A MINIMAX METHOD FOR FINDING SADDLE CRITICAL POINTS OF UPPER SEMI-DIFFERENTIABLE LOCALLY LIPSCHITZ CONTINUOUS FUNCTIONAL IN HILBERT SPACE AND ITS CONVERGENCE

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ABSTRACT. A minimax characterization for finding nonsmooth saddle critical points, i.e., saddle critical points of locally Lipschitz continuous functional, in Banach space is presented in [X. Yao and J. Zhou, *A local minimax characterization for computing multiple nonsmooth saddle critical points*, Math. Program., 104 (2005), no. 2-3, Ser. B, 749-760]. By this characterization, a descent-max method is devised. But, there is no numerical experiment and convergence result for the method. In this paper, to a class of locally Lipschitz continuous functionals, a minimax method for computing nonsmooth saddle critical points in Hilbert space will be designed. Numerical experiments will be carried out and convergence results will be established.

1. INTRODUCTION

Let B be a Banach space, B^* its dual space, $\langle , \rangle$ the dual relation, and $\| \cdot \|$ its norm. To a locally Lipschitz continuous functional $J : B \to \mathbb{R}$, the generalized gradient $\partial J(u)$ at $u \in B$ in the sense of Clarke [7] is defined as follows.

Definition 1.1. Let J be Lipschitz continuous near $u_0 \in B$. The generalized directional derivative $J^0(u_0, v)$ of J at u_0 in the direction of $v \in B$ is defined by

$$J^0(u_0, v) = \limsup_{\substack{u \to u_0 \\ t \downarrow 0}} \frac{J(u + tv) - J(u)}{t}.$$

The generalized gradient $\partial J(u_0)$ of J at u_0 is a subset of B^* given by

$$\partial J(u_0) = \{\zeta \in B^* | \langle \zeta, v \rangle \le J^0(u_0, v), \ \forall v \in B\}.$$

To convex functionals, we have the following definition for the subgradient.

Definition 1.2. Let $J : B \to \mathbb{R}$ be convex. The subgradient $\partial J(u_0)$ of J at u_0 is a subset of B^* given by

$$\partial J(u_0) = \{\zeta \in B^* | \langle \zeta, u - u_0 \rangle \le J(u) - J(u_0), \ \forall u \in B\}.$$

For convex functionals, the generalized gradient coincides with the subgradient.

According to Chang [5], a point $u_0 \in B$ is a critical point of a locally Lipschitz continuous functional J if and only if

$$0 \in \partial J(u_0).$$

If J is a C^1 functional, $\partial J(u_0) = \{\nabla J(u_0)\}$, i.e., $0 \in \partial J(u_0)$ becomes $\nabla J(u_0) = 0$, the well-known Euler-Lagrange equation. Critical points of a C^1 functional are called smooth critical points and critical points of a locally Lipschitz continuous functional are called nonsmooth critical points.

The classical optimization theory, nonsmooth analysis and calculus of variation study local maxima or minima. Traditional numerical methods in these areas are for finding local extremum points. Local extremum points are critical points. A critical point u_0 of J, that is not a local extremum point, is a saddle point, i.e., u_0 is a critical point and in any neighborhood $\mathcal{N}(u_0)$ of u_0, there are $v, w \in \mathcal{N}(u_0)$ such that

$$J(v) < J(u_0) < J(w).$$

In physical systems, saddle points appear as unstable equilibria or transient excited states.

The minimax principle, which characterizes a saddle point of J as a solution to

$$\min_{A \in \mathcal{A}} \max_{v \in A} J(v)$$

for some collection $\mathcal{A}$ of subsets A in B, is one of the most popular approaches in critical point theory. For smooth critical points, the Mountain Pass Lemma established in 1973 by Ambrosetti and Rabinowitz [2] set a milestone in contemporary critical point theory. Then, various saddle point theorems and linking theorems were established in the literature to prove existence of multiple critical points for various nonlinear problems; see [3], [12]. For nonsmooth critical points, in 1981, Chang [5] introduced the notion of nonsmooth critical points and obtained a nonsmooth version of the saddle point theorem of Rabinowitz. Kourogenis and Papageorgiou [9] generalized Chang's results and Kandilakis, Kourogenis and Papageorgiou [8] obtained a nonsmooth version of the Linking Theorem. All these saddle point theorems and linking theorems in the literature focus on the existence issue and they are not helpful to devise numerical algorithms for finding saddle critical points. For computing smooth saddle critical points, Li and Zhou [10] established a local minimax characterization of smooth saddle critical points in Hilbert space and designed a minimax algorithm based on the characterization. Then, Yao and Zhou [14] extended the local minimax characterization and the minimax algorithm in Hilbert space to a local minimax characterization and a minimax algorithm in Banach space. These two algorithms were successfully carried out to find smooth saddle critical points and convergence results were established; see [10], [11], [14], [15]. In 2005, Yao and Zhou [13] gave a local minimax characterization for nonsmooth saddle critical points and devised a descent-max method by the characterization. But, there is no numerical experiment and convergence result for the method. In this paper, to a class of locally Lipschitz continuous functionals, the local minimax characterization for nonsmooth saddle critical points in [13] will be reestablished in Hilbert space in another way. Then, a minimax method for finding nonsmooth saddle critical points will be designed. Numerical experiments will be carried out and convergence results will be obtained.

A typical example on application of nonsmooth critical point theory to partial differential equations is the Dirichlet problem,

$$(1.1) \qquad \begin{cases} -\Delta u(x) = f(x, u(x)), \ x \in \Omega, \\ u(x)|_{x \in \partial\Omega} = 0, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f(x,t)$ is a measurable function defined on $\Omega \times \mathbb{R}$ and for every $x \in \Omega$, $f(x,t)$ is locally bounded. The corresponding variational functional is

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x, u(x))dx,$$

where $F(x,t) = \int_0^t f(x,s)ds$. Denote $G(u) = \int_\Omega F(x, u(x))dx$. Then, according to Chang [5], we have the following theorems for the problem (1.1).

Theorem 1.3. *If $f(x,t)$ satisfies*

$$|f(x,t)| \leq C_1 + C_2|t|^\sigma$$

for $x \in \Omega \subset \mathbb{R}^n$ and $t \in \mathbb{R}$, where $0 < \sigma < \frac{n+2}{n-2}$ for $n \geq 3$, $\sigma > 0$ for $n = 1, 2$ and $C_1, C_2 > 0$ are two constants, then, $G(u) = \int_\Omega \int_0^{u(x)} f(x,s)dsdx$ is locally Lipschitz continuous in $L^{\sigma+1}(\Omega)$ and $H_0^1(\Omega)$ as well. In addition, if $\bar{f}(x,t)$ and $\hat{f}(x,t)$ are N-measurable (cf. [4]), where

$$\bar{f}(x,t) = \lim_{\delta \to 0} \text{ess inf}_{|s-t|<\delta} f(x,s)$$

and

$$\hat{f}(x,t) = \lim_{\delta \to 0} \text{ess sup}_{|s-t|<\delta} f(x,s),$$

then

$$\{\zeta(x)|\zeta \in \partial G(u)\} \subseteq [\bar{f}(x, u(x)), \hat{f}(x, u(x))] \ \ a.e.$$

in $L^{\sigma+1}(\Omega)$ and $H_0^1(\Omega)$ as well.

Remark 1.4. It is just for simplicity to assume $\sigma > 0$ for $n = 1, 2$ in Theorem 1.3, Theorem 1.5, Lemma 3.6, and for $m = 1, 2$ in Lemma 4.20, Lemma 4.21.

Theoretically, for the Dirichlet problem (1.1), people will find $u_0 \in H_0^1(\Omega)$ such that

(1.2) $$-\Delta u_0(x) \in [\bar{f}(x, u_0(x)), \hat{f}(x, u_0(x))]$$

for all $x \in \Omega$ [5]. By the inclusion $\{\zeta(x)|\zeta \in \partial G(u)\} \subseteq [\bar{f}(x, u(x)), \hat{f}(x, u(x))]$ in Theorem 1.3, to find such u_0, we can find a critical point of J. But, from a numerical point of view, this inclusion offers little information on $\partial G(u)$ and it is not helpful for computing critical points of J. Also in [5], Chang sharpened the conclusion.

Theorem 1.5. *If $f(x,t)$ is a Baire-measurable function defined on $\Omega \times \mathbb{R}$, is non-decreasing in t, and satisfies*

$$|f(x,t)| \leq C_1 + C_2|t|^\sigma$$

for $x \in \Omega \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where $0 < \sigma < \frac{n+2}{n-2}$ for $n \geq 3$, $\sigma > 0$ for $n = 1, 2$ and $C_1, C_2 > 0$ are two constants, then the functional $G(u) = \int_\Omega \int_0^{u(x)} f(x,s)dsdx$ is convex and

$$\partial G(u) = \{\zeta : \Omega \to \mathbb{R}|\zeta \text{ is measurable, } \zeta(x) \in [f(x, u(x) - 0), f(x, u(x) + 0)] \ \forall x \in \Omega\}$$

in $L^{\sigma+1}(\Omega)$ and $H_0^1(\Omega)$ as well.

Remark 1.6. The conclusions in this theorem and in Theorem 2.3 in [5] look different. If we read the proof of Theorem 2.3 in [5] carefully, it will be found that indeed Chang verified the conclusion in this theorem.

The equality

$$\partial G(u) = \{\zeta : \Omega \to \mathbb{R} | \zeta \text{ is measurable, } \zeta(x) \in [f(x, u(x)-0), f(x, u(x)+0)] \, \forall x \in \Omega\}$$

is much better than the inclusion

$$\{\zeta(x) | \zeta \in \partial G(u)(x)\} \subseteq [\bar{f}(x, u(x)), \hat{f}(x, u(x))].$$

It says that finding $u_0 \in H_0^1(\Omega)$ which satisfies (1.2) is equivalent to finding a critical point of J. On the other hand, this equality offers clear information on $\partial G(u)$ and it is helpful for calculating critical points of J. In Theorem 1.5,

$$J(u) = I(u) - G(u),$$

where $I(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx \in C^1(H_0^1(\Omega), \mathbb{R})$ and $G(u) = \int_\Omega \int_0^{u(x)} f(x, s)dsdx$ is convex on $H_0^1(\Omega)$. In this paper, upper semi-differentiable locally Lipschitz continuous functional will be defined in Banach space B first in Section 2. A locally Lipschitz continuous functional

$$J(u) = I(u) - G(u), \quad \forall u \in B,$$

where $I \in C^1(B, \mathbb{R})$ and $G : B \to \mathbb{R}$ is convex, is an upper semi-differentiable locally Lipschitz continuous functional on B. Then, to upper semi-differentiable locally Lipschitz continuous functionals, the local minimax characterization for nonsmooth saddle critical points in [13] will be reestablished in Hilbert space H and, according to this local minimax characterization, a minimax algorithm for capturing saddle critical points of upper semi-differentiable locally Lipschitz continuous functionals will be presented. In Section 3, the SC-condition will be defined first. Based on the SC-condition, the subsequence and sequence convergence for the minimax algorithm will be established. This is the first time that convergence results are obtained for a minimax algorithm to capture nonsmooth saddle points. In Section 4, this minimax algorithm will be implemented to solve numerical examples.

At the end of this section, let us recall some simple properties of the generalized directional derivative $J^0(u, v)$ and the generalized gradient $\partial J(u)$ to locally Lipschitz continuous functional J.

Proposition 1.7 ([7])**.** *Assume that J, W are locally Lipschitz continuous in B.*

(a) *For every $u \in B$, $\partial J(u)$ is a nonempty, convex and w^*-compact subset of B^*.*

(b) *If $|J(w) - J(v)| \le K\|w - v\|$ for all w, v in a neighborhood of $u \in B$, then $\partial J(u)$ is a bounded set in B^* with bound K.*

(c) *$\partial(J + W)(u) \subseteq \partial J(u) + \partial W(u)$ for every $u \in B$ and for $\lambda \in \mathbb{R}$, $\partial(\lambda J)(u) = \lambda \partial J(u)$ for every $u \in B$.*

(d) *If $u \in B$ is a local minimum or maximum point of J, then $0 \in \partial J(u)$.*

(e) *(Chain Rule) Let X be a Banach space. If $L : X \to B$ is strictly differential at v, i.e., there is a $D_s L(v) \in \mathcal{L}(X, B)$ such that for each $w \in X$,*

$$\lim_{\substack{v' \to v \\ t \downarrow 0}} \frac{L(v' + tw) - L(v')}{t} = \langle D_s L(v), w \rangle$$

and the convergence is uniform for w in compact sets, then $F = J \circ L$ has

$$\partial F(v) \subset \partial J(L(v)) \circ D_s L(v).$$

Equality holds if L maps every neighborhood of v to a set which is dense in a neighborhood of $L(v)$.

(f) $J^0(u, v) : B \times B \to \mathbb{R}$ *is upper semicontinuous.*

2. A MINIMAX METHOD IN HILBERT SPACE

Let H be a Hilbert space, $\langle , \rangle$ the inner product, $\| \cdot \|$ the norm introduced by the inner product $\langle , \rangle$ and L a closed subspace of H. Denote $[L, v]$ as the subspace spanned by L and $v \in H$ and $[u_1, u_2, ..., u_m]$ as the subspace spanned by $u_i \in H$, $i = 1, 2, ..., m$. For any subspace H' of H, $S_{H'}$ is the unit sphere in H' and for any closed subspace M of H, $H = M \oplus M^\perp$ is the orthogonal decomposition of H and $M^\perp$ is the orthogonal complement of M in H. For any $v \in H$ and $U \subset H$, $d(v, U) = \inf_{u \in U} \|v - u\|$, i.e., the distance between v and U.

To design a minimax algorithm, the following peak selection is important.

Definition 2.1. A set-valued mapping $P : S_{L^\perp} \to 2^H$ is the peak mapping of a locally Lipschitz continuous functional $J : H \to \mathbb{R}$ w.r.t. L if $\forall v \in S_{L^\perp}, P(v) = \{u \in [L, v] : u$ is a local maximum point of J in$[L, v]\}$. A single-valued mapping $p : S_{L^\perp} \to H$ is a peak selection of the locally Lipschitz continuous functional J w.r.t. L if $p(v) \in P(v)$, $\forall v \in S_{L^\perp}$. For a given $v \in S_{L^\perp}$, if p is locally defined in a neighborhood of v, we say that J has a local peak selection p at v.

The peak selections have the following property.

Lemma 2.2. *Assume that $L \subset H$ is a finite dimensional space, $v \in S_{L^\perp}$ and p is a local peak selection of a locally Lipschitz continuous functional J w.r.t. L at v. Then, there is $z \in \partial J(p(v))$ such that $z \perp [L, v]$.*

Proof. Since L is a finite dimension space, assume that $u_1, u_2, ..., u_m$ is a basis. Consider the composite functional

$$F(t_0, t_1, ..., t_m) = J(t_0 v + t_1 u_1 + \cdots + t_m u_m).$$

According to (e) in Proposition 1.7, we have

$$\partial F(t_0, t_1, ..., t_m) = \{(\langle \zeta, v \rangle, \langle \zeta, u_1 \rangle, ..., \langle \zeta, u_m \rangle) | \zeta \in \partial J(t_0 v + t_1 u_1 + \cdots + t_m u_m)\}.$$

Since p is a peak selection, by (d) in Proposition 1.7, $0 \in \partial F(t_0^v, t_1^v, ..., t_m^v)$, where $p(v) = t_0^v v + t_1^v u_1 + \cdots + t_m^v u_m$, i.e., there is $z \in \partial J(p(v))$ such that

$$\langle z, v \rangle = 0 \text{ and } \langle z, u_i \rangle = 0, \quad i = 1, ..., m,$$

i.e., $z \perp [L, v]$. $\qquad \square$

Remark 2.3. This conclusion is simple and important. To upper semi-differentiable locally Lipschitz continuous functionals which will be defined later, it will be used to reestablish the local minimax characterization for nonsmooth saddle critical points in [13] in Hilbert space. On the other hand, if $J \in C^1(H, \mathbb{R})$, $\partial J(p(v)) = \{\nabla J(p(v))\}$ and $\nabla J(p(v)) \perp [L, v]$ are obviously true.

We give two simple lemmas which will be used later.

Lemma 2.4. *In Banach space B, for any point v with $\|v\| = 1$, it holds that*

$$\|v - \frac{v + w}{\|v + w\|}\| \leq \frac{2\|w\|}{\|v + w\|}, \quad \forall w \in B.$$

Proof. Since $\|v\| = 1$, we have

$$
\begin{aligned}
\|v - \frac{v+w}{\|v+w\|}\| &= \|\frac{(\|v+w\|-1)v+w}{\|v+w\|}\| \le \frac{\|\|v+w\|-1\|\|v\|+\|w\|}{\|v+w\|} \\
&= \frac{\|\|v+w\|-\|v\|\|+\|w\|}{\|v+w\|} \le \frac{2\|w\|}{\|v+w\|}. \qquad \square
\end{aligned}
$$

Lemma 2.5. *For*

$$
J(u) = I(u) - G(u), \ \ \forall u \in B,
$$

where $I \in C^1(B, \mathbb{R})$ and $G : B \to \mathbb{R}$ is locally Lipschitz continuous on Banach space B,

$$
\partial J(u) = \{\nabla I(u) - g | g \in \partial G(u)\},
$$

where ∇I is the gradient of I.

