

## PERIODS OF SECOND KIND DIFFERENTIALS OF $(n, s)$ -CURVES

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*Dedicated to the 70th birthday  
of Victor Buchstaber*

ABSTRACT. Elliptic curves expressions for the periods of elliptic integrals of the second kind in terms of theta-constants have been known since the middle of the 19th century. In this paper we consider the problem of generalizing these results to curves of higher genera, in particular to a special class of algebraic curves, the so-called  $(n, s)$ -curves. It is shown that the representations required can be obtained by the comparison of two equivalent expressions for the projective connection, one due to Fay–Wirtinger and the other from Klein–Weierstrass. As a principle example, we consider the case of the genus two hyperelliptic curve, and a number of new Thomae and Rosenhain type formulae are obtained. We anticipate that our analysis for the genus two curve can be extended to higher genera hyperelliptic curves, as well as to other classes of  $(n, s)$  non-hyperelliptic curves.

### § 1. INTRODUCTION

We discuss the following problem, which is solved in particular cases: *Consider a curve  $\mathcal{C}$  of genus  $g > 1$  and its  $\mathfrak{a}$ - and  $\mathfrak{b}$ -periods of holomorphic differentials  $2\omega, 2\omega'$ . Let  $2\eta, 2\eta'$  be the periods of the differentials of the second kind conjugated to  $2\omega, 2\omega'$  according to the generalized Legendre relations*

$$(1.1) \quad \eta^T \omega = \omega^T \eta, \quad \eta^T \omega' - \omega^T \eta' = \frac{i\pi}{2}, \quad \eta'^T \omega' = \omega'^T \eta'.$$

*Express the periods of the differentials of the second kind in terms of the data, including  $\mathfrak{a}$ -periods,  $2\omega$ ,  $\theta$ -constants, depending on the Riemann period matrix  $\tau = \omega'/\omega$  and coefficients of the polynomial defining the curve  $\mathcal{C}$ .*

In the case of the elliptic curve  $y^2 = 4x^3 - g_2x - g_3$ , the question posed is answered by the Weierstrass formulae

$$(1.2) \quad \eta = -\frac{1}{12\omega} \sum_{k=2}^4 \frac{\vartheta_k''(0)}{\vartheta_k(0)} \quad \text{and equivalently,} \quad \eta = -\frac{1}{12\omega} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)},$$

where  $2\eta$  is the  $\mathfrak{a}$ -period of the elliptic differential of the second kind  $-x dx/y$ . For algebraic curves of higher genera, the *second period matrix*,  $(2\eta, 2\eta')$ , in the terminology of [28], appears naturally in the definition of the Riemann  $\theta$ -function; the possibility of expressing it in terms of the *first period matrix*,  $(2\omega, 2\omega')$ , and the  $\theta$ -constants is of theoretical interest. It is also useful for the implementation of calculations by computer algebra; e. g., the current versions of Maple/algcurves calculates only periods of differentials of the first kind. The  $\theta$ -constant representation of periods of differentials of the second kind is also important for defining the multi-dimensional  $\sigma$ -function which is currently of interest: the basic theory of the  $\sigma$ -function has been intensively developed over

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2010 *Mathematics Subject Classification*. Primary 32G15, 14K25, 30F30.

*Key words and phrases*. Moduli of algebraic curves, theta-constants, sigma-functions.

the last few decades by V. M. Buchstaber with co-workers; see the recent draft manuscript [13].

Multi-variative  $\sigma$ -functions were introduced by F. Klein, [23], who formulated the program of construction of Abelian functions on the basis of these functions in [24]. The expression of periods of the second kind, in terms of  $\theta$ -constants, and modular forms built on  $\theta$  constants, is a part of this program. Klein developed the theory in the cases of hyperelliptic curves and arbitrary genus 3 curves to realise his program in particular cases. Later the hyperelliptic theory was well documented by Baker [2] and especially the case of the genus two curve [3]. We present here an approach to solve the problem formulated above for the family of  $(n, s)$ -curves; these represent a natural generalization of the Weierstrass elliptic cubic to higher genera.

$(n, s)$ -curves are defined as follows. Let  $n$  and  $s$  be a pair of co-prime integers such that  $s > n \geq 2$ . Those non-negative natural numbers  $w_1, w_2, \dots, w_g$ , which cannot be represented in the form  $\alpha n + \beta s$  with non-negative integer  $\alpha$  and  $\beta$ , form the Weierstrass gap sequence of length  $g = (n-1)(s-1)/2$ . The  $(n, s)$ -curve is the non-degenerate plane curve of genus  $g$  given by the polynomial equation

$$(1.3) \quad f(x, y) = y^n - x^s - \sum_{\alpha, \beta} \lambda_{\alpha n + \beta s} x^\alpha y^\beta = 0$$

with  $\lambda_k \in \mathbb{C}$  and  $0 \leq \alpha < s-1, 0 \leq \beta < n-1$ .

The notion of  $(n, s)$ -curves was introduced in [11] and now attracts much interest. It is shown in [10] that models of  $(n, s)$ -curves are better adapted to the construction of multi-variable  $\sigma$ -functions than models suggested by Weierstrass and later models of mini-versal deformations of singularities of the form  $y^n = x^s$ . We also mention Klein [23], [24] and recent works presenting an effective description of multi-variate  $\sigma$ -functions [31] and [26]. In particular, the approach of Klein (for any Riemann surface of genus 3) and [26] (for an arbitrary Riemann surface of any genus) to the theory of higher genus sigma-functions is based on resolving the generalized Legendre relations in terms of theta-constants. Using  $(n, s)$ -curves, it is possible to develop the construction of Abelian functions and the associated integrable PDEs in terms of the  $\sigma$ -function of the trigonal curve, [16], [4], to develop the study of space curves [29], [1], to consider  $\tau$ -functions of integrable hierarchies as  $\sigma$ -functions, [30], [21], to develop the description of classical surfaces like Kummer, Coble surfaces [13], [15], to describe Jacobi inversion on the strata of non-hyperelliptic Jacobians [27], [7], to develop number-theoretical problems [8], [25], and others.

The present note aims to describe the moduli of the  $\sigma$ -function, but we anticipate that its content and area of applicability should be more general.

## § 2. THE METHOD

Let  $\mathcal{C}$  be a plane algebraic curve of genus  $g$  given by the polynomial equation  $f(x, y) = 0$ . Introduce the bi-differential  $\Omega(Q, R)$  on  $\mathcal{C} \times \mathcal{C} \ni (Q, R)$ , which is called the *canonical bi-differential of the second order* if it is

- symmetric:

$$(2.1) \quad \Omega(Q, R) = \Omega(R, Q),$$

- normalized at  $\mathfrak{a}$ -periods:

$$(2.2) \quad \oint_{\mathfrak{a}_k} \Omega(Q, R) = 0, \quad k = 1, \dots, g,$$

- and has the only pole of the second order along the diagonal. In other words it has the following expansion:

$$(2.3) \quad \Omega(Q, R) = \frac{d\xi(Q)d\xi(R)}{(\xi(Q) - \xi(R))^2} + \frac{1}{6}S(P) + \text{higher order terms},$$

where  $\xi(Q)$  and  $\xi(R)$  are local coordinates of points  $Q = (x, y)$  and  $R = (z, w)$  in the vicinity of a point  $P$ , respectively. The quantity  $S(P)$  is called the *holomorphic projective connection*.

