The World According to Wavelets: The Story of a Mathematical Technique in the Making

Reviewed by David Jerison

The World According to Wavelets:
The Story of a Mathematical Technique in the Making
Barbara Burke Hubbard
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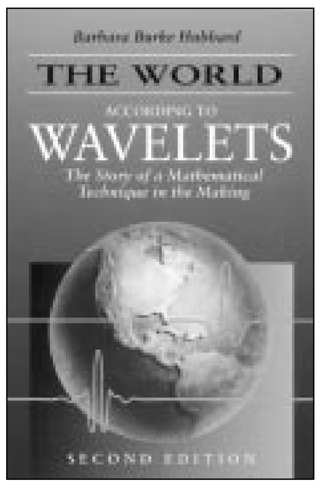
"There exists, certainly, a limit to what someone without the right background can understand of mathematics, but I am convinced that we are far from having reached that limit." With that declaration Barbara Burke Hubbard launches an ambitious history of wavelets aimed simultaneously at a popular and a technically literate audience. The first incarnation of this project was an article for a book on the frontiers of science written on commission from the National Academy Press. Thus the original purpose of the project, like that of the National Academy of Sciences, was to explain science and technology to policymakers. Since the book tells a compelling story of a mathematical innovation that is having a direct impact on the technology driving the Internet, one hopes that politicians will pay attention.

Wavelets are building blocks of function spaces that are more localized than Fourier series and integrals. The case for the utility of wavelets is even stronger now than when the second edition of the book was published in 1998. The single most impressive achievement of wavelets is their incorporation in engineering standards for image compression. At high ratios (greater than 20:1 for gray scale) images computed using the standard compression method (a discrete cosine transform and approximation of the Fourier coefficients) begin to look bad. The iconic image of Lenna, who graces three sets of pictures in the text, shows just how much better wavelets can preserve features that the eye cares about.



The first standard based on wavelets is "wavelet scalar quantization" (WSQ), adopted by the FBI in 1997 to encode fingerprints. The main difference between the first and second editions of the book is that the second edition finishes the story of WSQ and begins the story of new applications that surfaced after the first edition appeared. Soon there will be two much more widely applicable standards that were mentioned in the text but whose exact reliance on wavelets was not clearly established in time for publication: The next still-image compression standard known as JPEG2000 includes a wavelet option, and MPEG-4, the next video compression standard, will be entirely wavelet-based. In short, wavelets will play a role

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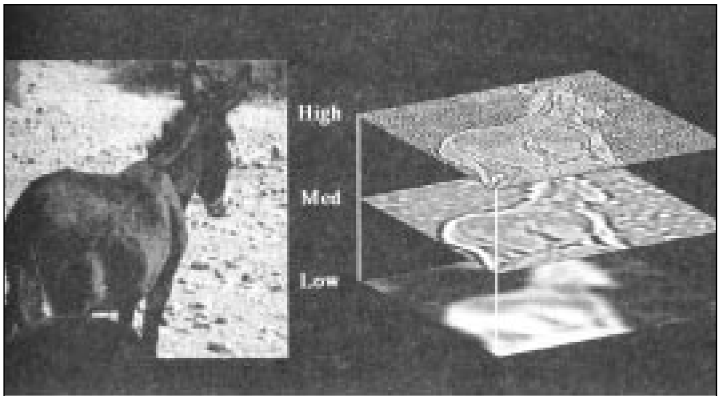

Figure 2. A "natural scene" and its decomposition into high, middle, and low frequencies, using a two-dimensional wavelet transform. The effectiveness of a transform cannot be understood or evaluated in isolation; it depends on the relationship between the transform and the properties of the data to be encoded. Natural scenes have certain statistical properties in common. For example, as shown in the above decomposition, they are redundant: many edges found in low frequencies also exist in middle or high frequencies. Wavelets with a certain narrow bandwidth of frequencies and certain narrow range of orientations can encode natural scenes concisely, with just a few coefficients. (Courtesy of David Field.)

in nearly all of our communications of pictures for the foreseeable future.