Proof. By (c) in Proposition 1.7,

$$
\partial J(u) \subseteq \partial I(u) + \partial(-G)(u) = \{\nabla I(u)\} - \partial G(u),
$$

i.e.,

$$
\partial J(u) \subseteq \{\nabla I(u) - g | g \in \partial G(u)\}.
$$

On the other hand, by (c) in Proposition 1.7 again,

$$
\partial G(u) \subseteq \partial I(u) + \partial(-J)(u) = \{\nabla I(u)\} - \partial J(u).
$$

This means that $\nabla I(u) - g_0 \in \partial J(u)$ for every $g_0 \in \partial G(u)$. Then,

$$
\{\nabla I(u) - g_0 | g_0 \in \partial G(u)\} \subseteq \partial J(u).
$$

Hence,

$$
\partial J(u) = \{\nabla I(u) - g | g \in \partial G(u)\}. \qquad \square
$$

In this section, we will establish a minimax characterization for upper semi-differentiable locally Lipschitz continuous functionals. First, we define upper and lower semi-differentiable locally Lipschitz continuous functionals.

Definition 2.6. In Banach space B, a locally Lipschitz continuous functional J is an upper semi-differentiable locally Lipschitz continuous functional if, for every $u \in B$, $z \in \partial J(u)$ implies, for w around u,

$$
J(w) - J(u) \le \langle z, w - u \rangle + F(w; u, z),
$$

where $F(w; u, z) = o(\|w - u\|)$ is an upper-bound functional around u and a locally Lipschitz continuous functional I is a lower semi-differentiable locally Lipschitz continuous functional if, for every $u \in B$, $z \in \partial I(u)$ implies, for w around u,

$$
I(w) - I(u) \ge \langle z, w - u \rangle + G(w; u, z),
$$

where $G(w; u, z) = o(\|w - u\|)$ is a lower-bound functional around u and $\langle, \rangle$ is the dual relation between B and its dual space B^*.

Remark 2.7. **(a)** If $J : B \to \mathbb{R}$ is Fréchet differentiable, then, to every $u \in B$, $z \in \partial J(u) = \{\nabla J(u)\}$ implies

$$
J(w) - J(u) = \langle z, w - u \rangle + o(\|w - u\|)
$$

for w around u, i.e.,

$$
(2.1) \qquad J(w) - J(u) \le \langle z, w - u \rangle + o(\|w - u\|)
$$

for w around u, and

$$
(2.2) \qquad J(w) - J(u) \ge \langle z, w - u \rangle + o(\|w - u\|)
$$

for w around u. To a locally Lipschitz continuous functional $J : B \to \mathbb{R}$, if, for every $u \in B$, $z \in \partial J(u)$ implies both (2.1) and (2.2), then J is Fréchet differentiable. Hence, a locally Lipschitz continuous functional $J : B \to \mathbb{R}$ can be considered as a semi-differentiable locally Lipschitz continuous functional if, for every $u \in B$, $z \in \partial J(u)$ implies (2.1) or for every $u \in B$, $z \in \partial J(u)$ implies (2.2).

(b) If $J \in C^1(B, \mathbb{R})$, J is an upper semi-differentiable locally Lipschitz continuous functional and also a lower semi-differentiable locally Lipschitz continuous functional.

The following lemma gives us an important class of upper semi-differentiable locally Lipschitz continuous functionals.

Lemma 2.8. *A locally Lipschitz continuous functional*

$$J(u) = I(u) - G(u), \ \ \forall u \in B,$$

where $I \in C^1(B, \mathbb{R})$ and $G : B \to \mathbb{R}$ is convex on Banach space B, is an upper semi-differentiable locally Lipschitz continuous functional on B.

Proof. Since $I \in C^1(B, \mathbb{R})$, to every $u \in B$,

$$I(w) - I(u) = \langle \nabla I(u), w - u \rangle + o(\|w - u\|)$$

for w around u and since G is convex in B, to every $u \in B$ and $g \in \partial G(u)$,

$$G(w) - G(u) \geq \langle g, w - u \rangle$$

for $w \in B$. By Lemma 2.5, for $z \in \partial J(u)$, there is $g_0 \in \partial G(u)$ such that $z = \nabla I(u) - g_0$. Thus,

$$
\begin{aligned}
J(w) - J(u) &= (I(w) - I(u)) - (G(w) - G(u)) \\
&\leq \ \langle \nabla I(u) - g_0, w - u \rangle + o(\|w - u\|) = \langle z, w - u \rangle + o(\|w - u\|)
\end{aligned}
$$

for w around u, i.e., J is a upper semi-differentiable locally Lipschitz continuous functional on B. $\square$

Remark 2.9. According to the verification of the lemma, it is clear that

$$F(w; u, z) = I(w) - I(u) - \langle \nabla I(u), w - u \rangle = o(\|w - u\|)$$

is an upper-bound functional of J around u.

To simplify the statement of mathematical justification and convergence results for our minimax algorithm, we need the definition for super-linear property of peak selection.

Definition 2.10. Let J be an upper semi-differentiable locally Lipschitz continuous functional on Hilbert space H and $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$. If p is a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ such that p is continuous at v and, for every $z \in \partial J(p(v)) \cap [L, v]^\perp$, there is an upper-bound functional $F(u; p(v), z)$ of J around $p(v)$ satisfying

$$\lim_{w \to v} \frac{|F(p(w); p(v), z)|}{\|w - v\|} = 0,$$

we say that the peak selection p has super-linear property at v. If p is a peak selection of J w.r.t. L defined on an open set $V \subset H$ and p has super-linear property at every $v \in V$, we say that the peak selection p has super-linear property in V.

Remark 2.11. **(a)** Theoretically, we can give a general definition by using "for a $z \in \partial J(p(v)) \cap [L, v]^\perp$" instead of "for every $z \in \partial J(p(v)) \cap [L, v]^\perp$". If we choose that special z in the general definition as z in Lemma 2.14, then, we still have Lemma 2.14 and Theorem 2.16. We can still use Lemma 2.14 as step-size rule to design a minimax algorithm. But, numerically, we prefer "for every $z \in \partial J(p(v)) \cap [L, v]^\perp$" since it is hard to find that special z in numerical computation. On the other hand, for establishing convergence results, i.e., Theorem 3.9, Theorem 3.12 and Corollary 3.14, we also prefer "for every $z \in \partial J(p(v)) \cap [L, v]^\perp$".

(b) Since $F(p(w); p(v), z) = o(\|p(w) - p(v)\|)$, the Lipschitz continuity of p around v, i.e.,

$$\|p(w) - p(v)\| \le l\|w - v\|$$

for w around v, where $l > 0$ is a constant, means that the peak selection p has super-linear property at v. Indeed, usually the Hölder continuity of p around v, i.e.,

$$\|p(w) - p(v)\| \le l\|w - v\|^\alpha$$

for w around v, where $l > 0$ is a constant and $0 < \alpha \le 1$ is close to 1, will guarantee that the peak selection p has super-linear property at v.

For $J \in C^1(H, \mathbb{R})$, we will prove that continuity of peak selection p of J w.r.t. a finite dimensional subspace $L \subset H$ at $v \in S_{L^\perp}$ implies super-linear property of p at v. First, we verify a simple and useful lemma.

Lemma 2.12. *Assume that H is a Hilbert space and $v \in H$ such that $\|v\| = 1$. Denote $M = \{tv | t \in \mathbb{R}\}$. Then, for every $u \in H$ with $\|u\| = 1$ and $\langle u, v \rangle > 0$, there is unique $d \in M^\perp$ with $\|d\| = 1$ and $s \ge 0$ such that*

$$u = \frac{1}{\sqrt{1 + s^2}}(v - sd).$$

Proof. Since H is a Hilbert space, there is $w \in M^\perp$ with $\|w\| = 1$ such that

$$(2.3) \qquad\qquad\qquad u = c_1 v + c_2 w.$$

Then, by $\|u\| = \|v\| = \|w\| = 1$ and $w \in M^\perp$, we know

$$(2.4) \qquad\qquad\qquad c_1^2 + c_2^2 = 1.$$

On the other hand, $\langle u, v \rangle > 0$, $\|v\| = 1$ and $w \in M^\perp$ mean $c_1 = \frac{\langle u, v \rangle}{\|v\|^2} = \langle u, v \rangle > 0$. Thus, there is $s \ge 0$ such that $c_1 = \frac{1}{\sqrt{1 + s^2}}$. If we set

$$d = \begin{cases} w, & \text{if } c_2 \le 0, \\ -w, & \text{if } c_2 > 0, \end{cases}$$

then, by (2.3) and (2.4),

$$u = \frac{1}{\sqrt{1 + s^2}}(v - sd).$$

Since $c_2 w$ is unique and $c_2 = \langle u, w \rangle$, d is unique. $\qquad\square$

By Lemma 2.12, we establish a relation between continuity and super-linear property of peak selection of $J \in C^1(H, \mathbb{R})$ as follows.

Lemma 2.13. *Assume that H is a Hilbert space, $J \in C^1(H, \mathbb{R})$, $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$ and p is a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$. If p is continuous at v, then p has super-linear property at v.*

Proof. In Hilbert space $L^\perp$, if $w \in S_{L^\perp}$ and $\|w - v\| < 1$, then

$$1 > \|w - v\|^2 = \|w\|^2 + \|v\|^2 - 2\langle w, v \rangle = 2 - 2\langle w, v \rangle, \quad i.e., \quad \langle w, v \rangle > \frac{1}{2}.$$

Thus, by Lemma 2.12, for every $w \in S_{L^\perp}$ with $\|w - v\| < 1$, there is $s \geq 0$ and $d \in [L, v]^\perp$ such that $\|d\| = 1$ and

$$w = \frac{1}{\sqrt{1 + s^2}}(v + sd).$$

Hence, for small $s > 0$,

$$J(p(w)) - J(p(v)) = J(p(\frac{v + sd}{\|v + sd\|})) - J(p(v))$$

$$\leq \quad J(p(\frac{v + sd}{\|v + sd\|})) - J(\frac{t_w}{\|v + sd\|}v + w_L) = \frac{t_w s}{\|v + sd\|}\langle \nabla J(\eta), d \rangle$$

$$= \quad \frac{t_w s}{\|v + sd\|}(\langle \nabla J(p(v)), d \rangle + \langle \nabla J(\eta) - \nabla J(p(v)), d \rangle)$$

$$= \quad \langle \nabla J(p(v)), p(w) - p(v) \rangle + \frac{t_w s}{\|v + sd\|}\langle \nabla J(\eta) - \nabla J(p(v)), d \rangle,$$

where $p(w) = t_w \frac{v + sd}{\|v + sd\|} + w_L$, $w_L \in L$, $\eta = \lambda p(w) + (1 - \lambda)(\frac{t_w}{\|v + sd\|}v + w_L)$ and $\lambda \in [0, 1]$, i.e.,

$$(2.5) \qquad |F(p(w); p(v), \nabla J(p(v)))| \leq \frac{|t_w|s}{\|v + sd\|}|\langle \nabla J(\eta) - \nabla J(p(v)), d \rangle|,$$

where

$$F(u; p(v), \nabla J(p(v)))$$
$$= \quad \begin{cases} D(u; p(v), \nabla J(p(v))), & \text{if } D(u; p(v), \nabla J(p(v))) > 0, \\ 0, & \text{if } D(u; p(v), \nabla J(p(v))) \leq 0 \end{cases}$$

is an upper-bound functional of J around $p(v)$ and

$$D(u; p(v), \nabla J(p(v))) = J(u) - J(p(v)) - \langle \nabla J(p(v)), u - p(v) \rangle.$$

On the other hand,

$$\|w - v\| = \|\frac{1}{\sqrt{1 + s^2}}(v + sd) - v\| = \|(\frac{1}{\sqrt{1 + s^2}} - 1)v + \frac{s}{\sqrt{1 + s^2}}d\|$$

$$= \quad \frac{1}{\sqrt{1 + s^2}}\|(1 - \sqrt{1 + s^2})v + sd\| = \frac{s}{\sqrt{1 + s^2}}\|\frac{s}{1 + \sqrt{1 + s^2}}v - d\|$$

$$= \quad \frac{s}{\sqrt{1 + s^2}}\sqrt{1 + \frac{s^2}{(1 + \sqrt{1 + s^2})^2}},$$

i.e.,

$$(2.6) \qquad\qquad\qquad \lim_{w - v} \frac{\|w - v\|}{s} = 1.$$

Hence, by (2.5) and (2.6),

$$\lim_{w \to v} \frac{|F(p(w); p(v), \nabla J(p(v)))|}{\|w - v\|} = 0,$$

i.e., p has super-linear property at v. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By peak selection, the following lemma can be verified. The conclusion will be used as a step-size rule for our minimax algorithm.

Lemma 2.14. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$. Assume that J is a upper semi-differentiable locally Lipschitz continuous functional and p is a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ such that $p(v)$ is not a critical point, p has super-linear property at v and $d(p(v), L) > 0$. Choose $z \in \partial J(p(v)) \cap [L, v]^\perp$. Denote $p(v) = t_v v + w_v$, where $w_v \in L$ and $w = -\mathrm{sign}(t_v)\frac{z}{\|z\|}$. Then, as $s > 0$ is small,*

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|,$$

where $v(s) = \dfrac{v + sw}{\|v + sw\|} = \dfrac{1}{\sqrt{1 + s^2}}(v + sw)$.

Proof. Since J is a upper semi-differentiable locally Lipschitz continuous functional and p is continuous at v, we have

$$J(p(v(s))) - J(p(v)) \leq \langle z, p(v(s)) - p(v)\rangle + F(p(v(s)); p(v), z)$$

for small $s > 0$, where $F(p(v(s)); p(v), z) = o(\|p(v(s)) - p(v)\|)$ is any upper-bound functional around $p(v)$. On the other hand, since $z \perp [L, v]$,

$$\langle z, p(v)\rangle = \langle z, t_v v + w_v\rangle = 0$$

and

$$\begin{aligned}
\langle z, p(v(s))\rangle &= \langle z, t_s v(s) + w_s\rangle \\
&= \langle z, t_s \frac{v + sw}{\|v + sw\|} + w_s\rangle = -\frac{\mathrm{sign}(t_v) t_s s\|z\|}{\|v + sw\|},
\end{aligned}$$

where $p(v(s)) = t_s v(s) + w_s$ and $w_s \in L$. According to the continuity of p at v and $d(p(v), L) > 0$,

$$\mathrm{sign}(t_v) = \mathrm{sign}(t_s)$$

for small $s > 0$. Since $\|v\| = 1$,

$$\|v + sw\| \leq 2$$

for small $s > 0$. Thus, we have

$$\begin{aligned}
J(p(v(s))) - J(p(v)) &\leq -\frac{\mathrm{sign}(t_v) t_s s\|z\|}{\|v + sw\|} + F(p(v(s)); p(v), z) \\
&\leq -\frac{1}{2}s|t_v|\|z\| + F(p(v(s)); p(v), z)
\end{aligned}$$

for small $s > 0$. By super-linear property of p at v, there is a upper-bound functional $F_0(p(v(s)); p(v), z)$ around $p(v)$ such that

$$\lim_{v(s)\to v} \frac{|F_0(p(v(s)); p(v), z)|}{\|v(s) - v\|} = 0.$$

Then, by (2.6),

$$\lim_{s\to 0} \frac{|F_0(p(v(s)); p(v), z)|}{s} = 0.$$

Therefore, since $p(v)$ is not a critical point and $d(p(v), L) > 0$, as $s > 0$ is small,

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|. \qquad \square$$

Remark 2.15. **(a)** The existence of $z \in \partial J(p(v)) \cap [L, v]^\perp$ is guaranteed by Lemma 2.2.

(b) Comparing Lemma 2.14 with Lemma 3 in [13], the difference is different elements of generalized gradient are used to decide descent direction. In this lemma, $z \in \partial J(p(v))$ with $z \perp [L, v]$ is used. In Lemma 3 in [13], $z \in \partial J(p(v))$ with $\|z\| = \min\{\|\zeta\| | \zeta \in \partial J(p(v))\}$ is used. If p is continuous at $v \in S_{L^\perp}$, for a sequence of generalized gradients $\{z_n\}$ such that $z_n \in \partial J(p(v_n)) \cap [L, v_n]^\perp$, $\{v_n\} \subset S_{L^\perp}$, $z_n \to z$ and $v_n \to v$, we have $z \in \partial J(p(v)) \cap [L, v]^\perp$. This property is crucial to the convergence of our minimax algorithm. For a sequence of generalized gradients $\{z_n\}$ such that $\|z_n\| = \min\{\|\zeta\| | \zeta \in \partial J(p(v_n))\}$, $\{v_n\} \subset S_{L^\perp}$, $z_n \to z$ and $v_n \to v$, we do have $z \in \partial J(p(v))$ under the assumption that p is continuous at $v \in S_{L^\perp}$. But, we do not know if $\|z\| = \min\{\|\zeta\| | \zeta \in \partial J(p(v))\}$.

(c) Comparing this lemma with Lemma 2.1 in [10], we assume super-linear property of p at v here instead of continuity of p at v in Lemma 2.1 in [10]. By Lemma 2.13, if $J \in C^1(H, \mathbb{R})$, these two assumptions are equivalent.

By Lemma 2.14, a local minimax characterization for nonsmooth saddle critical points in [13] can be reestablished to upper semi-differentiable locally Lipschitz continuous functionals as follows.