The bi-differential  $\Omega(Q, R)$  is uniquely defined by the conditions (2.1), (2.2), and (2.3). We realise it as follows. Let  $\theta$  be the standard  $\theta$ -function defined by its Fourier series

$$(2.4) \quad \theta(\mathbf{z}; \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp\{i\pi \mathbf{n}^T \tau \mathbf{n} + 2i\pi \mathbf{z}^T \mathbf{n}\},$$

where  $\tau$  is the Riemann period matrix of the normalised holomorphic differentials  $\mathbf{v} = (v_1, \dots, v_g)^T$ ,

$$(2.5) \quad \oint_{\mathfrak{a}_j} v_i = \delta_{i,j}, \quad \oint_{\mathfrak{b}_j} v_i = \tau_{i,j}, \quad i, j = 1, \dots, g.$$

Let  $\mathfrak{A}$  be a non-singular point of the  $\theta$ -divisor,  $\mathfrak{A} \in (\theta)$ , i. e.

$$\theta(\mathfrak{A}) = 0, \quad \text{grad } \theta(\mathfrak{A}) \neq 0.$$

Then

$$(2.6) \quad \Omega(Q, R) = d_x d_z \ln \theta \left( \mathfrak{A} + \int_R^Q \mathbf{v}; \tau \right), \quad Q = (x, y), \quad R = (z, w).$$

Such a realisation of  $\Omega(Q, R)$  yields the following representation of the holomorphic projective connection  $S(P)$ .

**The Fay–Wirtinger representation** of  $S(P)$  [33] is quoted by Fay [20, p. 19], where we denote  $S(P) = S_{FW}(P)$  in the form

$$(2.7) \quad S_{FW}(P) = \left\{ \int_{\bullet}^R H_{\mathfrak{A}}, P \right\} (P) + \frac{3}{2} \left( \frac{Q_{\mathfrak{A}}}{H_{\mathfrak{A}}} \right)^2 (P) - 2 \frac{T_{\mathfrak{A}}}{H_{\mathfrak{A}}} (P),$$

where  $\{\cdot, \cdot\}$  is the Schwartzian differential operator,

$$(2.8) \quad \begin{aligned} H_{\mathfrak{A}}(P) &= \sum_{i=1}^g \frac{\partial \theta}{\partial z_i}(\mathfrak{A}) v_i(P), \\ Q_{\mathfrak{A}}(R) &= \sum_{i,j=1}^g \frac{\partial^2 \theta}{\partial z_i \partial z_j}(\mathfrak{A}) v_i(P) v_j(P), \\ T_{\mathfrak{A}}(P) &= \sum_{i,j,k=1}^g \frac{\partial^3 \theta}{\partial z_i \partial z_j \partial z_k}(\mathfrak{A}) v_i(P) v_j(P) v_k(P). \end{aligned}$$

Here  $\mathfrak{A}$  is a non-singular point of the  $\theta$ -divisor,  $\mathfrak{A} \in (\theta)$ ,  $v_j$ ,  $j = 1, \dots, g$ , where the  $v_j$  are the normalized holomorphic differentials defined in (2.5).

Our method is based on a comparison of the Fay–Wirtinger representation of the projective connection  $S(P)$ , given above as  $S_{FW}(P)$ , and the equivalent representation of Klein and Weierstrass,  $S_{KW}(P)$ , described below.

**The Klein–Weierstrass representation** of the projective connection follows from the Klein–Weierstrass realisation of the bi-differential  $\Omega(Q, R)$  in algebraic form (see [2], [3] and the recent review [13]). Namely, let  $f(x, y) = 0$  be the equation of the

$(n, s)$ -curve  $\mathcal{C}$  of genus  $g$  with marked point  $P_0 = (\infty, \infty)$ . Then for arbitrary points  $(Q = (x, y), R = (z, w)) \in \mathcal{C} \times \mathcal{C}$ , the bi-differential  $\Omega(Q, R)$  is represented in the form

$$(2.9) \quad \Omega(Q, R) = \frac{\mathcal{F}(Q, R)}{(x-z)^2} \frac{dx}{f_y(x, y)} \frac{dz}{f_w(z, w)} + 2\mathbf{u}(Q)^T \mathcal{K}\mathbf{u}(R).$$

Here  $\mathcal{F}(Q, R)$  is a polynomial of its variables and  $\mathbf{u} = (u_1, \dots, u_g)^T$  is a vector of canonical holomorphic differentials computed at points  $Q$  and  $R$ .

In [31], Nakayashiki presented a description of the polynomial  $\mathcal{F}(Q, R)$  as a polynomial of  $(x, y; z, w)$  with the coefficients in homogeneous polynomials (with respect to the weights) of  $\lambda_i$  (=coefficients of  $f(x, y)$ ). Our approach is based on the explicit algebraic expression of  $\mathcal{F}(Q, R)$ , whose derivation is classically known and described as follows.

Let  $w_1 < w_2 < \dots < w_g$  be the Weierstrass gap sequence at the point  $P_0$  of the length  $g$ . Order the components of the vector  $\mathbf{u}$  of holomorphic differentials in such a way that the orders of vanishing at the point  $P_0$  of the holomorphic integrals are  $\text{ord}(\int u_k) = w_{g-k+1}$ . Represent the basis of canonical holomorphic differentials in the form

$$(2.10) \quad \mathbf{u}(x, y) = \frac{dx}{f_y(x, y)} \mathbf{U}(x, y),$$

with vector

$$\mathbf{U}(x, y) = (\mathcal{U}_1(x, y), \dots, \mathcal{U}_g(x, y))^T,$$

whose components are monomials constructed by the Weierstrass gap sequence. Introduce the vector

$$\mathcal{R}(x, y) = (\mathcal{R}_1(x, y), \dots, \mathcal{R}_g(x, y))^T$$

defining the conjugate meromorphic differential with the only pole at the point  $P_0$ .

$$(2.11) \quad \mathbf{r}(x, y) = \frac{dx}{f_y(x, y)} \mathcal{R}(x, y).$$

The vector  $\mathcal{R}(x, y)$  is constructed in such a way that the period matrices

$$\begin{aligned} 2\omega &= \left( \oint_{\mathbf{a}_j} u_i \right)_{i, j=1, \dots, g}, & 2\omega' &= \left( \oint_{\mathbf{b}_j} u_i \right)_{i, j=1, \dots, g}, \\ 2\eta &= - \left( \oint_{\mathbf{a}_j} r_i \right)_{i, j=1, \dots, g}, & 2\eta' &= - \left( \oint_{\mathbf{b}_j} r_i \right)_{i, j=1, \dots, g} \end{aligned}$$

satisfy the generalized Legendre relation

$$(2.12) \quad \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T = -\frac{i\pi}{2} \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix},$$

that was given in expanded form in (1.1).

The polynomial  $\mathcal{F}(Q, R) = \mathcal{F}(x, y; z, w)$  and vector  $\mathcal{R}(x, y)$  are found simultaneously within the following construction. Introduce vectors  $\phi(x, y)$  and  $\psi(x, y)$ :

$$(2.13) \quad \phi(x, y) = (y^{n-1}, y^{n-2}, \dots, 1)^T$$

and

$$(2.14) \quad \psi(x, y) = (1, \psi_1(x, y), \dots, \psi_{n-1}(x, y))^T,$$

where

$$\psi_k(x, y) = \left( \frac{f(x, y)}{y^{n-k}} \right)_+, \quad k = 1, \dots, n-1,$$

and the subscript  $+$  means that, after division, only monomials of non-negative power in  $y$  are taken into account in the sum.