Wavelets are also being used in various aspects of signal processing. Modems developed by Aware use wavelets as the basis for an orthogonal decomposition of the bandwidth of electrical signals over a telephone line. In this way the usable bandwidth is increased by a factor of 250 from 4KHz to 1MHz.

Besides their uses in image and signal processing, wavelets have influenced a large number of pure and applied mathematicians in such disparate fields as numerical analysis, computer vision, human vision, turbulence, and statistics—all of which are mentioned in the text. Happily, the book does not exaggerate the extent to which wavelets are central, as opposed to merely helpful, to these applications. Over and over Hubbard quotes scientists explaining the limitations of wavelets.

Some researchers have the romantic notion that wavelets and their many hybrids can act as filters that capture meaning in a way that resembles what humans do. For example, musicologists worked with Ronald Coifman to restore an 1889 recording of Brahms playing his own composition. David Field has incorporated wavelets in his theoretical models of human vision. According to Field, in "natural scenes" the variation of intensity is similar from close up and far away. This means that the image belongs to a function space in which each dyadic block of frequencies has roughly the same energy. My favorite figure in the text is Field's picture of a mule and its wavelet analysis at different levels of resolution (see above). It is at least plausible that the brain uses a similar mechanism to understand the scene. At any rate, one gets to see in vivid detail a decomposition of a function in this dyadic space. Another figure, provided by François Meyer, illustrates Ronald Coifman and Yves Meyer's "brushlets". (See the color version on the cover of the April 1998 *Notices*.) Using a clever choice of basis elements, brushlets decompose a photograph of an ape into a work of art in three different styles. One wonders whether some painters are equipped to see the world through one of these lenses.

The idea of wavelets came from attempts to improve the accuracy of analysis of seismic data used in oil prospecting. The windowed Fourier transform, in which the Fourier transform is truncated by a cut-off function, takes such analysis quite far, but in 1975 Jean Morlet, a geophysicist working for Elf-Aquitaine, wanted to do better. He had the idea of using a fixed wave shape to generate a family of test functions using dilations and translations. In this approach, unlike the windowed Fourier transform, the number of oscillations of each test function is the same. Translations and dilations form a two-parameter family, making the transform highly overdetermined. Morlet began an extensive collaboration with theoretical physicist Alex Grossmann, and in 1984 Grossman proved that with nearly any wave shape they could recover the signal exactly from their transform.[1] But in practice they wanted to recover the signal using a discrete sampling. In the simplest case, a single square wave (+1 on the left half of the unit interval and -1 on the right) translated by integers and dilated by powers of 2 generates a complete orthogonal system, the Haar basis. The problem was to find wavelets with more smoothness than the discontinuous square wave.

Yves Meyer heard about wavelets from a colleague in physics at the École Polytechnique. He promptly took a train to Marseille to talk to Grossmann. In 1985 Meyer succeeded in constructing smooth wavelets, setting off a furious spurt of development. Stéphane Mallat recognized that wavelet approximation was highly effective in image processing and formulated a general approximation scheme. A few weeks later, Ingrid Daubechies, inspired in part by Mallat's work, discovered a family of very explicit wavelets of immense practical importance. Daubechies's wavelets have compact support like the square wave, but with varying degrees of smoothness and varying numbers of oscillations. In a way reminiscent of but much deeper than the weightings of Simpson's rule and the higher order Newton-Cotes formulas for numerical integration, these wavelets have just the right cancellation properties to isolate features of signals that were previously not accessible to analysis. The (7,9) pair of Daubechies wavelets underlies the WSQ algorithm.