Theorem 2.16. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$. Assume that J is a upper semi-differentiable locally Lipschitz continuous functional and p is a local peak selection of J w.r.t. L at $v \in S_{L^\perp}$ such that*
  (a) *p has super-linear property at v and $d(p(v), L) > 0$, and*
  (b) *v is a local minimum point of $J(p(\cdot))$ on $S_{L^\perp}$.*
Then, $p(v)$ is a critical point of J.

Proof. If $p(v)$ is not a critical point of J, then by Lemma 2.14, as $s > 0$ is small,

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4} s |t_v| \|z\|,$$

where $v(s) = \dfrac{v + sw}{\|v + sw\|}$, $w = -\text{sign}(t_v)\dfrac{z}{\|z\|}$, $z \in \partial J(p(v)) \cap [L, v]^\perp$, $p(v) = t_v v + w_v$ and $w_v \in L$. It is a contradiction to assumption (b). $\qquad\square$

Remark 2.17. **(a)** Suppose that a critical point u is characterized by Theorem 2.16. Then, $u = p(v)$ means, for any neighborhood $\mathcal{N}(u)$ of u, there is $u_1 \in [L, v]$ such that $u_1 \in \mathcal{N}(u)$ and $J(u_1) < J(u)$ except that J is a constant functional around u on $[L, v]$, and, $u = p(v)$ and v is a local minimum point of $J(p(\cdot))$ on $S_{L^\perp}$ mean, in any neighborhood $\mathcal{N}(u)$ of u, there is $v_2 \in S_{L^\perp}$ around v such that $u_2 = p(v_2) \in \mathcal{N}(u)$ and $J(u_2) = J(p(v_2)) > J(p(v)) = J(u)$ except that $J(p(\cdot))$ is a constant functional around v on $S_{L^\perp}$. Therefore, except extreme cases, in any neighborhood $\mathcal{N}(u)$ of $u = p(v)$, there are $u_1, u_2 \in \mathcal{N}(u)$ such that $J(u_1) < J(u) < J(u_2)$, i.e., u is a saddle point of J.

(b) Comparing this theorem with Theorem 2.1 in [10], we assume super-linear property of p at v in this theorem instead of continuity of p at v in Theorem 2.1 in [10]. By Lemma 2.13, if $J \in C^1(H, \mathbb{R})$, these two assumptions are equivalent.

According to Lemma 2.14 and Theorem 2.16, we can design a minimax algorithm for capturing saddle critical points of an upper semi-differentiable locally Lipschitz continuous functional J. Assume that $u_1, u_2, ..., u_{n-1}$ are found critical points,

$L = [u_1, u_2, ..., u_{n-1}]$, $\lambda > 0$ is a constant and ε is a small positive number. The following is the flow chart for the algorithm.

Step 1. Choose $v_n^1 \in S_{L^\perp}$.

Step 2. Set $k = 1$ and solve for

$$u_n^k = p(v_n^k) = t_0^k v_n^k + t_1^k u_1 + \cdots + t_{n-1}^k u_{n-1}$$
$$= \arg\max \Big\{ J(t_0 v_n^k + t_1 u_1 + \cdots + t_{n-1} u_{n-1}) | t_i \in \mathbb{R}, i = 0, 1, ..., n-1 \Big\}.$$

Step 3. Find a descent direction $w_n^k = -\text{sign}(t_0^k) \frac{z_n^k}{\|z_n^k\|}$ at u_n^k, where $z_n^k \in \partial J(u_n^k) \cap [L, v_n^k]^\perp$.

Step 4. If $\|z_n^k\| < \varepsilon$, then output u_n^k, stop. Otherwise, do Step 5.

Step 5. For each $s > 0$, let $v_n^k(s) = \dfrac{v_n^k + s w_n^k}{\|v_n^k + s w_n^k\|}$ and use initial point $(t_0^k, t_1^k, ..., t_{n-1}^k)$ to solve for

$$p(v_n^k(s)) = \arg\max \Big\{ J(t_0 v_n^k(s) + \sum_{i=1}^{n-1} t_i u_i) | t_i \in \mathbb{R}, i = 0, 1, ..., n-1 \Big\},$$

then set $v_n^{k+1} = v_n^k(s_n^k)$ and $u_n^{k+1} = p(v_n^{k+1}) = t_0^{k+1} v_n^{k+1} + t_1^{k+1} u_1 + \cdots + t_{n-1}^{k+1} u_{n-1}$, where s_n^k satisfies

$$s_n^k = \max \Big\{ s = \frac{\lambda}{2^m} | m \in \mathbb{N}, J(p(v_n^k(s))) - J(p(v_n^k)) \leq -\frac{1}{4} |t_0^k| s \|z_n^k\| \Big\}.$$

Step 6. Update $k = k + 1$ and go to Step 3.

Remark 2.18. **(a)** In Step 1, we would like to choose $v_n^1 \in S_{L^\perp}$ such that it is an increasing-decreasing direction at u_{n-1}, i.e., $\phi(t) = J(u_{n-1} + t v_n^1)$ increases on $[0, \bar{t}]$ and decreases on $[\bar{t}, \hat{t}]$. Then, a local maximum point $\bar{t}$ of $\phi(t)$ can be used to construct $(t_0, t_1, ..., t_{n-2}, t_{n-1}) = (\bar{t}, 0, ..., 0, 1)$ as an initial point to calculate u_n^1.

(b) In Step 5, we usually set $0 < \lambda < 1$ to prevent the stepsize from being too large to lose search stability.

(c) Suppose that $u = 0$ is a found critical point (usually a minimum point) of J. Then, set $L = \{0\}$ and use the above algorithm to find a new critical point. Denote u_1 as a new critical point. Then, set $L = [0, u_1] = [u_1]$ and carry out the above algorithm to calculate a new critical point, u_2. Thus, set $L = [u_1, u_2]$ and implement the above algorithm to compute a new critical point, u_3, and so on. That is the usual way to use this minimax algorithm. If $u_0 \not\equiv 0$ is a found critical point (usually a minimum point) of J, then, $I(u) = J(u + u_0)$ has critical point $u = 0$. The above algorithm will be implemented to I to find critical points of J.

(d) In Step 5, from $m = 1$, we solve an n-dimensional optimization problem to get $p(v_n^k(s))$ and check if

$$(2.7) \qquad\qquad J(p(v_n^k(s))) - J(p(v_n^k)) \leq -\frac{1}{4} |t_0^k| s \|z_n^k\|$$

for $s = \frac{\lambda}{2^m}$, $m \in N$, one by one. As soon as we get $m_n^k \in N$ such that (2.7) holds for $s = \frac{\lambda}{2^{m_n^k}}$, we set $s_n^k = \frac{\lambda}{2^{m_n^k}}$, $v_n^{k+1} = v_n^k(s_n^k)$ and $u_n^{k+1} = p(v_n^{k+1})$. When n becomes larger, generally we have to do more calculation to solve the n-dimensional optimization problem for $p(v_n^k(s))$.

(e) Numerical experiments show that generally the iteration number in k will grow with n.

3. Convergence

Before establishing convergence for the minimax algorithm, we generalize it to an L-$\perp$ algorithm. First, L-$\perp$ selection should be defined to a locally Lipschitz continuous functional J and a closed subspace L in Hilbert space H.

Definition 3.1. A set-valued mapping $P : S_{L^\perp} \to 2^H$ is the L-$\perp$ mapping of a locally Lipschitz continuous functional J w.r.t. L if $\forall v \in S_{L^\perp}, P(v) = \{u \in [L, v] : \partial J(u) \cap [L, v]^\perp \neq \emptyset\}$. A single-valued mapping $p : S_{L^\perp} \to H$ is an L-$\perp$ selection of the locally Lipschitz continuous functional J w.r.t. L if $p(v) \in P(v)$, $\forall v \in S_{L^\perp}$. For a given $v \in S_{L^\perp}$, if p is locally defined in a neighborhood of v, we say that J has a local L-$\perp$ selection p at v.

Remark 3.2. If $L \subset H$ is a finite dimensional subspace, by Lemma 2.2, it is clear that a peak selection w.r.t. L is an L-$\perp$ selection w.r.t. L.

Then, it is easy to check that Lemma 2.14 and Theorem 2.16 are also correct to an L-$\perp$ selection p. By this generalization, an L-$\perp$ algorithm for capturing saddle critical points of a upper semi-differentiable locally Lipschitz continuous functional J can be designed. Assume that $u_1, u_2, ..., u_{n-1}$ are found critical points, $L = [u_1, u_2, ..., u_{n-1}, \lambda > 0$ is a constant and ε is a small positive number. The flow chart for the algorithm is given as follows.

Step 1. Choose $v_n^1 \in S_{L^\perp}$.
Step 2. Set $k = 1$ and find $u_n^k = p(v_n^k) = t_0^k v_n^k + t_1^k u_1 + \cdots + t_{n-1}^k u_{n-1}$ such that
$$\partial J(u_n^k) \cap [L, v_n^k]^\perp \neq \emptyset.$$

Step 3. Find a descent direction $w_n^k = -\text{sign}(t_0^k)\frac{z_n^k}{\|z_n^k\|}$ at u_n^k, where $z_n^k \in \partial J(u_n^k) \cap [L, v_n^k]^\perp$.
Step 4. If $\|z_n^k\| < \varepsilon$, then output u_n^k, stop. Otherwise, do Step 5.
Step 5. For each $s > 0$, let $v_n^k(s) = \dfrac{v_n^k + s w_n^k}{\|v_n^k + s w_n^k\|}$ and find $p(v_n^k(s)) = t_0(s)v_n^k(s) + \sum_{i=1}^{n-1} t_i(s)u_i$ such that
$$\partial J(p(v_n^k(s))) \cap [L, v_n^k(s)]^\perp \neq \emptyset,$$
then set $v_n^{k+1} = v_n^k(s_n^k)$ and $u_n^{k+1} = p(v_n^{k+1}) = t_0^{k+1}v_n^{k+1} + t_1^{k+1}u_1 + \cdots + t_{n-1}^{k+1}u_{n-1}$, where s_n^k satisfies
$$s_n^k = \max\left\{s = \frac{\lambda}{2^m} | m \in \mathbb{N}, J(p(v_n^k(s))) - J(p(v_n^k)) \leq -\frac{1}{4}|t_0^k|s\|z_n^k\|\right\}.$$

Step 6. Update $k = k + 1$ and go to Step 3.

To discuss convergence of the L-$\perp$ algorithm, we need the following nonsmooth version of PS condition [5] and SC-condition in Banach space B.

Definition 3.3. A locally Lipschitz continuous functional $J : B \to \mathbb{R}$ satisfies the PS condition if any $\{u_n\} \subset B$ such that $\{J(u_n)\}$ is bounded and $\zeta_n \to 0$, where $\zeta_n \in \partial J(u_n)$ and $\|\zeta_n\| = \min\{\|\zeta\| | \zeta \in \partial J(u_n)\}$, possesses a convergent subsequence.

Definition 3.4. A locally Lipschitz continuous functional $J : B \to \mathbb{R}$ satisfies the SC-condition if any $\{\zeta_n | \zeta_n \in \partial J(u_n)\}$, where $u_n \to u$, possesses a subsequence $\{\zeta_{n_i}\}$ such that $\zeta_{n_i} \to \zeta \in \partial J(u)$.

Remark 3.5. Since the algorithms are for upper semi-differentiable locally Lipschitz continuous functional J, generalized gradient is used instead of gradient. Then, the continuity of gradient to functional $J \in C^1(B, \mathbb{R})$ is lost. The SC-condition is defined instead for generalized gradient of locally Lipschitz continuous functional. It may be considered as a "subsequence-continuity-condition". The SC-condition is crucial to establish subsequence and sequence convergence for the algorithm. On the other hand, for functional $J \in C^1(B, \mathbb{R})$, the SC-condition is automatically satisfied.

For the SC-condition, the following lemma can be verified.

Lemma 3.6. *If $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $f(x, t)$ is a measurable function on $\Omega \times \mathbb{R}^n$ and $f(x, t)$ satisfies*

$$|f(x, t)| \le C_1 + C_2|t|^\sigma,$$

for $x \in \Omega$ and $t \in \mathbb{R}$, where $0 < \sigma < \frac{n+2}{n-2}$ for $n \ge 3$, $\sigma > 0$ for $n = 1, 2$ and $C_1, C_2 > 0$ are two constants, then the locally Lipschitz continuous functional $G(u) = \int_\Omega \int_0^{u(x)} f(x, s) ds dx$ satisfies the SC-condition in $H_0^1(\Omega)$.

Proof. First, according to Chang [5], $G(u) = \int_\Omega \int_0^{u(x)} f(x, s) ds dx$ is locally Lipschitz continuous in $L^{\sigma+1}(\Omega)$ and $H_0^1(\Omega)$ as well. Suppose that $u_n \to u$ in $H_0^1(\Omega)$. By the Sobolev embedding theorem, $u_n \to u$ in $L^{\sigma+1}(\Omega)$. By Theorem 2.2 in [5] and (b) in Proposition 1.7, $\{\zeta_n | \zeta_n \in \partial J(u_n)\}$ is bounded in $L^\eta(\Omega)$, where $\eta = \frac{\sigma+1}{\sigma}$. Since $L^\eta(\Omega)$ is reflexive, $\{\zeta_n\}$ has a subsequence which is weakly convergent to $\zeta \in \partial J(u)$. We assume that $\zeta_n \to \zeta$ weakly in $L^\eta(\Omega)$. Then, $\{\zeta_n\}$ must have a subsequence which is convergent to ζ in $H^{-1}(\Omega)$. Otherwise, there is $\epsilon > 0$ such that $\|\zeta_n - \zeta\|_{H^{-1}(\Omega)} \ge \epsilon$. This means there is $\{\xi_n\}$, where $\xi_n \in H_0^1(\Omega)$ and $\|\xi_n\|_{H_0^1(\Omega)} = 1$, and $\rho > 0$ such that

$$(3.1) \qquad \int_\Omega (\zeta_n(x) - \zeta(x))\xi_n(x) dx \ge \rho.$$

By the Rellich-Kondrachov theorem (cf. [1]), $\{\xi_n\}$ has a subsequence which is convergent to ξ in $L^{\sigma+1}(\Omega)$. We can assume $\xi_n \to \xi$. Then,

$$| \int_\Omega (\zeta_n(x) - \zeta(x))\xi_n(x) dx |$$

$$\le \quad \int_\Omega |(\zeta_n(x) - \zeta(x))(\xi_n(x) - \xi(x))| dx + | \int_\Omega (\zeta_n(x) - \zeta(x))\xi(x) dx |$$

$$\le \quad \|\zeta_n - \zeta\|_{L^\eta(\Omega)} \|\xi_n - \xi\|_{L^{\sigma+1}(\Omega)} + | \int_\Omega (\zeta_n(x) - \zeta(x))\xi(x) dx |$$

$$\le \quad (\|\zeta_n\|_{L^\eta(\Omega)} + \|\zeta\|_{L^\eta(\Omega)}) \|\xi_n - \xi\|_{L^{\sigma+1}(\Omega)} + | \int_\Omega (\zeta_n(x) - \zeta(x))\xi(x) dx |,$$

where $\eta = \frac{\sigma+1}{\sigma}$, means

$$\int_\Omega (\zeta_n(x) - \zeta(x))\xi_n(x) dx \to 0.$$

This is a contradiction to (3.1). Thus, $\{\zeta_n\}$ has a subsequence which is convergent to ζ in $H^{-1}(\Omega)$. Hence, by (f) in Proposition 1.7, the locally Lipschitz continuous functional $G(u) = \int_\Omega \int_0^{u(x)} f(x, s) ds dx$ satisfies the SC-condition in $H_0^1(\Omega)$. $\qquad \square$

Remark 3.7. A similar result can be established in Banach space $W_0^{1,q}(\Omega)$, $q > 1$.

To establish subsequence convergence, the following lemma is crucial.

Lemma 3.8. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$. Assume that J is a upper semi-differentiable locally Lipschitz continuous functional and p is a local L-$\perp$ selection of J w.r.t. L around $v \in S_{L^\perp}$ such that $p(v)$ is not a critical point, p has super-linear property at v and $d(p(v), L) > 0$. If $v_j \to v$ and $z_j \to z$, where $v_j \in S_{L^\perp}$, $z_j \in \partial J(p(v_j)) \cap [L, v_j]^\perp$ and $z \in \partial J(p(v)) \cap [L, v]^\perp$, then there is $s_0 = \frac{\lambda}{2^{m_0}}$, $m_0 \in \mathbb{N}$ such that, as j is large,*

$$J(p(v_j(s_0))) - J(p(v_j)) < -\frac{1}{4}s_0|t_j|\|z_j\|,$$

where $v_j(s_0) = \dfrac{v_j + s_0 w_j}{\|v_j + s_0 w_j\|}$, $w_j = -\mathrm{sign}(t_j)\frac{z_j}{\|z_j\|}$, $p(v_j) = t_j v_j + u_j$, $u_j \in L$.