Introduce the differential of the third kind  $\Pi_{P_1, P_2}(P)$ ,  $P = (x, y)$  with first order poles at two points  $P = P_1 = (x_1, y_1)$ ,  $P_2 = (x_2, y_2)$ , and residues  $\pm 1$ . It is given, in terms of the quantities introduced above, as

$$(2.15) \quad \Pi_{P_1, P_2}(P) = \frac{dx}{f_y(x, y)} \left\{ \frac{\psi^T(x_1, y_1)\phi(x, y)}{x - x_1} - \frac{\psi^T(x_2, y_2)\phi(x, y)}{x - x_2} \right\}.$$

To construct the bi-differential  $\Omega(Q, R)$  explicitly, we consider on  $\mathcal{C} \times \mathcal{C}$  the auxiliary bi-differential

$$(2.16) \quad \tilde{\Omega}(Q, R) = dz \frac{dx}{f_y(x, y)} \frac{\partial}{\partial z} \frac{\psi^T(z, w)\phi(x, y)}{x - z}, \quad Q = (x, y), R = (z, w).$$

It is straightforward to show that

$$\phi(x, y)^T \phi(x, Y) = \frac{f(x, Y) - f(x, y)}{Y - y};$$

therefore,  $\tilde{\Omega}(Q, R)$  has a pole of the second order along the diagonal as a form in  $x$ . Since this differential is holomorphic in  $(x, y)$  away from the diagonal, it has poles in the variable  $(z, w)$  at  $z = \infty$ . Restore the symmetry by setting

$$\tilde{\tilde{\Omega}}(Q, R) = \tilde{\Omega}(Q, R) + \mathcal{R}(z, w)^T \mathbf{U}(x, y) \frac{dx dz}{f_y(x, y) f_w(z, w)},$$

where the  $g$ -vector  $\mathbf{U}(x, y)$  is defined by the holomorphic differentials (2.10), and the  $g$ -vector  $\mathcal{R}(z, w)$  is found from the condition

$$(2.17) \quad \frac{\partial f(z, w)}{\partial w} \frac{\partial}{\partial z} \frac{\psi(z, w)^T \phi(x, y)}{x - z} - \frac{\partial f(x, y)}{\partial y} \frac{\partial}{\partial x} \frac{\psi(x, y)^T \phi(z, w)}{z - x} = \\ = \mathcal{R}^T(x, y) \mathbf{U}(z, w) - \mathcal{R}^T(z, w) \mathbf{U}(x, y).$$

As a result, the polynomial  $\mathcal{F}(x, y; z, w)$  is found from the relation

$$\frac{dx dz}{f_y(x, y)} \frac{\partial}{\partial z} \frac{\psi^T(z, w)\phi(x, y)}{x - z} + \mathcal{R}(z, w)^T \mathbf{U}(x, y) \frac{dx dz}{f_y(x, y) f_w(z, w)} = \\ = \frac{\mathcal{F}(x, y; z, w)}{(x - z)^2} \frac{dx}{f_y(x, y)} \frac{dz}{f_w(z, w)},$$

and the bi-differential  $\Omega(Q, R)$  is represented in the form

$$(2.18) \quad \Omega(Q, R) = \tilde{\tilde{\Omega}}(Q, R) + \mathcal{R}(z, w)^T \mathbf{U}(x, y) \frac{dx dz}{f_y(x, y) f_w(z, w)} + 2\mathbf{u}^T(Q) \varkappa \mathbf{u}(R).$$

Explicit formulae for the polynomials  $\mathcal{R}_j(x, y)$  are classically known for hyperelliptic curves [2] and found only recently for the family of  $(3, s)$ -trigonal curves (see review [13] where all necessary references are given). We note that the polynomials  $\mathcal{R}_j(x, y)$  are not uniquely defined, and an arbitrary polynomial built in  $\mathcal{U}_j(x, y)$  can be added without affecting the generalized Legendre condition (2.12).

The normalizing matrix  $\varkappa$  is given according to the construction as the symmetric matrix

$$(2.19) \quad \varkappa^T = \varkappa, \quad \text{and} \quad \varkappa = \eta(2\omega)^{-1}.$$

The period matrices  $2\eta$  and  $2\eta'$  are expressible in terms of  $\varkappa$  and  $\omega, \omega'$ :

$$(2.20) \quad \eta = 2\varkappa\omega, \quad \eta' = 2\varkappa\omega' - \frac{i\pi}{2}(\omega^{-1})^T.$$

Now we are in a position to define the fundamental  $\sigma$ -function of the  $(n, s)$ -curve  $\mathcal{C}$  of genus  $g = (n - 1)(s - 1)/2$ . To do that we introduce the  $\theta$ -function  $\theta[\varepsilon](\mathbf{z}; \tau)$  with characteristic

$$[\varepsilon] = \begin{bmatrix} \boldsymbol{\varepsilon}^T \\ \boldsymbol{\varepsilon}'^T \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \dots & \varepsilon_g \\ \varepsilon'_1 & \dots & \varepsilon'_g \end{bmatrix}$$

with half-integer  $\varepsilon_i, \varepsilon'_j = 0$  or  $1/2$  as the Fourier series

$$(2.21) \quad \begin{aligned} \theta[\varepsilon](\mathbf{z}; \tau) &= \exp\{i\pi \boldsymbol{\varepsilon}^T \tau \boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon}^T(\mathbf{z} + \boldsymbol{\varepsilon}')\} \theta(\mathbf{z} + \tau \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}') = \\ &= \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp\{i\pi(\mathbf{n} + \boldsymbol{\varepsilon})^T \tau(\mathbf{n} + \boldsymbol{\varepsilon}) + 2i\pi(\mathbf{n} + \boldsymbol{\varepsilon})^T(\mathbf{z} + \boldsymbol{\varepsilon}')\}. \end{aligned}$$

Here the Riemann matrix  $\tau = \omega'/\omega$  necessarily belongs to the Siegel upper-half space, i.e.  $\tau^T = \tau$  and  $\text{Im}\tau > 0$ . The  $\theta$ -function with characteristic  $[\varepsilon]$  is an even or odd function of  $\mathbf{z}$  whenever  $4\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}' = 0 \pmod{2}$  or  $4\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon}' = 1 \pmod{2}$ .

The vector of Riemann constants  $\mathbf{K}_{P_0}$  with the base point at  $P_0 = (\infty, \infty)$  is defined by the condition

$$(2.22) \quad \theta(\mathcal{A}(D) + \mathbf{K}_{P_0}) \equiv 0$$

for all divisors linearly equivalent to divisors  $D$  of degree  $g - 1$ ,  $D = P_1 + \dots + P_{g-1}$ , where  $\mathcal{A}(D)$  is the Abel map with base point  $P_0$ ,

$$\mathcal{A}(D) = \sum_{k=1}^{g-1} \int_{P_0}^{P_k} \mathbf{v}, \quad \mathbf{v} = (2\omega)^{-1} \mathbf{u}.$$

