is a prerequisite for many parts of the book.

The "strange curves" of the title are space-filling curves (or Peano curves), the "counting rabbits" of the title refers to the Fibonacci sequence, and the "other mathematical explorations" cover quite a bit of ground. Let us now list the major topics discussed in the book: Hamming codes, Shannon's theorem, Pick's theorem, Fermat's Little Theorem, space-filling curves, the probability of shared birthdays, the normal approximation to the binomial distribution, Stirling's formula, coin-weighing problems, testing of pooled blood samples, Fibonacci and Lucas numbers, partial fractions, Padé approximation, and the irrationality of e and π. A number of other topics are used as motivation, mentioned in passing, used as illustrations, and so on.



The author's writing style is informal, inviting, and clear. Some details of some arguments are stated as problems, but the solutions are provided. Not all problems are there to fill in gaps; some are there to give the reader a little more to think about (solutions are provided for those also). Every chapter has some problems stated for the reader to work on, as few as three, and as many as eight, but usually five. As in all human endeavors, there are some errors; as an existence proof, I note that the word "number" on page 194, line 10, should not be there. The errors are remarkably few, and the author is to be congratulated on a very careful job.

The first chapter, "Shannon's Free Lunch," is about codes. There are essentially no prerequisites until just before the end. The author begins with a discussion of the check-digit in the ISBN code. The idea of having check-digits is well known among mathematicians, but there are a lot of nonmathematicians who have never heard of such a thing. Everything in this chapter would be news to them. The first ISBN example is self-referential: The example is 0-691-11321-1, the ISBN of *Strange Curves, Counting Rabbits, and Other Mathematical*

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Explorations itself. That seems a tricky thing to arrange; I was very surprised by that example.

The issue in this chapter is the transmission of data on a noisy channel. The best possible rate of transmission on a noisy channel was characterized by Claude Shannon in 1948. The author refers to this characterization as “Shannon’s free lunch”. I don’t quite agree with the choice of metaphor. Shannon’s theorem tells us that there is necessarily a trade-off: If your channel gets noisier, your maximum rate of transmission gets slower. To me a free lunch would be no slowing in transmission rate. The first time Shannon’s theorem appears, it is not really stated, as “stated” is understood by a mathematician. Presumably, the first chapter is meant to be accessible, so a precise statement would be too off-putting. Later, the author does give a precise statement of Shannon’s theorem.

The second chapter, “Counting Dots,” is about Pick’s theorem. This is also a very accessible chapter. Pick’s theorem tells us that, for a simple polygon all the vertices of which are integer lattice points, the area of the polygonal region enclosed by the polygon is $I + B/2 - 1$, where I is the number of lattice points in the interior of the polygonal region and B is the number of lattice points lying on the boundary of the polygonal region, i.e., on the polygon itself. The result dates to 1899 and, we are told, is a staple of recreational mathematics. It was, in fact, the subject of that first lecture the author gave to school students, the lecture that ultimately led to this book being written. As an application, the author uses Pick’s theorem to give an elegant proof of the fact that, for relatively prime p and q, there exist integers a and b so that $ap - bq = 1$. That Pick’s theorem cannot hold in three dimensions is shown by a tetrahedral counterexample. The author tells us that there are some useful things that can be said about lattice points in polyhedra and that there is some current research on this topic, but, regrettably, he gives us no entrée to this current research.

The third chapter, “Fermat’s Little Theorem and Infinite Decimals,” begins with a look at decimal expansions of rational numbers. As we know, such decimal expansions either terminate or are infinitely recurring. The question one might not have thought about is “What is the period of recurrence in such a decimal expansion?” The most interesting cases will be those in which the denominator is prime, so we are led to consider the decimal expansion of k/p where p is prime and $1 \leq k \leq p - 1$. Since 2 and 5 are divisors of 10, they are special and, for these purposes, anomalous. But the other primes are seen, by example, to fit a nice pattern: For a given prime p, other than 2 or 5, the decimal expansion of k/p has the same period, independent of k, and the number of different types of decimal expansions is $p - 1$ divided by that period. With a conjecture in hand, the author leads the reader through a proof. In the course of proving the preceding facts it emerges that, for primes other than 2 or 5, $10^{p-1} - 1$ is divisible by p. Consequently, for all primes p, $10^p - 10$ is divisible by p. Since 10 is not particularly special, the reader has been led to Fermat’s Little Theorem: If p is a prime and a is an integer, then a^p is congruent to a modulo p.

The fourth chapter, “Strange Curves,” is about space-filling curves. The author constructs examples of such curves using geometric recursion illustrated with nice figures: He starts with a basic pattern in the unit square and then proceeds to subdivide the square and put a scaled, and possibly rotated, version of the basic pattern in each subsquare. The actual curve is the limit of appropriate parametrizations of those recursively constructed curves. Since the author wants to assume that the reader does not know the basic theorems about uniform convergence of continuous functions, there is a considerable bit of hand-waving. A credible job is done, and it might be a good refresher for a student who has taken advanced calculus but who has not thought about it lately. The first example of a space-filling curve was constructed by Peano in 1890. Peano’s paper, [PG], has no figures. It is a nice feature that the author shows us Peano’s original construction in a pictorial form (attributed to Moore and Schönflies).