Proof. By Lemma 2.14, as $s > 0$ is small,

$$J(p(v(s))) - J(p(v)) < -\frac{1}{4}s|t_v|\|z\|,$$

where $v(s) = \dfrac{v + sw}{\|v + sw\|}$, $w = -\mathrm{sign}(t_v)\frac{z}{\|z\|}$ and $p(v) = t_v v + u$, $u \in L$. Thus, since p is continuous at v, $v_j \to v$ and $z_j \to z$, there is $s_0 = \frac{\lambda}{2^{m_0}}$, $m_0 \in \mathbb{N}$ such that, as j is large,

$$J(p(v_j(s_0))) - J(p(v_j)) < -\frac{1}{4}s_0|t_j|\|z_j\|,$$

where $v_j(s_0) = \dfrac{v_j + s_0 w_j}{\|v_j + s_0 w_j\|}$, $w_j = -\mathrm{sign}(t_j)\frac{z_j}{\|z_j\|}$, $p(v_j) = t_j v_j + u_j$, $u_j \in L$. $\square$

First, we prove a subsequence convergence result for the L-$\perp$ algorithm.

Theorem 3.9. *If an upper semi-differentiable locally Lipschitz continuous functional J satisfies the PS condition and the SC-condition, $\lambda \in (0,1)$, $\{v_n^k\}$ is a sequence generated by the L-$\perp$ algorithm and the L-$\perp$ selection p of J satisfies:*

(a) p has super-linear property on $S_{L^\perp}$,
(b) $d(p(v_n^k), L) \geq \alpha > 0, \forall k = 1, 2, ...,$ and
(c) $\inf_{1 \leq k < \infty} J(p(v_n^k)) > -\infty$,

then,

(d) $\{v_n^k\}$ has a subsequence $\{v_n^{k_i}\}$ such that $u_n^{k_i} = p(v_n^{k_i})$ converges to a critical point of J,

(e) if a subsequence $v_n^{k_i} \to v_0$ as $i \to \infty$, then $u_0 = p(v_0)$ is a critical point of J.

Proof. (a) By the step-size rule, we have

$$(3.2) \qquad J(u_n^{k+1}) - J(u_n^k) \leq -\frac{1}{4}|t_0^k s_n^k|\|z_n^k\|,$$

for $k = 1, 2, ...$. On the other hand, by Lemma 2.4,

$$\|v_n^{k+1} - v_n^k\| = \left\| \frac{v_n^k + s_n^k w_n^k}{\|v_n^k + s_n^k w_n^k\|} - v_n^k \right\|$$

$$\leq \frac{2\|s_n^k w_n^k\|}{\|v_n^k + s_n^k w_n^k\|} \leq \frac{2s_n^k}{|1 - s_n^k|} \leq \frac{2s_n^k}{1 - \lambda}$$

and, by assumption (b),

$$(3.3) \qquad |t_0^k| \geq d(p(v_n^k), L) \geq \alpha.$$

Then,

$$(3.4) \qquad J(u_n^{k+1}) - J(u_n^k) \leq -\frac{1}{4}|t_0^k|s_n^k\|z_n^k\| \leq -\frac{(1-\lambda)}{8}\alpha\|v_n^{k+1} - v_n^k\|\|z_n^k\|.$$

Thus, there is a subsequence $z_n^{k_i} \to 0$. Otherwise, there is $\delta > 0$ such that $\|z_n^k\| \geq \delta$ for any k. From (3.4), we have

$$(3.5) \qquad J(u_n^{k+1}) - J(u_n^k) \leq -\frac{(1-\lambda)}{8}\alpha\delta\|v_n^{k+1} - v_n^k\|, \quad \forall k = 1, 2, ...$$

Adding up two sides of (3.5) gives

$$(3.6) \quad \lim_{k\to\infty} J(u_n^k) - J(u_n^1) = \sum_{k=1}^{\infty}[J(u_n^{k+1}) - J(u_n^k)] \leq -\frac{(1-\lambda)}{8}\alpha\delta\sum_{k=0}^{\infty}\|v_n^{k+1} - v_n^k\|,$$

i.e., $\{v_n^k\}$ is a Cauchy sequence. Thus $v_n^k \to \hat{v} \in S_{L^\perp}$. Since J satisfies the SC-condition and p is continuous, there is a subsequence $z_n^{k_i} \to z \in \partial J(p(\hat{v})) \cap [L, \hat{v}]^\perp$ with $z \neq 0$. On the other hand, by (3.3) and adding up two sides of (3.4), we have

$$\lim_{k\to\infty} J(u_n^k) - J(u_n^1) \leq -\frac{1}{4}\sum_{k=1}^{\infty}|t_0^k|s_n^k\|z_n^k\| \leq -\frac{1}{4}\alpha\delta\sum_{k=0}^{\infty}s_n^k,$$

or $s_n^k \to 0$ as $k \to \infty$. It contradicts Lemma 3.8. When there is a subsequence $z_n^{k_i} \to 0$, by the PS condition, we can assume $u_n^{k_i} = p(v_n^{k_i}) \to u_0$. By (f) in Proposition 1.7, $0 \in \partial J(u_0)$, i.e., u_0 is a critical point.

(b) Suppose $u_0 = p(v_0)$ is not a critical point. Then, there is $\delta > 0$ such that $\|z_n^{k_i}\| \geq \delta, i = 1, 2, ...$ By (3.2) and (3.3), we have

$$J(u_n^{k_i+1}) - J(u_n^{k_i}) \leq -\frac{1}{4}|t_0^{k_i}|s_n^{k_i}\|z_n^{k_i}\| < -\frac{1}{4}\alpha\delta s_n^{k_i}.$$

Since $\sum_{k=1}^{\infty}[J(u_n^{k+1}) - J(u_n^k)] = \lim_{k\to\infty} J(u_n^k) - J(u_n^1)$, it leads to $\lim_{i\to\infty}(J(u_n^{k_i+1}) - J(u_n^{k_i})) = 0$. Hence, $\lim_{i\to\infty} s_n^{k_i} = 0$. This contradicts Lemma 3.8. Thus, u_0 is a critical point. $\qquad\square$

Remark 3.10. Comparing this theorem with Theorem 3.2 in [11], we assume superlinear property of p on $S_{L^\perp}$ in this theorem instead of continuity of p on $S_{L^\perp}$ in Theorem 3.2 in [11]. By Lemma 2.13, if $J \in C^1(H, \mathbb{R})$, these two assumptions are equivalent.

To give a sequence convergence, the following Ekeland's variational principle [12] will be used first to prove an abstract existence-convergence result that is actually independent of the algorithm.

Lemma 3.11 (Ekeland's variational principle)**.** *Let X be a complete metric space and $J : X \to \mathbb{R} \cup \{+\infty\}$ be an upper semicontinuous functional bounded from below. Then, for any $\epsilon > 0$ and $x_0 \in X$ with $J(x_0) < +\infty$, there is $\bar{x} \in X$ such that*

$$J(\bar{x}) + \epsilon d(x_0, \bar{x}) \leq J(x_0) \ \ and \ \ J(x) + \epsilon d(x, \bar{x}) > J(\bar{x}) \ \ \forall x \in X \ and \ x \neq \bar{x}.$$

Then, sequence convergence results are presented as Corollary 3.14 and Theorem 3.17 by this abstract existence-convergence result. Denote $K = \{u \in B | 0 \in \partial J(u)\}$ and $K_c = \{u \in B | 0 \in \partial J(u), J(u) = c\}$, where J is a locally Lipschitz continuous functional in Banach space B. If J satisfies the PS condition, K_c is compact.

Theorem 3.12. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$ and $U = V \cap S_{L^\perp} \neq \emptyset$, where $V \subset H$ is open. Assume that an upper semi-differentiable locally Lipschitz continuous functional J satisfies the PS condition and an L-$\perp$ selection p of J satisfies:*

   (a) *p has super-linear property in U and p is continuous on $\bar{U}$, where $\bar{U}$ is the closure of U on $S_{L^\perp}$,*
   (b) *$\inf_{v \in U} d(p(v), L) \geq \alpha > 0$, and*
   (c) *$\inf_{v \in \partial \bar{U}} J(p(v)) > c = \inf_{v \in U} J(p(v)) > -\infty$, where $\partial \bar{U}$ is the boundary of $\bar{U}$ on $S_{L^\perp}$.*
Then, $K_c^p = p(U) \cap K_c \neq \emptyset$, where $p(U) = \{p(u) | u \in U\}$ and for any $\{v^k\} \subset U$ with $J(u^k) \to c$, where $u^k = p(v^k)$, we have

$$\lim_{k \to \infty} d(u^k, K_c^p) = 0.$$

Proof. Define

$$\hat{J}(p(v)) = \begin{cases} J(p(v)) & v \in \bar{U}, \\ +\infty & v \notin \bar{U}. \end{cases}$$

Then, $\hat{J}(p(\cdot))$ is upper semicontinuous and bounded from below on the complete metric space $S_{L^\perp}$. By assumption (c), either there is $v \in U$ such that $J(p(v)) = c$ or there is a sequence $\{v^k\} \subset U$ such that $J(p(v^k)) > c$ and $J(p(v^k)) \to c$. For the first case, $p(v) \in K_c^p = p(U) \cap K_c$ by Theorem 2.16. Of course, $K_c^p \neq \emptyset$. For the second case, denote $u^k = p(v^k)$. Applying Ekeland's variational principle to $\hat{J}(p(\cdot))$, for every $v^k \in U$ and $\delta^k = (J(u^k) - c)^{\frac{1}{2}}$, there is $\bar{v}^k \in S_{L^\perp}$ such that

$$(3.7) \qquad \hat{J}(p(\bar{v}^k)) - \hat{J}(p(v)) \;<\; \delta^k \|\bar{v}^k - v\| \quad \forall v \in S_{L^\perp},$$

$$(3.8) \qquad \hat{J}(p(\bar{v}^k)) - \hat{J}(p(v^k)) \;\leq\; -\delta^k \|\bar{v}^k - v^k\|.$$

By (3.8), the definition of $\hat{J}(p(\cdot))$ and $v^k \in U$, we have $\bar{v}^k \in \bar{U}$ and then, from (3.7) and (3.8),

$$(3.9) \qquad J(p(\bar{v}^k)) - J(p(v)) \;<\; \delta^k \|\bar{v}^k - v\| \quad \forall v \in U,$$

$$(3.10) \qquad J(p(\bar{v}^k)) - J(p(v^k)) \;\leq\; -\delta^k \|\bar{v}^k - v^k\|.$$

It follows

$$(3.11) \qquad c \leq J(p(\bar{v}^k)) \leq J(u^k) - \delta^k \|\bar{v}^k - v^k\|,$$

i.e,

$$(3.12) \qquad \|\bar{v}^k - v^k\| \leq \delta^k.$$

Then, by (3.11) and (3.12), $J(u^k) = J(p(v^k)) \to c$ implies $J(p(\bar{v}^k)) \to c$. Thus, by assumptions (b) and (c), we have $\bar{v}^k \in U$ and

$$(3.13) \qquad d(p(\bar{v}^k), L) \geq \alpha > 0$$

for large k. For these large k, if there is nonzero $\bar{z}^k \in \partial J(p(\bar{v}^k)) \cap [L, \bar{v}^k]^\perp$, then, when s is small, by Lemma 2.14,

$$J(p(\bar{v}^k(s))) - J(p(\bar{v}^k)) < -\frac{s}{4}|\bar{t}^k| \|\bar{z}^k\|,$$

where $\bar{v}^k(s) = \dfrac{\bar{v}^k + sw^k}{\|\bar{v}^k + sw^k\|} \in U$, $w^k = -\text{sign}(\bar{t}^k)\frac{\bar{z}^k}{\|\bar{z}^k\|}$, $p(\bar{v}^k) = \bar{t}^k \bar{v}^k + \bar{w}^k$ and $\bar{w}^k \in L$ and by Lemma 2.4,

$$\|v^k(s) - v^k\| = \|\frac{v^k + sw^k}{\|v^k + sw^k\|} - v^k\| \leq \frac{2\|sw^k\|}{\|v^k + sw^k\|} \leq \frac{2s}{1-s} \leq 4s,$$

i.e.,

$$J(p(\bar{v}^k(s))) - J(p(\bar{v}^k)) < -\frac{1}{16}\alpha\|\bar{z}^k\|\|\bar{v}^k(s) - \bar{v}^k\|,$$

by (3.13). Hence, by (3.9), we have

(3.14) $$\|\bar{z}^k\| < \frac{16}{\alpha}\delta^k,$$

which implies $\bar{z}^k \to 0$. Then, by the PS condition, $\{p(\bar{v}^k)\}$ has a convergent subsequence $\{p(\bar{v}^{k_j})\}$. Since $\{p(\bar{v}^{k_j})\}$ is convergent, $\bar{v}^{k_j} \in S_{L^\perp}$ and $\bar{w}^{k_j} \in L$, $\{t^{k_j}\bar{v}^{k_j}\}$ is convergent, i.e., we can assume $\{t^{k_j}\}$ and $\{\bar{v}^{k_j}\}$ are both convergent by (3.13). Denote $\bar{v} = \lim_{j\to\infty} \bar{v}^{k_j}$. By (f) in Proposition 1.7 and (3.14), $0 \in \partial J(p(\bar{v}))$, i.e., $p(\bar{v})$ is a critical point and by (3.12), $\bar{v} = \lim_{j\to\infty} v^{k_j}$. By assumption (a),

$$p(\bar{v}) = \lim_{j\to\infty} p(v^{k_j}) \text{ and } J(p(\bar{v})) = c$$

and then, by assumption (c), it is clear that

$$\bar{v} \in U \text{ and } \bar{u} = p(\bar{v}) \in K_c^p \neq \emptyset.$$

For any $\{v^k\} \subset U$ with $J(u^k) \to c$, where $u^k = p(v^k)$, let β be any limit point of $\{d(u^k, K_c^p)\}$ and $u^{k_j} = p(v^{k_j}) \in \{u^k\}$ such that $\lim_{j\to\infty} d(u^{k_j}, K_c^p) = \beta$. If there is a subsequence $\{u^{k_{j_i}}\} \subseteq \{u^{k_j}\}$ such that $J(u^{k_{j_i}}) = c$, then, by Theorem 2.16, $u^{k_{j_i}} \in K_c^p$, i.e., $\beta = 0$. Otherwise, we can assume $J(u^{k_j}) > c$, $j = 1, 2, ...$ Consider $\{u^{k_j}\}$ as $\{u^k\}$. To $\{u^{k_j}\}$, by using above argument for $\{u^k\}$, we know there is a subsequence $\{u^{k_{j_i}}\} \subseteq \{u^{k_j}\}$ such that $\bar{u} = \lim_{i\to\infty} p(v_{k_{j_i}}) \in K_c^p$, i.e., $\beta = 0$. Thus, every limit point of $\{d(u^k, K_c^p)\}$ is zero, i.e.,

$$\lim_{k\to\infty} d(u^k, K_c^p) = 0. \qquad \square$$

Remark 3.13. Comparing this theorem with Theorem 3.5 in [15], we assume superlinear property of p in U and continuity of p on $\bar{U}$, where $\bar{U}$ is the closure of U on $S_{L^\perp}$ in this theorem instead of continuity of p on $\bar{U}$, where $\bar{U}$ is the closure of U on $S_{L^\perp}$ in Theorem 3.5 in [15]. By Lemma 2.13, if $J \in C^1(H, \mathbb{R})$, these two assumptions are equivalent.

Corollary 3.14. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$ and $V_1, V_2 \subset H$ two open sets with $\emptyset \neq U_2 \equiv V_2 \cap S_{L^\perp} \subset V_1 \cap S_{L^\perp} \equiv U_1$. Assume that a upper semi-differentiable locally Lipschitz continuous functional J satisfies the PS condition and an L-$\perp$ selection p of J satisfies:*

   (a) *p has super-linear property in U_1,*
   (b) *$\inf_{v\in U_1} d(p(v), L) \geq \alpha > 0$ and $c = \inf_{v\in U_1} J(p(v)) > -\infty$,*
   (c) *there is $d > 0$ with*

$$\inf\{J(p(v))|v \in U_1, d(v, \partial U_1) \leq d\} = a > b = \sup\{J(p(v))|v \in U_2\},$$

where ∂U_1 is the boundary of U_1 on $S_{L^\perp}$,
   (d) *given $\{v_k\}$ such that $v_1 \in U_2$, $\|v_{k+1} - v_k\| < d$, $J(u_{k+1}) < J(u_k)$ and $\{u_k\}$ has a subsequence that converges to a critical point u_0, where $u_k = p(v_k)$, and*
   (e) *$K_c^p = p(U_1) \cap K \subset K_c$, where $p(U_1) = \{p(u)|u \in U_1\}$.*

Then,

$$\lim_{k \to \infty} d(u_k, K_c^p) = 0.$$

Proof. First, we prove that $v_k \in U_1$ and $d(v_k, \partial U_1) > d$, $k = 1, 2, ...$ In fact, if $v_k \in U_1$, $d(v_k, \partial U_1) > d$ and $J(u_k) \le b$, then, by assumption (d), $v_{k+1} \in U_1$ and $J(u_{k+1}) < b$, i.e., $v_{k+1} \in U_1$ and $d(v_{k+1}, \partial U_1) > d$ by assumption (c). Thus, since $v_1 \in U_2$, we have $v_k \in U_1$ and $d(v_k, \partial U_1) > d$ for $k = 1, 2, ...$, by assumption (c). Since $K_c^p = p(U_1) \cap K \subset K_c$ and $\{u_k\}$ has a subsequence which converges to a critical point u_0, we have $u_0 \in K_c^p \ne \emptyset$. Denote $U = \{v \in U_1 | d(v, \partial U_1) > d\}$. Then, by the monotonicity of $\{J(u_k)\}$, we have $J(u_k) \to c = \inf_{v \in U} J(p(v))$ as $k \to \infty$, and

$$\inf_{v \in \partial \bar{U}} J(p(v)) \ge a > b \ge J(p(v_1)) \ge c = \inf_{v \in U} J(p(v)),$$

where $\partial \bar{U}$ is the boundary of U on $S_{L^\perp}$. Thus, all assumptions of Theorem 3.12 are satisfied and the conclusion follows.   □

Remark 3.15. Comparing this corollary with Corollary 3.6 in [15], we assume super-linear property of p in U_1 in this theorem instead of continuity of p in U_1 in Corollary 3.6 in [15]. By Lemma 2.13, if $J \in C^1(H, \mathbb{R})$, these two assumptions are equivalent.