The vector of Riemann constants with base point  $P_0$  is a half-period if and only if there exists a holomorphic differential with a zero of order  $2g - 2$  at the point  $P_0$ ; see e.g., [19, p. 299]. Remarkably, the  $(n, s)$ -curve admits such a differential [31], namely the differential  $u_1 = dx/f_y(x, y)$  that has a zero of required order in the point  $P_0 = (\infty, \infty)$ . Indeed,

$$\frac{dx}{f_y(x, y)} \Big|_{\substack{x=1/\xi^n \\ y=1/\xi^s}} \sim \xi^{ns-n-s-1} d\xi = \xi^{2g-2} d\xi,$$

where we use the expression for the genus of the  $(n, s)$ -curve,  $g = (n - 1)(s - 1)/2$ . Therefore, the divisor  $(g - 1)P_0$  defines a spin structure, and the corresponding half-integer  $\theta$ -characteristic  $\gamma$  defines a  $\sigma$ -function, which is called the *fundamental  $\sigma$ -function of the  $(n, s)$ -curve*

$$(2.23) \quad \sigma(\mathbf{z}) = C\theta[\gamma]((2\omega)^{-1}\mathbf{z}; \tau) \exp\{\mathbf{z}^T \boldsymbol{\varkappa} \mathbf{z}\},$$

where  $\mathbf{z}$  is a vector from the Jacobi variety  $\text{Jac}(\mathcal{C}) = \mathbb{C}^g/2\omega \otimes 2\omega'$ ,  $C$  is a constant independent of the variable  $\mathbf{z}$ ; for details and another definition of  $\sigma(\mathbf{z})$  see, e.g., [13]. According to Klein [24], the quadratic form in the exponential  $\sum_{i,j} \varkappa_{i,j} z_i z_j$  should be presented in terms of  $\theta$ -constants; such an expression in terms of logarithmic derivatives of even  $\theta$ -constants,  $\partial \ln \theta[\varepsilon]/\partial \tau_{i,j}$ , was given in [24].

We remark that the characteristic  $[\gamma]$  can be odd and even, and the corresponding  $\sigma$ -function inherits this parity as a function of  $\mathbf{z}$ . This parity coincides with parity of the integer number  $(n^2 - 1)(s^2 - 1)/24$ ; see [12]. This last number represents the order of vanishing of  $\sigma(\mathbf{z})$  in  $\text{Jac}(\mathcal{C})$  over the variable  $\xi$ , when components of the vector  $\mathbf{z}$  are graded as  $z_k = \xi^{w_g - k + 1}$ , and  $(w_1, \dots, w_g)$  are the Weierstrass gap numbers at the point  $P_0 = (\infty, \infty)$ .

It is also useful to mention that, since  $(2g - 2)P_0$  belongs to the canonical class, it also follows ([26]) that the dimension of the moduli space of algebraic curves that can be represented as  $(n, s)$ -curves is equal to  $2g - 1$ ; that is the variety of such curves has

co-dimension  $g - 2$  in the space of moduli of all curves. In [26], the theory is constructed for arbitrary curves, whilst we concentrate here on the moduli problem of  $\sigma$ -functions of  $(n, s)$ -curves, finding various representations of the matrix  $\varkappa$  in terms of  $\theta$ -constants. The following result is important for the development of our argument.

**Proposition 2.1.** *Let the  $(n, s)$ -curve be given in the form*

$$(2.24) \quad y^n - a_{n-1}(x)y^{n-1} - \dots - a_1(x)y - a_0(x) = 0.$$

*The projective connection  $S_{KW}(x, y)$  at the point  $P = (x, y) \in \mathcal{C}$  and with the local coordinate  $\xi(P)$  is given as*

$$(2.25) \quad S_{KW}(x, y) = \{x, \xi\} d\xi^2 + \mathcal{T}(x, y) + 6\mathbf{r}^T(x, y)\mathbf{u}^T(x, y) + 12\mathbf{u}^T(x, y)\varkappa\mathbf{u}(x, y),$$

where

$$(2.26) \quad \mathcal{T}(x, y) = -\frac{1}{2f_y} [3y'' f_{yy} + 2y'^2 f_{yyy} + 6y' f_{yyx} + 6f_{yxx}] dx^2.$$

*Here the subscripts mean partial differentiation, the primes ' mean implicit differentiation of  $y$  as a function of  $x$  given by the equation of the curve,  $f(x, y) = 0$ , and  $\{\cdot, \cdot\}$  is the Schwartzian derivative,*

$$\{x(\xi), \xi\} = \frac{d^3 x(\xi)/d\xi^3}{dx(\xi)/d\xi} - \frac{3}{2} \left( \frac{d^2 x(\xi)/d\xi^2}{dx(\xi)/d\xi} \right)^2.$$

*Proof.* Currently we have only a proof by direct calculation for  $(n, s)$ -curves up to and including  $n = 10$ ; a more general proof is under investigation. Consider the representation (2.18) of the bi-differential  $\Omega(Q, R)$ . The last two terms in (2.18), after restriction to the diagonal  $Q = R$ , produce the last two terms (after multiplication by 6) in (2.25). The expansion of  $\tilde{\Omega}(Q, R)$  is of the form

$$\tilde{\Omega}(Q, R) = \frac{d\xi(Q) d\xi(R)}{(\xi(Q) - \xi(R))^2} + \{x(\xi), \xi\} d\xi(Q) d\xi(R) + \mathcal{T}(Q, R),$$

where the quantity  $\mathcal{T}(Q, R)$ , when restricted to the diagonal,  $Q = R = P$ , should be shown to be of the form (2.26). This can be done by direct calculation in the following way. Consider successive cases  $n = 2, 3, \dots$ , fixing the  $(n, s)$ -curve in the form (2.24). We

get:

$$n = 2: \quad f(x, y) = y^2 - a_1(x)y - a_0(x) = 0,$$

$$\mathcal{T} = -\frac{3}{f_y}(y'' - a_1'') dx^2,$$

$$n = 3: \quad f(x, y) = y^3 - a_2(x)y^2 - a_1(x)y - a_0(x) = 0,$$

$$\mathcal{T} = -\frac{3}{f_y}\{(3y - a_2)y'' + 2y'^2 - 2a_2'y' - a_1''\} dx^2,$$

$$n = 4: \quad f(x, y) = y^4 - a_3(x)y^3 - a_2(x)y^2 - a_1(x)y - a_0(x) = 0,$$

$$\mathcal{T} = -\frac{3}{f_y}\{(6y^2 - 3ya_3 - a_2)y'' + (8y - 2a_3)y'^2 - (6ya_3' + 2a_1')y' - 3y^2a_3'' - 2ya_2'' - a_1''\} dx^2,$$

$$n = 5: \quad f(x, y) = y^5 - \sum_{k=0}^4 a_k(x)y^k = 0,$$

$$\mathcal{T} = -\frac{3}{f_y}\{(10y^3 - 6a_4y^2 - 3a_3y - a_2)y'' + (20y^2 - 8a_4y - 2a_3)y'^2 - (12a_4y^2 + 6ya_3' + 2a_2')y' - 4a_4''y^3 - 3y^2a_3'' - 2ya_2'' - a_1''\} dx^2$$

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \end{array}$$

Analysing these formulae, we find the general representation of the term  $\mathcal{T}(P)$  given in the statement of the proposition. In the general case this result is currently just a conjecture, and a general proof is in the process of development.  $\square$

The procedure for the derivation of relations for periods of the differentials of the second kind is based on the uniqueness of the normalized bi-differential  $\Omega(Q, R)$  and is described as follows. Let  $\mathcal{C}$  be an  $(n, s)$ -curve. Let  $P$  be a point in the vicinity of  $P_0 = (\infty, \infty)$  where the local coordinate  $\xi(P)$  is given as  $x = 1/\xi^n$ . Let  $\mathfrak{A}$  be a non-singular point of the  $\theta$ -divisor,  $\mathfrak{A} \in (\theta)$ , satisfying the condition

$$(2.27) \quad H_{\mathfrak{A}}(P_0) \neq 0.$$

Expanding both parts of the equivalence

$$(2.28) \quad S_{FW}(P) \equiv S_{KW}(P)$$

in  $\xi(P)$ , and equating coefficients of corresponding powers, we obtain relations involving matrix elements of  $\varkappa$  and  $\theta$ -derivatives at the point  $\mathfrak{A}$ . In this way we obtain a compatible system of linear equations for the  $\varkappa_{i,j}$ .