[1] *This reproduced a 1964 result of A. P. Calderón—a fortunate duplication of effort, since it revealed how close the issue was to contemporary Fourier analysis and to the work of many pure mathematicians.*

We have long since stopped being surprised when pure mathematics has something significant to say about a practical problem, but it is still remarkable that a really good solution is migrating so quickly into all of our computers. Perhaps the most important lesson for policymakers is that the practical problem of finding oil had no physical relation to the processing of pictures. They should also take note that the ideas traveled from France to the United States immediately. Also worth examining is the path of discovery. It took physicists Morlet and Grossmann, speaking the language of Fourier analysis and quantum mechanics, to get Yves Meyer's attention. The discovery could conceivably have come about in a half dozen other ways. Hubbard mentions many instances of duplication and reinvention of known tools. In particular, in the image processing community, Burt and Adelson had already developed the so-called pyramid algorithm. But physicists have more longstanding and deeper ties to mathematics. The clear, accessible mathematical framework played an essential role in enabling researchers at each step of the way to formulate the right problems and spread their ideas.

In addition to being a case study of innovation, the book is a primer on wavelets, complete with references to dozens of books, academic articles, and software. In order to mitigate the impossible demands that technical details would place on the average reader, Hubbard defers those details. In the first half she describes without formulas (only words and graphs) what wavelets are, how they were discovered, and what they are good for. The second and larger portion explains Fourier series and transforms, the fast Fourier transform, continuous wavelet transforms, multiresolutions, and other terms of art that the first section explains metaphorically and by example. The appendices ratchet up the level even further with a description of the Riemann and Lebesgue integrals and proofs of Shannon's sampling theorem, Heisenberg's uncertainty principle, and completeness of Fourier series.

How did Hubbard decide she could get away with including complete proofs? One reason is that the book was originally published in France, an appropriate choice considering that much of the action in this drama played out in France. France is one of the few places in the world where future high school math teachers are required to learn about the Lebesgue integral and Fourier series. It is also the place where one can find a hotel owner who likes to recite all the French Fields Medal winners, all the more so since 1994. Another reason, as Hubbard explains, is that as a science writer who overcame math-avoidance by marrying a mathematician, she uses herself as her best test audience. Her threshhold for including high-powered material amounts to, "If I can understand it, so can you." On a more fundamental level, she has become one of us. Having been seduced by the clarity and insight that mathematical language and reasoning can bring, she wants to show her readers the real thing.

I enjoyed watching Hubbard walk the tightrope between the pitfalls of overly technical explanations and oversimplified ones. Her safety net is the second section with its definitions, formulas, and succinct explanations. Hubbard has gone to great lengths to make all parts of the book accessible in principle even to readers with the barest background. They can fill in gaps in their knowledge of trigonometry, complex exponentials, integrals, matrix arithmetic, and dot products. The notion that anybody could absorb all of these ideas on the spot is far-fetched, but most readers are going to have some gaps, and this comprehensive approach makes for a self-contained treatment. All this care and effort should pay off by inspiring some readers to learn more mathematics and by making it easier for them to do so.

I found only one mistake in the whole second section: The explanation of convolution by way of multiplication is carried out incorrectly. (The convolution of [4,2,6] with [3,2] is not [1,3,6,3,2], and the difference between convolution and multiplication is that in convolution one does *not* do any "carrying".) One more quibble: A few of the pictures are too small or poorly reproduced. In some cases their significance is subtle and can be grasped only after a close reading of the text and captions. With the added burden of deciphering a small picture, some readers may give up.

I recommend the book wholeheartedly to anyone interested in how mathematics works in practice. A well-motivated undergraduate can get a great deal out of it. Readers who want to follow more than half of the book will have to know about complex exponentials or be willing to learn. Less-experienced readers will avoid the second section and thus miss out on some of the best pictures and ideas. As a professional mathematician, I found the book frustrating or, more accurately, tantalizing. It goes so much further than one expects from a popular rendition that I began to hunger for an even deeper discussion than was feasible. But all readers can appreciate how vividly Hubbard portrays mathematical research. She has captured the authentic voice of mathematicians talking about their most exciting moments of discovery and about the difficulties and uncertainties they face.