To better understand Corollary 3.14, we give the following lemma.

Lemma 3.16. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$ and $V_1, V_2 \subset H$ two open sets with $\emptyset \ne U_2 \equiv V_2 \cap S_{L^\perp} \subset V_1 \cap S_{L^\perp} \equiv U_1$. Assume that an upper semi-differentiable locally Lipschitz continuous functional J satisfies the PS condition and the SC-condition, an L-$\perp$ selection p of J satisfies:*
  (a) p has super-linear property in U_1,
  (b) $\inf_{v \in U_1} d(p(v), L) \ge \alpha > 0$ and $c = \inf_{v \in U_1} J(p(v)) > -\infty$,
  (c) there is $d > 0$ with

$$\inf\{J(p(v)) | v \in U_1, d(v, \partial U_1) \le d\} = a > b = \sup\{J(p(v)) | v \in U_2\},$$

where ∂U_1 is the boundary of U_1 on $S_{L^\perp}$, and $\{v_n^k\}$ is a sequence generated by the L-$\perp$ algorithm according to the L-$\perp$ selection p with initial point $v_n^1 \in U_2$ and $\lambda \in (0, \frac{d}{d+2})$. Then, $\|v_n^{k+1} - v_n^k\| < d$, $J(u_n^{k+1}) < J(u_n^k)$ and $\{u_n^k\}$ has a subsequence that converges to a critical point u_n^0, where $u_n^k = p(v_n^k)$.

Proof. First, by Lemma 2.4, we have

$$\|v_n^{k+1} - v_n^k\| = \|\frac{v_n^k + s_n^k w_n^k}{\|v_n^k + s_n^k w_n^k\|} - v_n^k\| \le \frac{2\|s_n^k w_n^k\|}{\|v_n^k + s_n^k w_n^k\|} \le \frac{2s_n^k}{|1 - s_n^k|} \le \frac{2\lambda}{1 - \lambda}.$$

Then, $\lambda \in (0, \frac{d}{d+2})$ guarantees

$$\|v_n^{k+1} - v_n^k\| < d.$$

Since the L-$\perp$ algorithm is a descent method, we have

$$J(u_n^{k+1}) < J(u_n^k).$$

Consider $\{v_n^k\}$ as $\{v_k\}$. By exactly mimicking the verification of Corollary 3.14 to $\{v_k\}$, we can show that

$$\{v_n^k\} \subset U = \{v \in U_1 | d(v, \partial U_1) > d\},$$

where ∂U_1 is the boundary U_1 on $S_{L^\perp}$. Thus,

$$\overline{\{v_n^k\}} \subset U_1,$$

where $\overline{\{v_n^k\}}$ is the closure of $\{v_n^k\}$ on $S_{L^\perp}$. Then, by exactly mimicking the verification of Theorem 3.9, we can prove that $\{u_n^k\}$ has a subsequence that converges to a critical point u_n^0. $\qquad\square$

By Corollary 3.14 and Lemma 3.16, Theorem 3.17 immediately follows.

Theorem 3.17. *Let H be a Hilbert space with $H = L \oplus L^\perp$ for a finite dimensional subspace $L \subset H$ and $V_1, V_2 \subset H$ two open sets with $\emptyset \neq U_2 \equiv V_2 \cap S_{L^\perp} \subset V_1 \cap S_{L^\perp} \equiv U_1$. Assume that an upper semi-differentiable locally Lipschitz continuous functional J satisfies the PS condition and the SC-condition, an L-$\perp$ selection p of J satisfies*
   (a) *p has super-linear property in U_1,*
   (b) *$\inf_{v \in U_1} d(p(v), L) \geq \alpha > 0$ and $c = \inf_{v \in U_1} J(p(v)) > -\infty$,*
   (c) *there is $d > 0$ with*

$$\inf\{J(p(v)) | v \in U_1, d(v, \partial U_1) \leq d\} = a > b = \sup\{J(p(v)) | v \in U_2\},$$

where ∂U_1 is the boundary of U_1 on $S_{L^\perp}$, and
   (d) *$K_c^p = p(U_1) \cap K \subset K_c$, where $p(U_1) = \{p(u) | u \in U_1\}$,*
and $\{v_n^k\}$ is a sequence generated by the L-$\perp$ algorithm according to the L-$\perp$ selection p with initial point $v_n^1 \in U_2$ and $\lambda \in (0, \frac{d}{d+2})$. Then,

$$\lim_{k \to \infty} d(u_n^k, K_c^p) = 0.$$

4. NUMERICAL EXPERIMENT RESULTS

As we pointed out before, a typical example on application of nonsmooth critical point theory to partial differential equations is the Dirichlet problem (1.1), i.e.,

$$\begin{cases} -\Delta u(x) = f(x, u(x)), \ x \in \Omega, \\ u(x)|_{x \in \partial\Omega} = 0, \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$, $f(x, t)$ is a measurable function defined on $\Omega \times \mathbb{R}$ and for every $x \in \Omega$, $f(x, t)$ is locally bounded. The corresponding variational functional on $H_0^1(\Omega)$ is

$$(4.1) \qquad J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x, u(x)) dx,$$

where $F(x, t) = \int_0^t f(x, s) ds$. Denote $G(u) = \int_\Omega F(x, u(x)) dx$. By Theorem 1.3, to find a $u_0 \in H_0^1(\Omega)$ such that

$$-\Delta u_0(x) \in [\bar{f}(x, u_0(x)), \hat{f}(x, u_0(x))]$$

for all $x \in \Omega$, we can find a critical point of J (4.1). As a numerical example, we set

$$(4.2) \qquad f(x, t) = \begin{cases} c\alpha |x|^r |t|^{\alpha-2} t, \ \text{if } |t| > 1, \\ c\beta |x|^r |t|^{\beta-2} t, \ \text{if } |t| \leq 1, \end{cases}$$

where $c > 0$, $\alpha > \beta > 2$, $\alpha < \frac{2m}{m-2}$ for $m \geq 3$, $r \geq 0$ and $|x|$ is the Euclidean norm of $x \in \mathbb{R}^m$. If $c = 1$, $\alpha = \beta > 2$ and $r = 0$ in (4.2), the Dirichlet problem (1.1) is the Lane-Emden equation and if $c = 1$, $\alpha = \beta > 2$ and $r > 0$ in (4.2), the Dirichlet problem (1.1) is the Hénon equation. These two equations were used as numerical examples for finding smooth saddle critical points [6], [10], [11]. If $r \geq 0$, $\alpha > \beta > 2$ and $\alpha < \frac{2m}{m-2}$ for $m \geq 3$, J (4.1) is locally Lipschitz continuous and it is not differentiable everywhere. If $c > 0$, $r \geq 0$, $\alpha > \beta > 2$ and $\alpha < \frac{2m}{m-2}$ for

$m \geq 3$, it is easy to check that $f(x,t)$ in (4.2) satisfies all assumptions in Theorem 1.5. Thus,

$$(4.3) \qquad F(x,t) = \begin{cases} c|x|^r|t|^\alpha, & \text{if } |t| > 1, \\ c|x|^r|t|^\beta, & \text{if } |t| \leq 1, \end{cases}$$

and

$$(4.4) \qquad \partial G(u) = \{\zeta : \Omega \to \mathbb{R} | \zeta \text{ is measurable, } \zeta(x) \in f_-^+(x, u(x)) \ \forall x \in \Omega\},$$

where

$$\begin{aligned} f_-^+(x, u(x)) &= [f(x, u(x) - 0), f(x, u(x) + 0)] \\ &= \begin{cases} \{c\alpha|x|^r|u(x)|^{\alpha-2}u(x)\}, & \text{if } |u(x)| > 1, \\ \{c\beta|x|^r|u(x)|^{\beta-2}u(x)\}, & \text{if } |u(x)| < 1, \\ [c\beta|x|^r, c\alpha|x|^r], & \text{if } u(x) = 1, \\ [-c\alpha|x|^r, -c\beta|x|^r], & \text{if } u(x) = -1. \end{cases} \end{aligned}$$

$f(x,t)$ in (4.2) with $\alpha = 6$, $\beta = 4$, $r = 0$ and $c = 1$ is an example in [13].

Before we discuss the details on numerical experiment, we establish several lemmas for J (4.1).

Lemma 4.1. *Assume that $L = \{0\}$, $f(x,t)$ is a measurable function defined on $\Omega \times \mathbb{R}$ and satisfies:*
(a)
$$|f(x,t)| \leq C_1 + C_2|t|^\sigma$$
for $x \in \Omega$ and $t \in \mathbb{R}$, where $0 < \sigma < \frac{m+2}{m-2}$ for $m \geq 3$, $\sigma > 0$ for $m = 1, 2$ and $C_1, C_2 > 0$ are two constants,
(b) *$G(tv) \geq 0$ and $G(tv) = o(t^2)$ as $t \to 0$ and $t \to \infty$, for every $v \in S_{L^\perp}$.*
Then, the peak mapping
$$P(v) = \{tv | t > 0 \text{ such that } J(tv) = \max_{s \geq 0} J(sv)\}$$
is well defined on $S_{L^\perp}$, i.e., $P(v) \neq \emptyset$ for each $v \in S_{L^\perp}$.

Proof. First, similar to Lemma 3.6, assumption (a) guarantees that J (4.1) is locally Lipschitz continuous on $H_0^1(\Omega)$. Then, denote $I(t) = J(tv)$ for $v \in S_{L^\perp}$. Since $G(tv) = o(t^2)$ as $t \to 0$, we have

$$I(t) = t^2 \Big(\frac{1}{2} \int_\Omega |\nabla v(x)|^2 dx - \frac{G(tv)}{t^2} \Big) = t^2 \Big(\frac{1}{2} - \frac{G(tv)}{t^2} \Big) > 0$$

as $t > 0$ is small and since $G(tv) \geq 0$ and $G(tv) = o(t^2)$ as $t \to \infty$, we have

$$I(t) = t^2 \Big(\frac{1}{2} - \frac{G(tv)}{t^2} \Big) \to -\infty$$

as $t \to +\infty$. Thus, by the continuity of $I(t)$ and $I(0) = 0$, there is at least one $t_v > 0$ such that t_v is a local maximum point of $I(t)$, i.e.,

$$P(v) = \{tv | t > 0 \text{ such that } J(tv) = \max_{s \geq 0} J(sv)\} \neq \emptyset. \qquad \square$$

Remark 4.2. For our numerical example, by (4.2),

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha-1},$$

where $C_1 = c\beta b^r$, $C_2 = c\alpha b^r$ and b is a bound of Ω and by (4.3), $G(u) = \int_\Omega F(x, u(x)) dx$ satisfies that $G(tv) \geq 0$ and $G(tv) = o(t^2)$ as $t \to 0$ and $t \to \infty$, for every $v \in S_{L^\perp}$.

Lemma 4.3. *Assume that $L = \{0\}$, J (4.1) is locally Lipschitz continuous on $H_0^1(\Omega)$, the peak mapping $P(v) = \{tv | t > 0 \text{ such that } J(tv) = \max_{s \geq 0} J(sv)\}$ is well defined on $S_{L^\perp}$, $F(x, t)$ is regular (cf. [7]) at every $t \in \mathbb{R}$ for $x \in \Omega$ and satisfies that $\frac{\partial F(x,t)}{t}$ is monotone in the sense that for any $|t_2| > |t_1| > 0$, any $\delta_1 F(x, t_1) \in \partial F(x, t_1)$ and $\delta_2 F(x, t_2) \in \partial F(x, t_2)$,*

$$\frac{\delta_2 F(x, t_2)}{t_2} > \frac{\delta_2 F(x, t_1)}{t_1} \quad \text{for } x \in \Omega \text{ and } x \neq x_1, ..., x_k,$$

where $x_1, ..., x_k \in \Omega$ and $\partial F(x, t)$ is generalized gradient of $F(x, \cdot)$ at t for fixed $x \in \Omega$. Then, the peak mapping

$$P(v) = \{tv | t > 0 \text{ such that } J(tv) = \max_{s \geq 0} J(sv)\}$$

is single-valued for every $v \in S_{L^\perp}$, i.e., for every $v \in S_{L^\perp}$, $P(v) = \{p(v)\}$, where $p(v) = t_v v$ for some $t_v > 0$.

Proof. Since $F(x, t)$ is regular at every $t \in \mathbb{R}$ for $x \in \Omega$, we have

$$\partial I(t) = \{t - \int_\Omega \delta F(x, tv(x)) v(x) dx | \delta F(x, tv(x)) \in \partial F(x, s)|_{s=tv(x)}\},$$

where $I(t) = J(tv)$ for every $v \in S_{L^\perp}$. Hence, $0 \in \partial I(t_v)$, where $t_v v \in P(v)$, i.e., there is $\delta F(x, t_v v(x)) \in \partial F(x, s)|_{s=t_v v(x)}$ such that

$$\int_\Omega \frac{\delta F(x, t_v v(x))}{t_v v(x)} v^2(x) dx = 1.$$

By monotonicity of $\frac{\partial F(x,t)}{t}$, if $t_0 > t_v$,

$$\int_\Omega \frac{\delta F(x, t_0 v(x))}{t_0 v(x)} v^2(x) dx > \int_\Omega \frac{\delta F(x, t_v v(x))}{t_v v(x)} v^2(x) dx = 1$$

and if $0 < t_0 < t_v$,

$$\int_\Omega \frac{\delta F(x, t_0 v(x))}{t_0 v(x)} v^2(x) dx < \int_\Omega \frac{\delta F(x, t_v v(x))}{t_v v(x)} v^2(x) dx = 1.$$

Thus, such $t_v > 0$ is unique, i.e, the peak mapping P is single-valued for every $v \in S_{L^\perp}$. $\qquad\square$

Remark 4.4. **(a)** To our numerical example,

$$(4.5) \qquad \partial F(x, t) = \begin{cases} \{c\alpha |x|^r |t|^{\alpha-2} t\}, & \text{if } |t| > 1, \\ \{c\beta |x|^r |t|^{\beta-2} t\}, & \text{if } |t| < 1, \\ [c\beta |x|^r, c\alpha |x|^r], & \text{if } t = 1, \\ [-c\alpha |x|^r, -c\beta |x|^r], & \text{if } t = -1. \end{cases}$$

Hence, for any $|t_2| > |t_1|$, any $\delta_1 F(x, t_1) \in \partial F(x, t_1)$ and $\delta_2 F(x, t_2) \in \partial F(x, t_2)$,

$$\frac{\delta_2 F(x, t_2)}{t_2} > \frac{\delta_2 F(x, t_1)}{t_1} \quad \text{for } x \in \Omega \text{ and } x \neq 0.$$

(b) Lemma 4.1 and Lemma 4.3 guarantee that

$$\inf_{v \in S_{L^\perp}} J(p(v)) \geq 0,$$

where p is the peak selection of J (4.1) w.r.t. $L = \{0\}$ with $p(v) = tv$, $t > 0$ for $v \in S_{L^\perp}$, under assumptions in Lemma 4.1 and Lemma 4.3. Indeed, if the peak mapping of J (4.1) w.r.t. a finite dimensional space $L \subset H_0^1(\Omega)$,

$$P(v) = \{tv + w | t > 0 \text{ and } w \in L \text{ such that } J(tv + w) = \max_{s \geq 0, u \in L} J(sv + u)\},$$

is well defined, then,

$$\inf_{v \in S_{L^\perp}} J(p(v)) \geq 0,$$

where p is a peak selection of J (4.1) w.r.t. L such that $p(v) \in P(v)$ for every $v \in S_{L^\perp}$, under assumptions in Lemma 4.1 and Lemma 4.3.

Lemma 4.5. *Assume that $L = \{0\}$, J (4.1) is locally Lipschitz continuous on $H_0^1(\Omega)$, the peak mapping $P(v) = \{tv | t > 0$ such that $J(tv) = \max_{s \geq 0} J(sv)\}$ is well defined on $S_{L^\perp}$, $P(v_0) = \{u_0\}$ where $v_0 \in S_{L^\perp}$, $t = 0$ is not a maximum point of $J(tv_0)$ on $\{t \in \mathbb{R} | t \geq 0\}$, $F(x, t) = F_1(x)F_2(t)$, $0 \leq F_1(x) \leq C$ for every $x \in \Omega$ where $C > 0$ is a constant, $F_2(t)$ is regular at every $t \in \mathbb{R}$ and satisfies that, for every $\delta F_2(t) \in \partial F_2(t)$,*

$$\delta F_2(t)t \geq a_1|t|^\mu + a_2,$$

where $\mu > 2$, $\mu < \frac{2m}{m-2}$ for $m \geq 3$ and $a_1 > 0$, a_2 are two constants. Then, any peak selection $p(v)$ is continuous at v_0, where $p(v) \in P(v)$ for $v \in S_{L^\perp}$ and $p(v)$ is a global maximum point of J (4.1) on $\{tv | t \geq 0\}$ for $v \in S_{L^\perp}$ around v_0.