*Remark.* The Weierstrass formula (1.2) can be easily obtained by expanding both sides of the Weierstrass representation of the  $\sigma$ -function in terms of the Jacobian  $\theta$ -function using the Weierstrass series for  $\sigma(u)$  with coefficients given recursively. This method can be generalized to higher genera curves in the cases where  $\sigma$ -expansions are known. But only isolated cases of such expansions are elaborated: the Buchstaber–Leykin recursion for genus two  $\sigma$ -functions [9], calculation of the first few terms of  $\sigma$ -expansions of (3, 4)-curves presented in [16], (3, 5) [4], (3, 7), and (3, 8)-curves are given in [18] and (4, 5)-curves in [17]. We see an advantage of the method proposed, since the derivation is reduced to examining a series over one complex variable, the local coordinate, whilst the generalization of the Weierstrass method leads to a series in many variables.

In the next section, we will demonstrate how the approach works in the case of the hyperelliptic curve.

§ 3. HYPERELLIPTIC CURVE

We realise the hyperelliptic curve  $\mathcal{C}$  of genus  $g$  in the form

$$(3.1) \quad y^2 = 4x^{2g+1} + \lambda_{2g}x^{2g} + \lambda_{2g-1}x^{2g-1} + \dots + \lambda_0, \quad \lambda_i \in \mathbb{C}.$$

The basis of holomorphic and meromorphic differentials can be fixed as

$$(3.2) \quad \begin{aligned} u_i &= \frac{x^{i-1}}{y} dx, \quad i = 1, \dots, g, \\ r_i &= \sum_{k=j}^{2g+1-j} (k+1-j)\lambda_{k+1+j} \frac{x^k dx}{4y}, \quad j = 1, \dots, g. \end{aligned}$$

The matrices of their  $\mathbf{a}$ - and  $\mathbf{b}$ -periods,  $2\omega, 2\eta$ , and  $2\omega', 2\eta'$  satisfy the generalized Legendre relation (2.12). As we already noted, the meromorphic differentials  $r_i$  are not uniquely determined and a linear combination of holomorphic differentials can be added without violation of the relation (2.12). This basis in the space of meromorphic differentials was introduced by Baker [2] and we will call it the *Baker basis*.

The normalized canonical bi-differential has the form

$$(3.3) \quad \Omega(x, y; z, w) = \frac{2yw + F(x, z)}{4(x-z)^2} \frac{dx dz}{y w} + 2 \sum_{i,j=1}^g x^{i-1} z^{j-1} \varkappa_{i,j} \frac{dx dz}{y w}.$$

Here  $F(x, z)$  is the *Kleinian 2-polar*, with  $F(x, x) = 2y^2$ , given by the formula

$$(3.4) \quad F(x, z) = \sum_{k=0}^g x^k z^k (\lambda_{2k} + \lambda_{2k+1}(x+z)).$$

The symmetric matrix

$$(3.5) \quad \varkappa = \begin{pmatrix} \varkappa_{1,1} & \dots & \varkappa_{1,g} \\ \dots & \dots & \dots \\ \varkappa_{1,g} & \dots & \varkappa_{g,g} \end{pmatrix} = \eta(2\omega)^{-1},$$

and therefore the period matrices of differentials of the second kind  $2\eta, 2\eta'$ , are computed by the known  $\varkappa$  as in (2.20).

When the bi-differential is found in the form (3.3), then the fundamental  $\sigma$ -function can be defined as in (2.23), and the Klein–Weierstrass  $\wp$ -functions introduced:

$$(3.6) \quad \begin{aligned} \wp_{ij}(\mathbf{z}) &= -\frac{\partial^2}{\partial z_i \partial z_j} \ln \sigma(\mathbf{z}), \quad i, j = 1, \dots, g, \\ \wp_{ijk}(\mathbf{z}) &= -\frac{\partial^3}{\partial z_i \partial z_j \partial z_k} \ln \sigma(\mathbf{z}), \quad i, j, k = 1, \dots, g, \quad \text{etc.} \end{aligned}$$

The solution of the Jacobi inversion problem found by Baker (see [13]) looks remarkably simple in the  $\wp$ -variables:

$$(3.7) \quad \begin{aligned} x^g - \wp_{gg}(\mathbf{z})x^{g-1} - \wp_{g, g-1}(\mathbf{z})x^{g-2} - \dots - \wp_{1g}(\mathbf{z}) &= 0, \\ y_k = \wp_{ggg}(\mathbf{z})x_k^{g-1} + \wp_{g-1, gg}(\mathbf{z})x_k^{g-2} + \dots + \wp_{2, gg}(\mathbf{z})x_k + \wp_{1, gg}(\mathbf{z}), \\ k &= 1, \dots, g. \end{aligned}$$

**Proposition 3.1.** *Let  $\xi(P)$  be the local coordinate of the point  $P$ . The projective connection is given by the formula*

$$(3.8) \quad S_{KW}(x, y) = \{x(\xi), \xi\} d\xi^2 - \frac{3}{2} \frac{y''(x)}{y(x)} dx^2 + 6\mathbf{u}^T(x, y)\mathbf{r}(x, y) + 12\mathbf{u}^T(x, y)\varkappa\mathbf{u}(x, y).$$

*Proof.* The formula (3.8) represents a particular case of (2.25) or can be derived in an analogous way.  $\square$

Consider some further special cases.

**3.1. Elliptic curve.** As already mentioned, the formula for  $\varkappa = \eta/(2\omega)$  in the case of an elliptic curve can be obtained by the expansion of  $\sigma(u)$ . But we will demonstrate how the method works for the elliptic curve

$$(3.9) \quad y^2 = 4x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0$$

and the infinite point  $P_0$ , where the local coordinate of a point  $P$  is introduced as  $x = 1/\xi^2 \sim \infty$ .

$$\frac{S_{KW}(P)}{(d\xi)^2} \sim -\frac{3}{4}\lambda_2 + 12\varkappa + \left(-\frac{3}{2}\lambda_1 + \frac{9}{32}\lambda_2^2 - 3\varkappa\lambda_2\right)\xi^2 + O(\xi^4).$$

To expand  $S_{FW}(P)$ , we choose

$$\mathfrak{a} = \frac{1}{2} + \frac{\tau}{2} \in (\theta)$$

and write the expression for  $S_{FW}$  in terms of Jacobian  $\theta$ -functions. We have for the quantities (2.8):

$$H_{\mathfrak{a}}(P) = -\frac{d\xi}{2\omega}\vartheta'_1(0)e^{-i\pi\tau/4}, \quad Q_{\mathfrak{a}}(P) = 0, \quad T_{\mathfrak{a}}(P) = -\frac{d\xi}{8\omega^3}\vartheta'''_1(0)e^{-i\pi\tau/4}.$$

Taking this into account, we write

$$\frac{S_{FW}}{(d\xi)^2} \sim -\frac{1}{4}\lambda_2 - \frac{1}{2\omega^2}\frac{\vartheta'''_1(0)}{\vartheta'_1(0)} + \left(\frac{5}{32}\lambda_2^2 - \frac{3}{2}\lambda_1 + \frac{\lambda_2}{8\omega^2}\frac{\vartheta'''_1(0)}{\vartheta'_1(0)}\right)\xi^2 + O(\xi^4).$$

Equating coefficients of expansions in (2.28), we get

$$(3.10) \quad \varkappa = \frac{\eta}{2\omega} = \frac{1}{24}\lambda_2 - \frac{1}{24\omega^2}\frac{\vartheta'''_1(0)}{\vartheta'_1(0)}.$$

At  $\lambda_2 = 0$ , the Weierstrass formula (1.2) follows. We can check that we get again (3.10) on equating coefficients of the  $\xi^2$  term. In the next section, we develop the same expansion procedure for the genus two curve, omitting details of the underlying computer algebra calculations.