The next chapter, “Shared Birthdays, Normal Bells,” begins a series of three chapters relating to probability theory. The chapter starts with the shared birthday problem. The question is “Given a group of n people, what is the probability that at least two have the same birthday, i.e., celebrate their birthdays on the same day of the year?” To solve this problem sensibly one must shift to finding the probability of the complement. At this point the reader must know about logarithms and how to use calculus to estimate them. The average person is now out of the readership.

The theme of the chapter then shifts a bit to coin-tossing, the normal approximation to the binomial distribution (with equal probabilities of success and failure), and the Central Limit Theorem. Highlights are a wonderful figure showing the annual rainfall at Kew Gardens, in London, for the years 1697–1987 and a method for computing the Gaussian integral that was new to me. No attempt is made to prove the Central Limit Theorem, but a convincing case is made for the normal distribution providing a good approximation to the binomial distribution.

Now that the gauntlet of calculus has been thrown down, the chapter “Stirling Works” tackles a derivation of Stirling’s formula. The arguments feel natural and are very nicely presented. They might be heavy going for the popular reader, even the popular reader who did well in calculus.

In chapter 7, "Spare Change, Pools of Blood," the level of mathematical sophistication eases up a bit. The main point of the chapter is solving a practical medical problem: Given a blood test for a relatively rare condition, could one test a group of patients more efficiently by applying the test to pooled samples of blood? By pooling the blood to be tested, there is the chance of ruling out, with just one test, the presence of the condition in the entire group of patients whose blood was in that pooled sample. Such a blood test was, in fact, developed by the author's brother-in-law, and that test is sensitive enough to detect the condition in a pooled sample from 100 patients.

Before addressing the medical problem, the author explores the coin-weighing problem often seen in recreational mathematics, since the ideas from solving the coin-weighing problem turn out to be relevant to the pooled-blood-sample problem: The coin-weighing problem here is to determine the minimum number of weighings needed to find a coin of greater weight in a group of other coins all of which have the same weight. The solution of the coin-weighing problem very nicely motivates the "binary protocol" for the blood testing: First test a pooled sample from all the patients. If that test is negative, you are done. If the test is positive, form two equally sized groups and test each of those pooled samples. Any negative test tells you that that group does not have the condition. Any positive test tells you to divide the group into halves and test the halves. A slight refinement of the analysis shows that one ought to divide the group to be tested into subgroups of size $1/p$, where p is the probability of the condition, before running any tests. With this improvement, the author shows that the expected number of tests required for N patients is $Np[1 + \log_2(1/p)]$, and he shows that, even if the preceding value could be improved by using a different protocol, it cannot be improved by any more than a factor of two. For simplicity, the issue of false test results is not addressed. Subsequent chapters do not rely on any topics from this chapter.

Chapter 8, "Fibonacci's Rabbits Revisited," is about the Fibonacci numbers, no surprise there, and the Lucas numbers. Of course, we have all read some things about the Fibonacci numbers. Typically there is a heavy dose of examples from nature. Nature is fine, but in this chapter the author gives us some meatier mathematics of the Fibonacci numbers. For instance, we see in what sense the Fibonacci ratios (i.e., the ratios of successive Fibonacci numbers) are best approximations to the Golden Ratio. Another gem is the discussion of the fact that, for a prime number p, the pth Lucas number minus 1 is divisible by p, a fact that turns out to be an analogue of Fermat's Little Theorem. In this chapter, the author introduces continued fractions and matrices, tools that will continue to be used as we approach the proofs of irrationality in the last chapter.

The penultimate chapter, "Chasing the Curve," has nothing to do with the earlier strange curves in chapter 4. The topic here is, rather, Padé approximation. While a Taylor approximation is a polynomial chosen so that it and its first few derivatives agree with the given function at the given point, a Padé approximation is a rational function chosen so that it and its first few derivatives agree with the given function at the given point. Now, one thing the Taylor approximation has going for it is that finding the next one is not that much more work. At first glance, that seems not to be the case with Padé approximation, but, for certain examples, when one thinks in terms of continued fractions, the situation looks better. The continued fraction for $\tan x$ is developed in some detail. For the exponential function and for the arctangent function, the treatment is lightly sketched. These continued fractions are seen again in the next chapter.

The final chapter, "Rational and Irrational," is devoted to proofs of irrationality. It is a bit ironic, after all the reader has been through up to this point, to see the classical Pythagorean proof of the irrationality of $\sqrt{2}$ given in full detail. But that is a mere moment. The author's main method for showing the irrationality of a given number α is to show that if there is a sequence of rational numbers $p_n/q_n \neq \alpha$ which converges too rapidly to α, then α is irrational. For instance, if p_n/q_n converges to α, never equals α, and $(p_n/q_n - \alpha)q_n \to 0$, then α is irrational. To see this, one argues by contradiction. Supposing $\alpha = P/Q$, then we see that

$$(p_n/q_n - P/Q)q_n = \frac{p_nQ - q_nP}{Q}$$

must converge to 0 but never equal 0. Since the numerator is an integer and the denominator is fixed, we have a contradiction. The author applies the preceding argument to show that e is irrational. A generalization of that approach, combined with the continued fraction expansion for $\tan x$, is used to show that π is irrational.

In summary, this book gives a lively and carefully written treatment of a number of interesting topics. The material should be fully accessible to mathematics majors. Those who have taken calculus (and remember some of it) should be able to follow the ideas, if not all the details. The range of topics is wide, so even the experienced mathematician may learn something new.

References

[PG] Sur une courbe, qui remplit toute une aire plane, *Mathematische Annalen* **36** (1890), 157–160.