Proof. Since $F_2(t)$ is regular at every $t \in \mathbb{R}$, we have

$$(4.6) \qquad \partial I(t) = \{t - \int_\Omega F_1(x)\delta F_2(tv(x))v(x)dx | \delta F_2(tv(x)) \in \partial F_2(s)|_{s=tv(x)}\},$$

where $I(t) = J(tv)$ for $v \in S_{L^\perp}$. Suppose that $v_n \to v_0$. Denote that $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v_0) = t_0 v_0$, $t_0 > 0$. Then, $0 \in \partial I_n(t_n)$, where $I_n(t) = J(tv_n)$. By (4.6), there is $\delta F_2(t_n v_n(x)) \in \partial F_2(s)|_{s=t_n v_n(x)}$ such that

$$t_n - \int_\Omega F_1(x)\delta F_2(t_n v_n(x))v_n(x)dx = 0.$$

Thus,

$$1 = \frac{1}{t_n^2} \int_\Omega F_1(x)\delta F_2(t_n v_n(x))t_n v_n(x)dx \geq \frac{1}{t_n^2} \int_\Omega F_1(x)(a_1|t_n v_n(x)|^\mu + a_2)dx,$$

i.e.,

$$(4.7) \qquad 1 \geq a_1\|p(v_n)\|^{\mu-2} \int_\Omega F_1(x)|v_n(x)|^\mu dx - \|p(v_n)\|^{-2}C|\Omega|a_2,$$

where $|\Omega|$ is the measure of Ω. Suppose that $\{t_n\}$ is unbounded. Then, there is $\{t_{n_k}\}$ such that $t_{n_k} \to +\infty$, i.e., $\|p(v_{n_k})\| \to +\infty$. Since $v_n \to v_0$, this contradicts (4.7). Hence, $\{t_n\}$ is bounded. Without loss of generality, we can assume that $t_n \to \bar{t}$. Then, since $p(v_n)$ is a global maximum point of J (4.1) on $\{tv_n | t \geq 0\}$ for large n, $\bar{t}v_0$ is a maximum point of J (4.1) on $\{tv_0 | t \geq 0\}$. On the other hand, Since $P(v_0) = \{u_0\}$ and $t = 0$ is not a maximum point of $J(tv_0)$ on $\{t \in \mathbb{R} | t \geq 0\}$, $u_0 = t_0 v_0 = \bar{t}v_0$. Hence, $\bar{t} = t_0$, i.e., $p(v_n) \to p(v_0)$, i.e., $p(v)$ is continuous at v_0. $\qquad \square$

Remark 4.6. **(a)** For our numerical example, $F_1(x) = |x|^r$, $r > 0$,

$$F_2(t) = \begin{cases} |t|^\alpha, & \text{if } |t| > 1, \\ |t|^\beta, & \text{if } |t| \leq 1, \end{cases}$$

and

$$\partial F_2(t) = \begin{cases} \{c\alpha|t|^{\alpha-2}t\}, & \text{if } |t| > 1, \\ \{c\beta|t|^{\beta-2}t\}, & \text{if } |t| < 1, \\ [c\beta, c\alpha], & \text{if } t = 1, \\ [-c\alpha, -c\beta], & \text{if } t = -1. \end{cases}$$

Thus, for every $\delta F_2(t) \in \partial F_2(t)$,

$$\delta F_2(t)t \geq a_1|t|^\beta + a_2,$$

where $a_1 = c\beta$ and $a_2 = 0$.

(b) For our numerical example, Lemma 4.1, Lemma 4.3 and Lemma 4.5 guarantee that the peak selection $p(v) = tv$, where $t > 0$ and $J(tv) = \max_{s \geq 0} J(sv)$, is continuous on $S_{L^\perp}$, where $L = \{0\}$.

Lemma 4.5 can be extended to the following theorem and the theorem can be verified in the same way as Theorem 4.1 in [10]. On the other hand, this theorem extends Theorem 4.1 in [10].

Theorem 4.7. *Assume that $L \subset H_0^1(\Omega)$ is a finite dimensional space, J (4.1) is locally Lipschitz continuous on $H_0^1(\Omega)$, the peak mapping of J (4.1) w.r.t. L,*

$$P(v) = \{tv + w | t > 0 \text{ and } w \in L \text{ such that } J(tv + w) = \max_{s \geq 0, u \in L} J(sv + u)\},$$

is well defined and $P(v_0) = \{u_0\}$ where $v_0 \in S_{L^\perp}$, $(0, w)$ is not a maximum point of $J(tv_0 + u)$ on $\{t \in \mathbb{R} | t \geq 0\} \times L$ for any $w \in L$, $F(x, t) = F_1(x)F_2(t)$, $0 \leq F_1(x) \leq C$ for $x \in \Omega$ where $C > 0$ is a constant, $F_2(t)$ is regular at every $t \in \mathbb{R}$ and satisfies that, for every $\delta F_2(t) \in \partial F_2(t)$,

$$\delta F_2(t)t \geq a_1|t|^\mu + a_2, \ \forall x \in \Omega,$$

where $\mu > 2$, $\mu < \frac{2m}{m-2}$ for $m \geq 3$ and $a_1 > 0$, a_2 are two constants. Then, any peak selection $p(v)$ is continuous at v_0, where $p(v) \in P(v)$ for $v \in S_{L^\perp}$ and $p(v)$ is a global maximum point of J (4.1) on $\{tv + w | t \geq 0, w \in L\}$ for $v \in S_{L^\perp}$ around v_0.

Lemma 4.8. *Assume that $L = \{0\}$, J (4.1) is locally Lipschitz continuous on $H_0^1(\Omega)$, the peak mapping $P(v) = \{tv | t > 0 \text{ such that } J(tv) = \max_{s \geq 0} J(sv)\}$ is well defined on $S_{L^\perp}$, $F(x, t)$ is regular at every $t \in \mathbb{R}$ for $x \in \Omega$ and satisfies that, for every $\delta F(x, t) \in \partial F(x, t)$ where $\partial F(x, t)$ is generalized gradient of $F(x, \cdot)$ at t for fixed $x \in \Omega$,*

$$(4.8) \qquad \delta F(x, t)t \leq at^2 + \sum_{i=1}^{k} b_i|t|^{\rho_i}, \ \forall x \in \Omega,$$

where $0 \leq a < \lambda_1$, λ_1 is the first eigenvalue of Δ on Ω, $\rho_1, ..., \rho_k > 2$, $\rho_1, ..., \rho_k < \frac{2m}{m-2}$ for $m \geq 3$ and $\rho_1, ..., \rho_k, b_1 > 0, ..., b_k > 0$ are constants. Then, there is $\alpha > 0$ such that $d(p(v), L) \geq \alpha$, where $p(v)$ is a peak selection such that $p(v) \in P(v)$ for $v \in S_{L^\perp}$.

Proof. Since $F(x, t)$ is regular at every $t \in \mathbb{R}$ for $x \in \Omega$, we have

$$\partial I(t) = \{t - \int_\Omega \delta F(x, tv(x))v(x)dx | \delta F(x, tv(x)) \in \partial F(x, s)|_{s=tv(x)}\},$$

where $I(t) = J(tv)$ for $v \in S_{L\perp}$. On the other hand, according to (4.8), we have

$$\int_\Omega \delta F(x, u(x))u(x)dx \leq \int_\Omega (a(u(x))^2 + \sum_{i=1}^k b_i |u(x)|^{\rho_i})dx$$

$$\leq a\lambda_1^{-1} \int_\Omega |\nabla u(x)|^2 dx + \sum_{i=1}^k b_i c_i (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{\rho_i}{2}}$$

$$= (a\lambda_1^{-1} + \sum_{i=1}^k b_i c_i \|u\|^{\rho_i-2})\|u\|^2,$$

where $u = tv$, $\delta F(x, u(x)) \in \partial F(x, s)|_{s=u(x)}$ and $c_1, ..., c_k$ are constants. Thus, as $t > 0$ is small,

$$\delta I(t) \geq (1 - a\lambda_1 + \sum_{i=1}^k b_i c_i t^{\rho_i-2})t > 0$$

for any $\delta I(t) \in \partial I(t)$. Hence, there is $\alpha > 0$ such that $d(p(v), L) = \|p(v)\| \geq \alpha$ for any peak selection $p(v)$ such that $p(v) \in P(v)$ for $v \in S_{L\perp}$. $\square$

Remark 4.9. To our numerical example, $\partial F(x, t)$ is given by (4.5). Hence, for every $\delta F(x, t) \in \partial F(x, t)$,

$$\delta F(x, t)t \leq at^2 + b_1 |t|^\beta + b_2 |t|^\alpha, \ \forall x \in \Omega,$$

where $a = 0$, $b_1 = c\beta b^r$, $b_2 = c\alpha b^r$ and b is a bound of Ω. Hence, to our numerical example, there is $\alpha > 0$ such that

$$d(p(v), L) \geq \alpha,$$

where p is the peak selection of J (4.1) w.r.t. $L = \{0\}$ with $p(v) = tv$, $t > 0$ for $v \in S_{L\perp}$.

To super-linear property of peak selection to J (4.1) with $f(x, t)$ (4.2), we have Theorem 4.18. For verifying this theorem, we need Lemma 4.10–Lemma 4.16 to get Theorem 4.17. Then, Theorem 4.18 can be obtained by Theorem 4.17.

Lemma 4.10. *Assume that $L = \{0\}$, $f(x, t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L\perp}$, p is continuous at $v \in S_{L\perp}$ and $\{v_n\} \subset S_{L\perp}$ satisfies $\|v_n - v\| \leq 1$, $v_n \to v$ and*

$$(4.9) \qquad \frac{|t_n - t|}{\|v_n - v\|^{\beta_0}} \to +\infty,$$

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where $p(v_n) = t_n v_n$, $t_n > 0$, $p(v) = tv$, $t > 0$ and $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. Denote

$$A_1^n = \{x \in \Omega ||tv(x)| > 1, |t_n v_n(x)| > 1\},$$

$$A_2^n = \{x \in \Omega ||tv(x)| > 1, |t_n v_n(x)| = 1\}, \quad A_3^n = \{x \in \Omega ||tv(x)| > 1, |t_n v_n(x)| < 1\},$$

$$B_1^n = \{x \in \Omega ||tv(x)| = 1, |t_n v_n(x)| > 1\},$$

$$B_2^n = \{x \in \Omega ||tv(x)| = 1, |t_n v_n(x)| = 1\}, \quad B_3^n = \{x \in \Omega ||tv(x)| = 1, |t_n v_n(x)| < 1\},$$

$$C_1^n = \{x \in \Omega ||tv(x)| < 1, |t_n v_n(x)| > 1\},$$

$$C_2^n = \{x \in \Omega ||tv(x)| < 1, |t_n v_n(x)| = 1\}, \quad C_3^n = \{x \in \Omega ||tv(x)| < 1, |t_n v_n(x)| < 1\}.$$

Thus,

(a) if $A_2^n \cup A_3^n \cup B_3^n \neq \emptyset$ for $n = 1, 2, ...$, then, either there is $\{t_{n_k}\}$ such that $t > t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t < t_{n_k}$ and

$$\int_{\Omega_{n_k}} |t_{n_k} v_{n_k}(x)| dx \leq C s_{n_k}^{\beta_0},$$

where $\Omega_{n_k} = A_2^{n_k} \cup A_3^{n_k} \cup B_3^{n_k}$,

(b) if $B_2^n \neq \emptyset$ for $n = 1, 2, ...$, then, either there is $\{t_{n_k}\}$ such that $t > t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t < t_{n_k}$ and

$$\int_{B_2^{n_k}} |t_{n_k} v_{n_k}(x)| dx \leq C s_{n_k}^{\beta_0},$$

where $v_{n_k} = \frac{1}{\sqrt{1 + s_{n_k}^2}}(v - s_{n_k} d_{n_k})$, $\|d_{n_k}\| = 1$, $\langle d_{n_k}, v \rangle = 0$ and $s_{n_k} \geq 0$.

Proof. (a) Since $\|v_n - v\| \leq 1$ and

$$\|v_n - v\|^2 = \langle v_n - v, v_n - v \rangle = \|v_n\|^2 + \|v\|^2 - 2\langle v_n, v \rangle = 2 - 2\langle v_n, v \rangle,$$

we have $\langle v_n, v \rangle \geq \frac{1}{2}$. By Lemma 2.12, there is unique d_n with $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$ and $s_n \geq 0$ such that

$$(4.10) \qquad\qquad v_n = \frac{1}{\sqrt{1 + s_n^2}}(v - s_n d_n).$$

Then,

$$v_n - v = -\frac{s_n^2}{(\sqrt{1 + s_n^2})(1 + \sqrt{1 + s_n^2})}v - \frac{s_n}{\sqrt{1 + s_n^2}}d_n,$$

i.e.,

$$\|v_n - v\| = \frac{s_n}{\sqrt{1 + s_n^2}}\sqrt{\frac{s_n^2}{(1 + \sqrt{1 + s_n^2})^2} + 1}.$$

Thus,

$$\lim_{n \to \infty} \frac{\|v_n - v\|}{s_n} = 1.$$

Hence, by (4.9),

$$(4.11) \qquad\qquad D_n = \frac{|t_n - t|}{s_n^{\beta_0}} \to +\infty.$$

By (4.10), we have

$$
t_n v_n
$$

$$
= \frac{1}{\sqrt{1+s_n^2}}(t_n v - t_n d_n s_n) = (t_n v - t_n d_n s_n) + (\frac{1}{\sqrt{1+s_n^2}} - 1)(t_n v - t_n d_n s_n)
$$

$$
= tv - (t - t_n)v - t_n d_n s_n - \frac{(t_n v - t_n d_n s_n)s_n^2}{f(s_n)},
$$

i.e.,

$$
(4.12) \qquad t_n v_n = tv - (t - t_n)v - t_n d_n s_n - \frac{t_n v s_n^2}{f(s_n)} + \frac{t_n d_n s_n^3}{f(s_n)},
$$

where $f(s_n) = \sqrt{1+s_n^2}(1 + \sqrt{1+s_n^2})$.

For every $x \in A_2^n \cup A_3^n$, we have either $tv(x) > 1$, $|t_n v_n(x)| \leq 1$ or $tv(x) < -1$, $|t_n v_n(x)| \leq 1$. If $tv(x) > 1$, $|t_n v_n(x)| \leq 1$ and $t < t_n$, then, by (4.11) and (4.12), there is $n_1 > 0$ such that n_1 is independent of x and as $n > n_1$,

$$
-\frac{1}{2}(t - t_n)v(x) - 2t_n |d_n(x)| s_n
$$

$$
\leq -(t - t_n)v(x) - t_n d_n(x) s_n - \frac{t_n v(x) s_n^2}{f(s_n)} - t_n |d_n(x)| s_n
$$

$$
\leq -(t - t_n)v(x) - t_n d_n(x) s_n - \frac{t_n v(x) s_n^2}{f(s_n)} + \frac{t_n d_n(x) s_n^3}{f(s_n)}
$$

$$
= t_n v_n(x) - tv(x) < 0,
$$

i.e.,

$$
|t_n - t||v(x)| < 4t_n |d_n(x)| s_n.
$$

If $tv(x) < -1$, $|t_n v_n(x)| \leq 1$ and $t < t_n$, then, by (4.11) and (4.12), there is $n_2 > 0$ such that n_2 is independent of x and as $n > n_2$,

$$
-\frac{1}{2}(t - t_n)v(x) + 2t_n |d_n(x)| s_n
$$

$$
> -\frac{1}{2}(t - t_n)v(x) - t_n d_n(x) s_n + \frac{t_n d_n(x) s_n^3}{f(s_n)}
$$

$$
> -(t - t_n)v(x) - t_n d_n(x) s_n - \frac{t_n v(x) s_n^2}{f(s_n)} + \frac{t_n d_n(x) s_n^3}{f(s_n)}
$$

$$
= t_n v_n(x) - tv(x) > 0,
$$

i.e.,

$$
|t_n - t||v(x)| < 4t_n |d_n(x)| s_n.
$$

Thus, $\forall x \in A_2^n \cup A_3^n$, if $t < t_n$ and $n > \max(n_1, n_2)$, we have

$$
|t_n - t||v(x)| < 4t_n |d_n(x)| s_n.
$$

Similarly, $\forall x \in B_3^n$, if $t < t_n$, there is $n_3 > 0$ such that n_3 is independent of x and as $n > n_3$, we have

$$
|t_n - t||v(x)| < 4t_n |d_n(x)| s_n.
$$

Hence, $\forall x \in A_2^n \cup A_3^n \cup B_3^n$, if $t < t_n$ and $n > \max(n_1, n_2, n_3)$, we have

$$
(4.13) \qquad |t_n - t||v(x)| < 4t_n |d_n(x)| s_n.
$$

The inequality (4.13) means, if $t < t_n$ and n is large,

$$(4.14) \qquad\qquad |t_n - t| \int_{\Omega_n} |v(x)| dx < 4 t_n s_n \int_{\Omega_n} |d_n(x)| dx,$$

where $\Omega_n = A_2^n \cup A_3^n \cup B_3^n$. By (4.14), if $t < t_n$ and n is large,