**3.2. Genus two curve.** As the principal example in this paper, consider a genus two hyperelliptic curve of the form (3.1):

$$(3.11) \quad y^2 = 4x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5),$$

where  $e_1, \dots, e_5$  are finite branch points.

Fix a homology basis  $\mathfrak{a}_1, \mathfrak{a}_2; \mathfrak{b}_1, \mathfrak{b}_2$  and calculate the  $2 \times 2$  period matrices of the holomorphic and meromorphic differentials  $2\omega, 2\omega', 2\eta, 2\eta'$ . Also introduce the Riemann period matrix  $\tau$  and the necessarily symmetric matrix  $\varkappa$ ,

$$\tau = \omega'\omega^{-1}, \quad \tau^T = \tau, \quad \text{Im } \tau > 0, \quad \text{and} \quad \varkappa = 2\eta\omega^{-1}, \quad \varkappa^T = \varkappa.$$

Denote

$$(3.12) \quad (2\omega)^{-1} = (\mathbf{U}, \mathbf{V}),$$

where  $\mathbf{U} = (U_1, U_2)^T$  and  $\mathbf{V} = (V_1, V_2)^T$  are 2-column vectors called *winding vectors*. We will use below the directional derivatives denoted by

$$\begin{aligned} \partial_{\mathbf{U}}\Psi(\mathbf{z}) &= U_1 \frac{\partial}{\partial z_1} \Psi(z_1, z_2) + U_2 \frac{\partial}{\partial z_2} \Psi(z_1, z_2), \\ \partial_{\mathbf{V}}\Psi(\mathbf{z}) &= V_1 \frac{\partial}{\partial z_1} \Psi(z_1, z_2) + V_2 \frac{\partial}{\partial z_2} \Psi(z_1, z_2), \end{aligned}$$

where  $\Psi(\mathbf{z})$  is a function of two complex variables. Higher derivatives  $\partial_{\mathbf{U}^2}$ ,  $\partial_{\mathbf{U}\mathbf{V}}$  are defined accordingly.

Beside the canonical  $\theta$ -function,  $\theta$ -functions with half-integer characteristics will be used to formulate our result

$$\begin{aligned} \theta[\varepsilon](\mathbf{z}; \tau) &= \theta \begin{bmatrix} \varepsilon^T \\ \varepsilon'^T \end{bmatrix} (\mathbf{z}; \tau) = \theta \begin{bmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon'_1 & \varepsilon'_2 \end{bmatrix} (z_1, z_2; \tau) = \\ (3.13) \quad &= \sum_{\mathbf{n} \in \mathbb{Z}^2} \exp\{i\pi(\mathbf{n} + \varepsilon)^T \tau(\mathbf{n} + \varepsilon) + 2i\pi(\mathbf{z} + \varepsilon')(\mathbf{n} + \varepsilon)\} \end{aligned}$$

with characteristics  $\varepsilon_i, \varepsilon'_i = 0$  or  $1/2$ . We will also use the  $\theta$ -relation for characteristics  $[\varepsilon]$  and  $[\rho]$ :

$$(3.14) \quad \theta[\varepsilon](\mathbf{z} + \tau\rho + \rho'; \tau) = \theta[\varepsilon + \rho](\mathbf{z}; \tau) \exp\{-i\pi\rho^T \tau\rho - 2i\pi\rho^T \mathbf{z} - 2i\pi(\rho' + \varepsilon')^T \rho\}.$$

There are sixteen  $\theta$ -functions with characteristics, six of which are odd functions of  $\mathbf{z}$ , and 10 of which are even. We denote the corresponding odd (even) characteristics as  $[\delta_1], \dots, [\delta_6]$  ( $[\varepsilon_1], \dots, [\varepsilon_{10}]$ ). We will also call the half-period  $\tau\varepsilon + \varepsilon'$  odd or even whenever its characteristic is odd or even.

The  $\theta$ -functions with characteristics and zero argument are called  $\theta$ -constants. There are 10 non-vanishing even  $\theta$ -constants  $\theta[\varepsilon_i] = \theta[\varepsilon_i](\mathbf{0}; \tau)$ ,  $i = 1, \dots, 10$ . The six derivative odd  $\theta$ -constants satisfy

$$\theta_1[\delta_i] = \left. \frac{\partial \theta[\delta_i](z_1, z_2; \tau)}{\partial z_1} \right|_{\mathbf{z}=\mathbf{0}}, \quad \theta_2[\delta_i] = \left. \frac{\partial \theta[\delta_i](z_1, z_2; \tau)}{\partial z_2} \right|_{\mathbf{z}=\mathbf{0}},$$

$\theta_1[\delta_i], \theta_2[\delta_i] \neq 0$ , simultaneously for all  $i = 1, \dots, 6$ , i. e.,  $\text{grad } \theta[\delta_i] \neq 0$ .

The following half-period is odd:

$$\mathfrak{A}_i = \tau \begin{pmatrix} \delta_{i,1} \\ \delta_{i,2} \end{pmatrix} + \begin{pmatrix} \delta'_{i,1} \\ \delta'_{i,2} \end{pmatrix}, \quad \mathfrak{A}_i \in (\theta)$$

and belongs to the  $\theta$ -divisor,  $(\theta)$ . According to the Riemann vanishing theorem, it is represented in the form

$$\mathfrak{A}_i = (2\omega)^{-1} \int_{(\infty, \infty)}^{(e_i, 0)} \mathbf{u} + \mathbf{K}_{P_0},$$

where  $\mathbf{K}_{P_0}$  is the vector of Riemann constants with base at  $P_0 = (\infty, \infty)$ .

The 10 even half-periods are represented in the form

$$\mathfrak{A}_{i,j} = (2\omega)^{-1} \left( \int_{(\infty, \infty)}^{(e_i, 0)} \mathbf{u} + \int_{(\infty, \infty)}^{(e_j, 0)} \mathbf{u} \right) + \mathbf{K}_{P_0}, \quad 1 \leq i < j \leq 5,$$

and one can denote the corresponding even characteristic as  $[\varepsilon_{i,j}]$ .

There are formulae due to Bolza [6] (see also [14]), which express the branch points  $e_i$  in terms of derivative  $\theta$ -constants and also which find the correspondence between

branch points and odd characteristics:

$$(3.15) \quad e_i \leftrightarrow [\delta_i]: \quad e_i = -\frac{\partial_U \theta[\delta_i]}{\partial_V \theta[\delta_i]}, \quad i = 1, \dots, 5.$$

Because  $e_6 = \infty$ , the characteristic  $[\delta_6]$  is the characteristic of the vector of Riemann constants,  $[\gamma] = [\delta_6]$ . Therefore only five half-periods  $\mathbf{a}_i$  satisfy the condition (2.27).

We emphasise that the procedure described here allows us to find the vector of Riemann constants and the correspondence between half periods and branch points, using Maple/algcurves software. It is not necessary to visualise the homology basis generated by the Tretkoff–Tretkoff algorithm and enciphered in the Maple program. Our procedure allows us to represent the matrix  $\varkappa = 2\eta\omega^{-1}$  in the following form [14]:

**Proposition 3.2.** *For any pair of integers  $i, j$ ,  $1 \leq i < j \leq 5$ , one can write 10 relations for each of the 10 even characteristics  $[\varepsilon_{i,j}]$ :*

$$(3.16) \quad \varkappa = -\frac{1}{2} \begin{pmatrix} e_i e_j (e_k + e_m + e_n) + e_k e_m e_n & -e_i e_j \\ -e_i e_j & e_i + e_j \end{pmatrix} - \frac{1}{2} (2\omega)^{-1T} \frac{1}{\theta[\varepsilon_{i,j}]} \begin{pmatrix} \theta_{1,1}[\varepsilon_{i,j}] & \theta_{1,2}[\varepsilon_{i,j}] \\ \theta_{1,2}[\varepsilon_{i,j}] & \theta_{2,2}[\varepsilon_{i,j}] \end{pmatrix} (2\omega)^{-1},$$

with  $k \neq m \neq n \neq i \neq j \in \{1, \dots, 5\}$  and  $\theta_{r,s}[\varepsilon] = \frac{\partial^2 \theta[\varepsilon]}{\partial z_r \partial z_s}$ .

*Proof.* Let

$$\wp_{mn}(\mathbf{z}) = -\frac{\partial^2}{\partial z_m \partial z_n} \ln \sigma(\mathbf{z}), \quad m, n = 1, 2.$$

Let  $\mathbf{z}^{(i,j)}$  be the Abelian image of two branch points,  $e_i, e_j$

$$\mathbf{z}^{(i,j)} = \int_{P_0}^{(e_i, 0)} \mathbf{u} + \int_{P_0}^{(e_j, 0)} \mathbf{u}.$$

Then Baker’s solution of the Jacobi inversion problem (3.7) leads to the equalities

$$\begin{aligned} \wp_{22}(\mathbf{z}^{(i,j)}) &= e_i + e_j, \\ \wp_{12}(\mathbf{z}^{(i,j)}) &= -e_i e_j, \\ \wp_{11}(\mathbf{z}^{(i,j)}) &= \frac{F(e_i, e_j)}{4(e_i - e_j)^2} = e_i e_j (e_k + e_m + e_n) + e_k e_m e_n, \end{aligned}$$

where  $i \neq j \neq k \neq m \neq n \in \{1, \dots, 5\}$ . Taking into account the definition of the  $\sigma$ -function (2.23) in terms of  $\theta$ -functions, as well as the  $\theta$ -relation, (3.14), we find

$$\wp_{11}(\mathbf{z}^{(i,j)}) = -2\varkappa_{n,m} - \frac{\partial_U^2 \theta_{1,1}[\varepsilon_{i,j}]}{\theta[\varepsilon_{i,j}]}$$

and similar expressions for  $\wp_{12}(\mathbf{z}^{(i,j)})$ ,  $\wp_{22}(\mathbf{z}^{(i,j)})$ . Solving the above equations with respect to  $\varkappa_{1,1}$ , we get (3.16). □

The formulae (3.16) represent the generalization of the Weierstrass formulae

$$(3.17) \quad 2\eta\omega = -2e_1\omega^2 - \frac{1}{2} \frac{\vartheta_2''}{\vartheta_2} = -2e_2\omega^2 - \frac{1}{2} \frac{\vartheta_3''}{\vartheta_3} = -2e_3\omega^2 - \frac{1}{2} \frac{\vartheta_4''}{\vartheta_4}$$

to the genus two hyperelliptic curve. Recall that in the Weierstrass theory,  $e_1 + e_2 + e_3 = 0$ , so adding these three formulae gives the first of (1.2).

For typographical convenience, we will use a shorter notation for directional derivatives

$$\begin{aligned} \partial_U \theta[\varepsilon] &= \Theta_1[\varepsilon], & \partial_V \theta[\varepsilon] &= \Theta_2[\varepsilon], & \partial_{U^2} \theta[\varepsilon] &= \Theta_{1,1}[\varepsilon], \\ \partial_{UV} \theta[\varepsilon] &= \Theta_{1,2}[\varepsilon], & \partial_{V^2} \theta[\varepsilon] &= \Theta_{2,2}[\varepsilon], & \text{etc.} \end{aligned}$$

Also,  $\theta[\varepsilon] = \Theta[\varepsilon]$  at even  $[\varepsilon]$ . Summing (3.16) over 10 even characteristics, and using the above notation for directional derivatives, we get

$$(3.18) \quad \varkappa = \frac{1}{80} \begin{pmatrix} 4\lambda_2 & \lambda_3 \\ \lambda_3 & 4\lambda_4 \end{pmatrix} - \frac{1}{20} \sum_{10 \text{ even } \varepsilon} \frac{1}{\Theta[\varepsilon]} \begin{pmatrix} \Theta_{1,1}[\varepsilon] & \Theta_{1,2}[\varepsilon] \\ \Theta_{1,2}[\varepsilon] & \Theta_{2,2}[\varepsilon] \end{pmatrix}.$$

The representation (3.18) can be compared to the formula of Korotkin–Shramchenko [26], derived for a general algebraic curve. We emphasise that (3.18) is written in the Baker basis (3.2), which enables us to write, in simpler form, the differential equations for the  $\wp$ -symbols, the multi-dimensional generalisations of the Weierstrass  $\wp$ -function.

Now we are in the position to present generalisations of the Weierstrass formula (1.2) to genus two curves. All the following relations are obtained by the expansion procedure described above, exemplified in the case of the elliptic curve.

**Proposition 3.3.** *Denote according to the Bolza formula*

$$\frac{\Theta_1[\delta_i]}{\Theta_2[\delta_i]} = -e_i,$$

where  $e_1, \dots, e_5$  are branch points of the curve  $\mathcal{C}$ . Then the entries to the matrix  $\varkappa$  have the form

$$(3.19) \quad \varkappa_{2,2} = \frac{1}{24} \lambda_4 - \frac{1}{6} e_i - \frac{1}{6} \frac{\Theta_{2,2,2}[\delta_i]}{\Theta_2[\delta_i]},$$

$$(3.20) \quad \varkappa_{1,2} = -\frac{1}{24} \lambda_4 e_i - \frac{1}{3} e_i^2 - \frac{1}{12} e_i \frac{\Theta_{2,2,2}[\delta_i]}{\Theta_2[\delta_i]} - \frac{1}{4} \frac{\Theta_{1,2,2}[\delta_i]}{\Theta_2[\delta_i]},$$

$$(3.21) \quad \varkappa_{1,1} = -\frac{\lambda_3}{8} e_i - \frac{5}{24} \lambda_4 e_i^2 - \frac{7}{6} e_i^3 - \frac{1}{2} \frac{\Theta_{1,1,2}[\delta_i]}{\Theta_2[\delta_i]} - \frac{1}{2} e_i \frac{\Theta_{1,2,2}[\delta_i]}{\Theta_2[\delta_i]} - \frac{1}{6} e_i^2 \frac{\Theta_{2,2,2}[\delta_i]}{\Theta_2[\delta_i]}.$$