$$
\begin{aligned}
\frac{|t_n - t|}{s_n^{\beta_0}} &< 4 s_n^{-2\alpha_0} t_n \int_{\Omega_n} |d_n(x)| dx \Big(\frac{\int_{\Omega_n} |v(x)| dx}{s_n^{\beta_0}} \Big)^{-1} \\
&\leq 4 s_n^{-2\alpha_0} t_n \Big(\int_{\Omega_n} |d_n(x)|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\Omega_n} 1^2 dx \Big)^{\frac{1}{2}} \Big(\frac{\int_{\Omega_n} |v(x)| dx}{s_n^{\beta_0}} \Big)^{-1} \\
&\leq 4 s_n^{-2\alpha_0} t_n t \Big(\int_{\Omega_n} |d_n(x)|^2 dx \Big)^{\frac{1}{2}} |\Omega_n|^{\frac{1}{2}} \Big(\frac{\int_{\Omega_n} |tv(x)| dx}{s_n^{\beta_0}} \Big)^{-1} \\
&\leq 4 s_n^{-2\alpha_0} t_n t \lambda_1^{-1} |\Omega_n|^{\frac{1}{2}} \Big(\frac{\int_{\Omega_n} |tv(x)| dx}{s_n^{\beta_0}} \Big)^{-1},
\end{aligned}
$$

where λ_1 is the first eigenvalue of Δ on Ω and $|\Omega_n|$ is the measure of Ω_n, and by (4.11) and (4.14), if $t < t_n$ and n is large,

$$
\begin{aligned}
|\Omega_n| &= \int_{\Omega_n} dx \leq \int_{\Omega_n} |tv(x)| dx < 4 t_n t \frac{s_n^{\beta_0}}{|t_n - t|} s_n^{1-\beta_0} \int_{\Omega_n} |d_n(x)| dx \\
&= 4 s_n^{\frac{1}{2} - \alpha_0} t_n t D_n^{-1} \int_{\Omega_n} |d_n(x)| dx \leq 4 s_n^{\frac{1}{2} - \alpha_0} t_n t D_n^{-1} \lambda_1^{-1} |\Omega_n|^{\frac{1}{2}},
\end{aligned}
$$

i.e.,

$$|\Omega_n| \leq s_n^{1-2\alpha_0}.$$

Then, if $t < t_n$ and n is large,

$$\frac{|t_n - t|}{s_n^{\beta_0}} \leq 4 s_n^{\frac{1}{2} - 3\alpha_0} t_n t \lambda_1^{-1} \Big(\frac{\int_{\Omega_n} |tv(x)| dx}{s_n^{\beta_0}} \Big)^{-1},$$

i.e.,

$$\frac{\int_{\Omega_n} |tv(x)| dx}{s_n^{\beta_0}} \leq 4 s_n^{\frac{1}{2} - 3\alpha_0} t_n t \lambda_1^{-1} \Big(\frac{|t_n - t|}{s_n^{\beta_0}} \Big)^{-1} = 4 s_n^{\frac{1}{2} - 3\alpha_0} t_n t \lambda_1^{-1} D_n^{-1}.$$

Thus, since $\alpha_0 \in (0, \frac{1}{6}]$, by (4.11), if $t < t_n$ and n is large,

$$(4.15) \qquad\qquad \int_{\Omega_n} |tv(x)| dx \leq D s_n^{\beta_0},$$

where D is a constant. By (4.10) and (4.15), if $t < t_n$ and n is large, then

$$
\begin{aligned}
\int_{\Omega_n} |t_n v_n(x)| dx &= \frac{t_n}{\sqrt{1 + s_n^2}} \int_{\Omega_n} |\sqrt{1 + s_n^2} v_n(x)| dx \\
&\leq \frac{t_n}{\sqrt{1 + s_n^2}} \Big(\int_{\Omega_n} |\sqrt{1 + s_n^2} v_n(x) + s_n d_n(x)| dx + \int_{\Omega_n} |s_n d_n(x)| dx \Big) \\
&\leq \frac{t_n}{\sqrt{1 + s_n^2}} \Big(t^{-1} \int_{\Omega_n} |tv(x)| dx + s_n |\Omega_n|^{\frac{1}{2}} \Big(\int_{\Omega_n} |d_n(x)|^2 dx \Big)^{\frac{1}{2}} \Big) \\
&\leq \frac{t_n}{\sqrt{1 + s_n^2}} \Big(t^{-1} D s_n^{\beta_0} + s_n |\Omega_n|^{\frac{1}{2}} \lambda_1^{-1} \Big) \leq C s_n^{\beta_0},
\end{aligned}
$$

where $C > 0$ is a constant. Hence, either there is $\{t_{n_k}\}$ such that $t > t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t < t_{n_k}$ and

$$\int_{\Omega_{n_k}} |t_{n_k} v_{n_k}(x)| dx \le C s_{n_k}^{\beta_0},$$

where $\Omega_{n_k} = A_2^{n_k} \cup A_3^{n_k} \cup B_3^{n_k}$.

(b) The conclusion can be verified in the same way as (a). $\qquad \square$

Lemma 4.11. *Assume that $L = \{0\}$, $f(x,t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \le 1$, $v_n \to v$ and*

$$\frac{|t_n - t|}{\|v_n - v\|^{\beta_0}} \to +\infty,$$

where $p(v_n) = t_n v_n$, $t_n > 0$, $p(v) = tv$, $t > 0$ and $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. A_i^n, B_i^n, C_i^n, $i = 1, 2, 3$, are defined in Lemma 4.10. Thus,
   (a) *if $C_1^n \cup C_2^n \cup B_1^n \ne \emptyset$ for $n = 1, 2, ...$, then, either there is $\{t_{n_k}\}$ such that $t < t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t > t_{n_k}$ and*

$$\int_{\Omega_{n_k}} |tv(x)| dx \le C s_{n_k}^{\beta_0},$$

where $\Omega_{n_k} = C_1^{n_k} \cup C_2^{n_k} \cup B_1^{n_k}$,
   (b) *if $B_2^n \ne \emptyset$ for $n = 1, 2, ...$, then, either there is $\{t_{n_k}\}$ such that $t < t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t > t_{n_k}$ and*

$$\int_{B_2^{n_k}} |tv(x)| dx \le C s_{n_k}^{\beta_0},$$

where $v_{n_k} = \frac{1}{\sqrt{1+s_{n_k}^2}}(v - s_{n_k} d_{n_k})$, $\|d_{n_k}\| = 1$, $\langle d_{n_k}, v \rangle = 0$ and $s_{n_k} \ge 0$.

Proof. (a) Similar to Lemma 4.10, there is unique d_n with $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$ and $s_n \ge 0$ such that

$$(4.16) \qquad v_n = \frac{1}{\sqrt{1+s_n^2}}(v - s_n d_n)$$

and

$$(4.17) \qquad D_n = \frac{|t_n - t|}{s_n^{\beta_0}} \to +\infty.$$

By (4.16),

$$v = \sqrt{1+s_n^2}\, v_n + s_n d_n.$$

Then,

$$tv = \sqrt{1+s_n^2}\, t v_n + t d_n s_n = (t v_n + t d_n s_n) + (\sqrt{1+s_n^2} - 1) t v_n,$$

i.e.,

$$(4.18) \qquad tv = t_n v_n + (t - t_n) v_n + t d_n s_n + \frac{t v_n s_n^2}{1 + \sqrt{1+s_n^2}}.$$

For every $x \in C_1^n \cup C_2^n$, we have either $|tv(x)| < 1$, $t_n v_n(x) \geq 1$ or $|tv(x)| < 1$, $t_n v_n(x) \leq -1$. If $|tv(x)| < 1$, $t_n v_n(x) \geq 1$ and $t > t_n$, then, by (4.17) and (4.18), there is $n_1 > 0$ such that n_1 is independent of x and as $n > n_1$,

$$
\begin{aligned}
(t - t_n)v_n(x) + td_n(x)s_n &\leq (t - t_n)v_n(x) + td_n(x)s_n + \frac{tv_n(x)s_n^2}{1 + \sqrt{1 + s_n^2}} \\
&= tv(x) - t_n v_n(x) < 0,
\end{aligned}
$$

i.e.,

$$
|t_n - t||v_n(x)| < t|d_n(x)|s_n.
$$

If $|tv(x)| < 1$, $t_n v_n(x) < -1$ and $t > t_n$, then, (4.17) and (4.18), there is $n_2 > 0$ such that n_2 is independent of x and as $n > n_2$,

$$
\begin{aligned}
\frac{1}{2}(t - t_n)v_n(x) + td_n(x)s_n &\geq (t - t_n)v_n(x) + td_n(x)s_n + \frac{tv_n(x)s_n^2}{1 + \sqrt{1 + s_n^2}} \\
&= tv(x) - t_n v_n(x) > 0,
\end{aligned}
$$

i.e.,

$$
|t_n - t||v_n(x)| < 2t|d_n(x)|s_n.
$$

Thus, $\forall x \in C_1^n \cup C_2^n$, if $t > t_n$ and $n > \max(n_1, n_2)$, we have

$$
|t_n - t||v_n(x)| < 2t|d_n(x)|s_n.
$$

Similarly, $\forall x \in B_1^n$, if $t > t_n$, there is $n_3 > 0$ such that n_3 is independent of x and as $n > n_3$, we have

$$
|t_n - t||v_n(x)| < 2t|d_n(x)|s_n.
$$

Hence, $\forall x \in C_1^n \cup C_2^n \cup B_1^n$, if $t > t_n$ and $n > \max(n_1, n_2, n_3)$, we have

$$
(4.19) \qquad |t_n - t||v_n(x)| < 2t|d_n(x)|s_n.
$$

The inequality means, if $t > t_n$ and n is large, then

$$
(4.20) \qquad |t_n - t| \int_{\Omega_n} |v_n(x)|dx < 2ts_n \int_{\Omega_n} |d_n(x)|dx,
$$

where $\Omega_n = C_1^n \cup C_2^n \cup B_1^n$. By (4.20), if $t > t_n$ and n is large,

$$
\begin{aligned}
\frac{|t_n - t|}{s_n^{\beta_0}} &< 2s_n^{-2\alpha_0}t \int_{\Omega_n} |d_n(x)|dx \Big(\frac{\int_{\Omega_n} |v_n(x)|dx}{s_n^{\beta_0}}\Big)^{-1} \\
&\leq 2s_n^{-2\alpha_0}t\Big(\int_{\Omega_n} |d_n(x)|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\Omega_n} 1^2 dx\Big)^{\frac{1}{2}}\Big(\frac{\int_{\Omega_n} |v_n(x)|dx}{s_n^{\beta_0}}\Big)^{-1} \\
&\leq 2s_n^{-2\alpha_0}t_n t\Big(\int_{\Omega_n} |d_n(x)|^2 dx\Big)^{\frac{1}{2}}|\Omega_n|^{\frac{1}{2}}\Big(\frac{\int_{\Omega_n} |t_n v_n(x)|dx}{s_n^{\beta_0}}\Big)^{-1} \\
&\leq 2s_n^{-2\alpha_0}t_n t\lambda_1^{-1}|\Omega_n|^{\frac{1}{2}}\Big(\frac{\int_{\Omega_n} |t_n v_n(x)|dx}{s_n^{\beta_0}}\Big)^{-1},
\end{aligned}
$$

where λ_1 is the first eigenvalue of Δ on Ω and $|\Omega_n|$ is the measure of Ω_n, and by (4.17) and (4.20),

$$
\begin{aligned}
|\Omega_n| &= \int_{\Omega_n} dx \leq \int_{\Omega_n} |t_n v_n(x)|dx < 2t_n t \frac{s_n^{\beta_0}}{|t_n - t|}s_n^{1-\beta_0}\int_{\Omega_n} |d_n(x)|dx \\
&= 2s_n^{\frac{1}{2}-\alpha_0}t_n t D_n^{-1}\int_{\Omega_n} |d_n(x)|dx \leq 2s_n^{\frac{1}{2}-\alpha_0}t_n t D_n^{-1}\lambda_1^{-1}|\Omega_n|^{\frac{1}{2}},
\end{aligned}
$$

i.e.,

$$|\Omega_n| \le s_n^{1-2\alpha_0}.$$

Then,

$$\frac{|t_n - t|}{s_n^{\beta_0}} \le 2s_n^{\frac{1}{2}-3\alpha_0} t_n t \left(\frac{\int_{\Omega_n} |t_n v_n(x)| dx}{s_n^{\beta_0}} \right)^{-1},$$

i.e.,

$$\frac{\int_{\Omega_n} |t_n v_n(x)| dx}{s_n^{\beta_0}} \le 2s_n^{\frac{1}{2}-3\alpha_0} t_n t \left(\frac{|t_n - t|}{s_n^{\beta_0}} \right)^{-1} = 2s_n^{\frac{1}{2}-3\alpha_0} t_n t D_n^{-1}.$$

Thus, since $\alpha_0 \in (0, \frac{1}{6}]$, by (4.17), if $t > t_n$ and n is large, then

$$(4.21) \qquad \int_{\Omega_n} |t_n v_n(x)| dx \le D s_n^{\beta_0},$$

where $D > 0$ is a constant. By (4.16) and (4.21), if $t > t_n$ and n is large, then

$$\int_{\Omega_n} |tv(x)| dx \le \int_{\Omega_n} |t(v(x) - s_n d_n(x))| dx + \int_{\Omega_n} |ts_n d_n(x)| dx$$

$$\le \quad tt_n^{-1} \sqrt{1 + s_n^2} \int_{\Omega_n} |t_n v_n(x)| dx + t|\Omega_n| s_n \left(\int_{\Omega_n} |d_n(x)|^2 dx \right)^{\frac{1}{2}}$$

$$\le \quad tt_n^{-1} D \sqrt{1 + s_n^2} s_n^{\beta_0} + t|\Omega_n| s_n \le C s_n^{\beta_0},$$

where $|\Omega_n|$ is the measure of Ω_n and $C > 0$ is a constant. Hence, either there is $\{t_{n_k}\}$ such that $t < t_{n_k}$ or there is $\{t_{n_k}\}$ and a constant $C > 0$ such that $t > t_{n_k}$ and

$$\int_{\Omega_{n_k}} |tv(x)| dx \le C s_{n_k}^{\beta_0},$$

where $\Omega_{n_k} = C_1^{n_k} \cup C_2^{n_k} \cup B_1^{n_k}$.

(b) The conclusion can be proved in the same way as (a). □

Lemma 4.12. *Assume that $L = \{0\}$, $f(x,t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \le 1$, $v_n \to v$. Denote $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v) = tv$, $t > 0$. A_i^n, B_i^n, C_i^n, $i = 1,2,3$, are defined in Lemma 4.10. Then,*
  (a)

$$G_0^n + G_1^n + G_2^n + G_3^n = A_n,$$

where

$$A_n = A_n^+ + A_n^0 + A_n^-$$

with

$$A_n^+ = c\alpha t_n^\alpha \int_{\Omega_n^+} |x|^r U^n(\alpha)(x) dx, \quad A_n^0 = c(\beta + k_n) t_n^\beta \int_{\Omega_n^0} |x|^r U^n(\beta)(x) dx,$$

$$A_n^- = c\beta t_n^\beta \int_{\Omega_n^-} |x|^r U^n(\beta)(x) dx, \quad U^n(\gamma) = |v_n|^\gamma - |v|^\gamma \ for \ \gamma > 0,$$

and

$$\Omega_n^+ = A_1^n \cup B_1^n \cup C_1^n, \quad \Omega_n^0 = A_2^n \cup B_2^n \cup C_2^n, \quad \Omega_n^- = A_3^n \cup B_3^n \cup C_3^n,$$

$k_n \in [0, \alpha - \beta]$,

$$G_0^n = G_0^{n^+} + G_0^{n^0} + G_0^{n^-} - f^n(2)$$

with

$$G_0^{n^+} = cf^n(\alpha) \int_{\Omega_n^+} \alpha |x|^r |v(x)|^\alpha dx, \quad G_0^{n^0} = cf^n(\beta) \int_{\Omega_n^0} (\beta + k_n)|x|^r |v(x)|^\beta dx$$

and

$$G_0^{n^-} = cf^n(\beta) \int_{\Omega_n^-} \beta |x|^r |v(x)|^\beta dx, \quad f^n(\gamma) = t^\gamma - t_n^\gamma \ \text{for } \gamma > 0,$$

$$G_1^n = I_1^n + I_2^n + I_3^n$$

with

$$I_1^n = c\alpha \int_{A_2^n} |x|^r |tv(x)|^\alpha dx - c(\beta + k_n) \int_{A_2^n} |x|^r |tv(x)|^\beta dx \geq 0,$$

$$I_2^n = c\alpha \int_{A_3^n} |x|^r |tv(x)|^\alpha dx - c\beta \int_{A_3^n} |x|^r |tv(x)|^\beta dx \geq 0,$$

and

$$I_3^n = ck \int_{B_3^n} |x|^r |tv(x)|^\beta dx \geq 0,$$

$k \in [0, \alpha - \beta]$,

$$G_2^n = J_1^n + J_2^n + J_3^n$$

with

$$J_1^n = c\beta \int_{C_1^n} |x|^r |tv(x)|^\beta dx - c\alpha \int_{C_1^n} |x|^r |tv(x)|^\alpha dx,$$

$$J_2^n = -ck_n \int_{C_2^n} |tv(x)|^\beta dx,$$

and

$$J_3^n = c(\beta + k) \int_{B_1^n} |x|^r |tv(x)|^\beta dx - c\alpha \int_{B_1^n} |x|^r |tv(x)|^\alpha dx,$$

and

$$G_3^n = c(k - k_n) \int_{B_2^n} |x|^r |tv(x)|^\beta dx,$$

(b) *there is a constant $D > 0$ such that, for every $\bar{\Omega} \subseteq \Omega \subset \mathbb{R}^m$, $\gamma > 1$ for $m = 1, 2$ and $1 < \gamma \leq \frac{2m}{m-2}$ for $m \geq 3$,*

(4.22)
$$\left| \int_{\bar{\Omega}} |x|^r U^n(\gamma)(x) dx \right| \leq D s_n,$$

as n is large and there is a constant $\bar{D} > 0$ such that

$$|A_n| \leq \bar{D} s_n,$$

where $v_n = \frac{1}{\sqrt{1+s_n^2}}(v - s_n d_n)$, $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$ and $s_n \geq 0$.