*Proof.* The proof is based on the expansion procedure of (2.28) with subsequent elimination of matrix elements  $\varkappa_{i,j}$  from the relations obtained. We omit here rather cumbersome computer algebra details.  $\square$

Summing up over all five representations of  $\varkappa_{2,2}$ , we get

$$(3.22) \quad \varkappa_{2,2} = \frac{1}{20} \lambda_4 - \frac{1}{30} \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{2,2,2}[\delta]}{\Theta_2[\delta]},$$

where we exclude from the summation the characteristic  $[\delta_6]$  of the vector of Riemann constants for which  $\Theta_2[\gamma] = 0$ . Analogously for  $\varkappa_{1,2}$  and  $\varkappa_{1,1}$  we get

$$(3.23) \quad \varkappa_{1,2} = \frac{1}{40} \lambda_3 - \frac{1}{800} \lambda_4^2 - \frac{1}{20} \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{1,2,2}[\delta]}{\Theta_2[\delta]} + \frac{1}{1200} \lambda_4 \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{2,2,2}[\delta]}{\Theta_2[\delta]}$$

and

$$(3.24) \quad \begin{aligned} \varkappa_{1,1} &= \frac{3}{40} \lambda_2 - \frac{1}{400} \lambda_4 \lambda_3 + \frac{1}{8000} \lambda_4^3 - \frac{1}{10} \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{1,1,2}[\delta]}{\Theta_2[\delta]} + \\ &+ \frac{1}{200} \lambda_4 \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{1,2,2}[\delta]}{\Theta_2[\delta]} - \frac{1}{12000} \lambda_4^2 \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{2,2,2}[\delta]}{\Theta_2[\delta]}. \end{aligned}$$

When  $\lambda_4 = 0$ , the  $\varkappa$ -matrix takes the simpler form

$$(3.25) \quad \varkappa = \frac{1}{40} \begin{pmatrix} 3\lambda_2 & \lambda_3 \\ \lambda_3 & 0 \end{pmatrix} - \frac{1}{20} \sum_{5 \text{ odd } [\delta]} \frac{1}{\Theta_2[\delta]} \begin{pmatrix} 2\Theta_{1,1,2}[\delta] & \Theta_{1,2,2}[\delta] \\ \Theta_{1,2,2}[\delta] & \frac{2}{3}\Theta_{2,2,2}[\delta] \end{pmatrix}.$$

**3.3. Certain  $\theta$ -constant relations.** Comparing the formulae (3.18) derived in [14], and the above formula, we conclude:

**Proposition 3.4.** *Let  $\mathcal{C}$  be the genus two hyperelliptic curve with branch point at infinity and realised in the form*

$$y^2 = 4x^5 + \lambda_4x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0, \quad \lambda_i \in \mathbb{C}.$$

*Let  $2\omega$  be the matrix of  $\mathbf{a}$ -periods of holomorphic differentials. Then the following relation holds:*

$$(3.26) \quad \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{2,2,2}[\delta]}{\Theta_2[\delta]} = \frac{3}{2} \sum_{10 \text{ even } \varepsilon} \frac{\Theta_{2,2}[\varepsilon]}{\Theta[\varepsilon]}.$$

*There exists necessarily one odd characteristic  $[\delta]$  for which  $\Theta_2[\delta] = 0$  and the summation on the left-hand side over the odd  $[\delta]$  runs over the remaining five.*

This is a generalization of the Weierstrass formula,

$$\frac{\vartheta_1'''(0)}{\vartheta_1''(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}.$$

Comparing the expressions (3.18) and (3.23), (3.24) in the same way, we find that when  $\lambda_4 = 0$ :

$$(3.27) \quad 4 \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{1,2,2}[\delta]}{\Theta_2[\delta]} - 4 \sum_{10 \text{ even } \varepsilon} \frac{\Theta_{1,2}[\varepsilon]}{\Theta[\varepsilon]} = \lambda_3$$

and

$$(3.28) \quad 4 \sum_{5 \text{ odd } [\delta]} \frac{\Theta_{1,1,2}[\delta]}{\Theta_2[\delta]} - 2 \sum_{10 \text{ even } \varepsilon} \frac{\Theta_{1,1}[\varepsilon]}{\Theta[\varepsilon]} = \lambda_2.$$

The formulae (3.27) and (3.28) express the parameters of the curve, in this case—the symmetric combination of branch points—in terms of sums of theta constants which are symmetric with respect to characteristics. These formulae can be interpreted as a new kind of Thomae-type formulae. The derivation of such classes of relations in the case of non-hyperelliptic curves would be of interest.

The above formulae and (3.19)–(3.21) lead to various generalisations of the Jacobi derivative formula. E.g. subtracting the two expressions (3.19) written for different indices  $i$  and  $j$ , we get (using the classical *Rosenhain derivative formula* [32] for simplification)

$$(3.29) \quad \pm\pi^2 \det(2\omega)^{-1} \Theta[\varepsilon_p] \Theta[\varepsilon_q] \Theta[\varepsilon_r] \Theta[\varepsilon_s] = \Theta_{2,2,2}[\delta_i] \Theta_2[\delta_j] - \Theta_{2,2,2}[\delta_j] \Theta_2[\delta_i],$$

where  $[\delta_i], [\delta_j]$  are two arbitrary odd characteristics from the set of  $[\delta_1], \dots, [\delta_6]$ , and four even characteristics,  $[\varepsilon_p], [\varepsilon_q], [\varepsilon_r], [\varepsilon_s]$  are of the form  $[\delta_i] + [\delta_j] + [\delta_k] \pmod 2$  where  $k \in \{1, 2, 3, 4, 5, 6\} \setminus \{i, j\}$ . This formula can be interpreted as a *Higher Rosenhain derivative formula*.

New interesting generalisations of the Jacobi derivative formula were recently found by Grushevsky and Salvati Manni [22], who also presented a detailed list of references for the other known generalizations in their paper. We do not discuss here the relevance

of the formulae obtained here to the results [22] but plan to consider this question in a separate publication.

Concluding, we note that the procedures described here, of the derivation of formulae of the form (3.19)–(3.21), work in all cases when the Klein–Weierstrass algebraic representation of the bi-differential  $\Omega(Q, R)$  is known. Therefore, the next cases that could be analyzed are the cases of  $(3, s)$ -trigonal curves,

$$(3.30) \quad y^3 - a_2(x)y^2 - a_1(x)y - a_0(x) = 0$$

with appropriate polynomials  $a_1(x)$  and  $a_0(x)$ . Analytic expressions for the basic meromorphic differentials can be found in [13], [16] whilst expressions for the projective connection  $S_{KW}(P)$  are given in the course of the proof of Proposition 2.1.

#### ACKNOWLEDGEMENTS

Two of the authors, JCE and VZE, are grateful to S. Grushevsky for a stimulating discussion on the possibilities of deriving Thomae-type formulae that express parameters of the curves, such as coefficients of defining polynomial  $\lambda$ , in terms of  $\theta$ -constants during a meeting in ICMS, Edinburgh, in October 2012. The authors are grateful to V. Buchstaber, S. Grushevsky, D. Korotkin, and A. Nakayashiki who agreed to read a draft of this paper before its publication and made remarks that we took into account in the final version. KE and VZE gratefully acknowledge the Deutsche Forschungsgemeinschaft (DFG) for financial support within the framework of the DFG Research Training group 1620 Models of gravity.

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Originally published in English