Proof. (a) Since $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v) = tv$, $t > 0$, there are $k, k_n \in [0, \alpha - \beta]$ such that

$$t^2 - c \int_{\Omega^+} \alpha f(x) dx - c \int_{\Omega^0} (\beta + k) g(x) dx - c \int_{\Omega^-} \beta g(x) dx = 0,$$

where $f(x) = |x|^r |tv(x)|^\alpha$, $g(x) = |x|^r |tv(x)|^\beta$, $\Omega^+ = \{x \in \Omega | |tv(x)| > 1\}$, $\Omega^0 = \{x \in \Omega | |tv(x)| = 1\}$ and $\Omega^- = \{x \in \Omega | |tv(x)| < 1\}$, and

$$t_n^2 - c \int_{\Omega_n^+} \alpha f_n(x) dx - c \int_{\Omega_n^0} (\beta + k_n) g_n(x) dx - c \int_{\Omega_n^-} \beta g_n(x) dx = 0,$$

where $f_n(x) = |x|^r |t_n v_n(x)|^\alpha$ and $g_n(x) = |x|^r |t_n v_n(x)|^\beta$, i.e.,

$$t_n^2 - c \int_{\Omega_n^+} \alpha \bar{f}_n(x) dx - c \int_{\Omega_n^0} (\beta + k_n) \bar{g}_n(x) dx - c \int_{\Omega_n^-} \beta \bar{g}_n(x) dx = A_n,$$

where $\bar{f}_n(x) = |x|^r |t_n v(x)|^\alpha$ and $\bar{g}_n(x) = |x|^r |t_n v(x)|^\beta$.

By the first and third equality, after simple calculation, we can find that the conclusion is correct.

(b) Indeed, as n is large,

$$\left| \int_{\bar{\Omega}} |x|^r U^n(\gamma)(x) dx \right| = \left| \int_{\bar{\Omega}} |x|^r |v_n(x)|^\gamma dx - \int_{\bar{\Omega}} |x|^r |v(x)|^\gamma dx \right|$$

$$= \gamma (\phi(v_n, v, \eta))^{\gamma-1} \left| \left(\int_{\bar{\Omega}} |x|^r |v_n(x)|^\gamma dx \right)^{\frac{1}{\gamma}} - \left(\int_{\bar{\Omega}} |x|^r |v(x)|^\gamma dx \right)^{\frac{1}{\gamma}} \right|$$

$$\leq \gamma (\phi(v_n, v, \eta))^{\gamma-1} \left(\int_{\bar{\Omega}} |x|^r |v_n(x) - v(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$$

$$\leq 2^{\gamma-1} \gamma \left(\int_{\bar{\Omega}} |x|^r |v(x)|^\gamma dx \right)^{\frac{\gamma-1}{\gamma}} \left(\int_{\bar{\Omega}} |x|^r |v_n(x) - v(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$$

$$\leq 2^{\gamma-1} \gamma \left(\int_{\Omega} |x|^r |v(x)|^\gamma dx \right)^{\frac{\gamma-1}{\gamma}} \left(\int_{\Omega} |x|^r |v_n(x) - v(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$$

$$\leq 2^{\gamma-1} M \gamma \left(\int_{\Omega} |v(x)|^\gamma dx \right)^{\frac{\gamma-1}{\gamma}} \left(\int_{\Omega} |v_n(x) - v(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$$

$$\leq 2^{\gamma-1} C^\gamma M \gamma \|v\|^{\gamma-1} \|v_n - v\| \leq (2C)^\gamma M \gamma \|v\|^{\gamma-1} \frac{s_n}{\sqrt{1+s_n^2}} \|d_n\|$$

$$= (2C)^\gamma M \gamma \|v\|^{\gamma-1} \frac{s_n}{\sqrt{1+s_n^2}},$$

where $\phi(v_n, v, \eta) = \eta \left(\int_{\bar{\Omega}} |x|^r |v_n(x)|^\gamma dx \right)^{\frac{1}{\gamma}} + (1-\eta) \left(\int_{\bar{\Omega}} |x|^r |v(x)|^\gamma dx \right)^{\frac{1}{\gamma}}$, $\eta \in [0,1]$,

$$M = \sup_{x \in \Omega} |x|^r,$$

and $C > 0$ is a constant, i.e.,

$$\left| \int_{\bar{\Omega}} |x|^r U^n(\gamma)(x) dx \right| \leq D s_n,$$

where $D > 0$ is a constant. Hence, as n is large,

$$|A_n| \leq |A_n^+| + |A_n^0| + |A_n^-| \leq c(\alpha t_n^{\alpha_1} + (\beta + k_n) t_n^\beta + \beta t_n^\beta) D s_n,$$

i.e., there is a constant $\bar{D} > 0$ such that $|A_n| \leq \bar{D} s_n$. $\square$

Lemma 4.13. *Assume that $L = \{0\}$, $f(x,t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \leq 1$, $v_n \to v$. Denote $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v) = tv$, $t > 0$. A_i^n, B_i^n, C_i^n, $i = 1,2,3$, are defined in Lemma 4.10. Then,*
(a)

$$H_0^n + H_1^n + H_2^n + H_3^n = -B_n,$$

where

$$B_n = B_n^+ + B_n^0 + B_n^-$$

with

$$B_n^+ = c\alpha t^\alpha \int_{\Omega^+} |x|^r U^n(\alpha)(x) dx, \quad B_n^0 = c(\beta + k)t^\beta \int_{\Omega^0} |x|^r U^n(\beta)(x) dx,$$

$$B_n^- = c\beta t^\beta \int_{\Omega^-} |x|^r U^n(\beta)(x) dx, \quad U^n(\gamma) = |v_n|^\gamma - |v|^\gamma \ for \ \gamma > 0,$$

and

$$\Omega^+ = A_1^n \cup A_2^n \cup A_3^n, \quad \Omega^0 = B_1^n \cup B_2^n \cup B_3^n, \quad \Omega^- = C_1^n \cup C_2^n \cup C_3^n,$$

$k \in [0, \alpha - \beta]$,

$$H_0^n = H_0^{n^+} + H_0^{n^0} + H_0^{n^-} - f^n(2)$$

with

$$H_0^{n^+} = cf^n(\alpha) \int_{\Omega^+} \alpha |x|^r |v_n(x)|^\alpha dx, \quad H_0^{n^0} = cf^n(\beta) \int_{\Omega^0} (\beta + k)|x|^r |v_n(x)|^\beta dx,$$

and

$$H_0^{n^-} = cf^n(\beta) \int_{\Omega^-} \beta |x|^r |v_n(x)|^\beta dx, \quad f^n(\gamma) = t_n^\gamma - t^\gamma \ for \ \gamma > 0,$$

$$H_1^n = I_1^n + I_2^n + I_3^n$$

with

$$I_1^n = c\alpha \int_{C_1^n} |x|^r |t_n v_n(x)|^\alpha dx - c\beta \int_{C_1^n} |x|^r |t_n v_n(x)|^\beta dx \geq 0,$$

$$I_2^n = ck_n \int_{C_2^n} |x|^r |t_n v_n(x)|^\beta dx \geq 0,$$

and

$$I_3^n = c\alpha \int_{B_1^n} |x|^r |t_n v_n(x)|^\alpha dx - c(\beta + k) \int_{B_1^n} |x|^r |t_n v_n(x)|^\beta dx \geq 0,$$

$k_n \in [0, \alpha - \beta]$,

$$H_2^n = J_1^n + J_2^n + J_3^n$$

with

$$J_1^n = c\beta \int_{A_3^n} |x|^r |t_n v_n(x)|^\beta dx - c\alpha \int_{A_3^n} |x|^r |t_n v_n(x)|^\alpha dx,$$

$$J_2^n = -ck \int_{B_3^n} |x|^r |t_n v_n(x)|^\beta dx,$$

and

$$J_3^n = c(\beta + k_n) \int_{A_2^n} |x|^r |t_n v_n(x)|^\beta dx - c\alpha \int_{A_2^n} |x|^r |t_n v_n(x)|^\alpha dx,$$

and

$$H_3^n = c(k_n - k) \int_{B_2^n} |x|^r |t_n v_n(x)|^\beta dx,$$

(b) *there is a constant $\bar{D} > 0$ such that*

$$|B_n| \leq \bar{D} s_n,$$

where $v_n = \frac{1}{\sqrt{1+s_n^2}}(v - s_n d_n)$, $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$ and $s_n > 0$.

Proof. (a) Since $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v) = tv$, $t > 0$, there are $k, k_n \in [0, \alpha - \beta]$ such that

$$t^2 - c \int_{\Omega^+} \alpha f(x) dx - c \int_{\Omega^0} (\beta + k) g(x) dx - c \int_{\Omega^-} \beta g(x) dx = 0,$$

where $f(x) = |x|^r |tv(x)|^\alpha$ and $g(x) = |x|^r |tv(x)|^\beta$, i.e.,

$$t^2 - c \int_{\Omega^+} \alpha \bar{f}_n(x) dx - c \int_{\Omega^0} (\beta + k) \bar{g}_n(x) dx - c \int_{\Omega^-} \beta \bar{g}_n(x) dx = -B_n,$$

where $\bar{f}_n(x) = |x|^r |tv_n(x)|^\alpha$ and $\bar{g}_n(x) = |x|^r |tv_n(x)|^\beta$, and

$$t_n^2 - c \int_{\Omega_n^+} \alpha f_n(x) dx - c \int_{\Omega_n^0} (\beta + k_n) g_n(x) dx - c \int_{\Omega_n^-} \beta g_n(x) dx = 0,$$

where $f_n(x) = |x|^r |t_n v_n(x)|^\alpha$, $g_n(x) = |x|^r |t_n v_n(x)|^\beta$, $\Omega_n^+ = A_1^n \cup B_1^n \cup C_1^n$, $\Omega_n^0 = A_2^n \cup B_2^n \cup C_2^n$ and $\Omega_n^- = A_3^n \cup B_3^n \cup C_3^n$.

By the second and third equality, after simple computation, we should know that the conclusion is true.

(b) By (4.22), as n is large,

$$|B_n| \leq |B_n^+| + |B_n^0| + |B_n^-| \leq c(\alpha t^{\alpha_1} + (\beta + k) t^{\alpha_2} + \beta t^\beta) D s_n,$$

i.e., there is a constant $\bar{D} > 0$ such that $|B_n| \leq \bar{D} s_n$. $\square$

Lemma 4.14. *Assume that $L = \{0\}$, $f(x, t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \leq 1$, $v_n \to v$. Denote $p(v_n) = t_n v_n$, $t_n > 0$, $p(v) = tv$, $t > 0$ and $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. A_i^n, B_i^n, C_i^n, $i = 1, 2, 3$, are defined in Lemma 4.10. If $A_2^n \cup A_3^n \cup B_3^n \neq \emptyset$ and $t > t_n$ for $n = 1, 2, ...$, then, there is a constant $L_h > 0$ such that*

$$|t_n - t| \leq L_h \|v_n - v\|^{\beta_0}.$$

Proof. Suppose that there is not a constant $L_h > 0$ such that

$$|t_n - t| \leq L_h \|v_n - v\|^{\beta_0}.$$

Then, there is $\{v_{n_l}\}$ such that

$$\lim_{l \to \infty} \frac{|t_{n_l} - t|}{\|v_{n_l} - v\|^{\beta_0}} = +\infty.$$

Thus, similar to Lemma 4.10,

$$\text{(4.23)} \qquad \frac{|t_{n_l} - t|}{s_{n_l}^{\beta_0}} \to +\infty,$$

where $v_{n_l} = \frac{1}{\sqrt{1 + s_{n_l}^2}} (v - s_{n_l} d_{n_l})$, $\|d_{n_l}\| = 1$, $\langle d_{n_l}, v \rangle = 0$ and $s_{n_l} \geq 0$.

First, by Lemma 4.11, there are following four cases. In these four cases, $\{n_{l'}\}$ represents a subsequence of $\{n_l\}$.

The first case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} \neq \emptyset$, $B_2^{n_{l'}} \neq \emptyset$,

$$\text{(4.24)} \qquad \int_{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}} |tv(x)| dx \leq D s_{n_{l'}}^{\beta_0} \quad \text{and} \quad \int_{B_2^{n_{l'}}} |tv(x)| dx \leq D s_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The second case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} \neq \emptyset$, $B_2^{n_{l'}} = \emptyset$ and

$$\int_{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}} |tv(x)|dx \leq Ds_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The third case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} = \emptyset$, $B_2^{n_{l'}} \neq \emptyset$ and

$$\int_{B_2^{n_{l'}}} |tv(x)|dx \leq Ds_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The fourth case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} = \emptyset$ and $B_2^{n_{l'}} = \emptyset$.

We only discuss the first case and the remaining three cases can be handled in a similar way. By (4.24),

$$|J_1^{n_{l'}}| = |c\beta \int_{C_1^{n_{l'}}} |x|^r |tv(x)|^\beta dx - c\alpha \int_{C_1^{n_{l'}}} |x|^r |tv(x)|^\alpha dx|$$

$$\leq c\beta M \int_{C_1^{n_{l'}}} |tv(x)|dx + c\alpha M \int_{C_1^{n_{l'}}} |tv(x)|dx \leq c(\beta + \alpha)MDs_{n_{l'}}^{\beta_0},$$

$$|J_2^{n_{l'}}| = |-ck_{n_{l'}} \int_{C_2^{n_{l'}}} |x|^r |tv(x)|^\beta dx| \leq ck_{n_{l'}} M \int_{C_2^{n_{l'}}} |tv(x)|dx \leq ck_{n_{l'}} MDs_{n_{l'}}^{\beta_0},$$

and

$$|J_3^{n_{l'}}| = |c(\beta + k) \int_{B_1^{n_{l'}}} |x|^r |tv(x)|^\beta dx - c\alpha \int_{B_1^{n_{l'}}} |x|^r |tv(x)|^\alpha dx|$$

$$\leq c(\beta + k)M \int_{B_1^{n_{l'}}} |tv(x)|dx + c\alpha M \int_{B_1^{n_{l'}}} |tv(x)|dx \leq c(\beta + \alpha + k)MDs_{n_{l'}}^{\beta_0},$$

where $M = \sup_{x \in \Omega} |x|^r$. These three inequalities mean that

$$(4.25) \qquad |G_2^{n_{l'}}| \leq |J_1^{n_{l'}}| + |J_2^{n_{l'}}| + |J_3^{n_{l'}}| \leq c(2\alpha + 2\beta + k_{n_{l'}} + k)MDs_{n_{l'}}^{\beta_0}.$$

By (4.24),

$$|G_3^{n_{l'}}| = |c(k - k_{n_{l'}}) \int_{B_2^{n_{l'}}} |x|^r |tv(x)|^\beta dx| \leq c|k - k_{n_{l'}}|M \int_{B_2^{n_{l'}}} |tv(x)|dx,$$

i.e.,

$$(4.26) \qquad |G_3^{n_{l'}}| \leq c|k - k_{n_{l'}}|MDs_{n_{l'}}^{\beta_0}.$$

Then, by (4.25), (4.26) and Lemma 4.12,

$$G_0^{n_{l'}} \leq G_0^{n_{l'}} + G_1^{n_{l'}} = A_{n_{l'}} - G_2^{n_{l'}} - G_3^{n_{l'}} \leq |A_{n_{l'}}| + |G_2^{n_{l'}}| + |G_3^{n_{l'}}|$$

$$\leq \bar{D}s_{n_{l'}} + c(2\alpha + 2\beta + k_{n_{l'}} + k)MDs_{n_{l'}}^{\beta_0} + c|k - k_{n_{l'}}|MDs_{n_{l'}}^{\beta_0} \leq \bar{C}s_{n_{l'}}^{\beta_0},$$

where $\bar{C}, \bar{D} > 0$ are two constants, i.e.,

$$(4.27) \qquad t - t_{n_{l'}} \leq \frac{\bar{C}}{G^{n_{l'}}} s_{n_{l'}}^{\beta_0},$$

where $G^{n_{l'}} = G^+_{n_{l'}} + G^0_{n_{l'}} + G^-_{n_{l'}} - (t + t_{n_{l'}})$ with

$$G^+_{n_{l'}} = c\alpha(\eta_1 t + (1-\eta_1)t_{n_{l'}})^{\alpha-1} \int_{\Omega^+_{n_{l'}}} \alpha|x|^r |v(x)|^\alpha dx,$$

$$G^0_{n_{l'}} = c\beta(\eta_2 t + (1-\eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^0_{n_{l'}}} (\beta + k_{n_{l'}})|x|^r |v(x)|^\beta dx,$$

and

$$G^-_{n_{l'}} = c\beta(\eta_2 t + (1-\eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^-_{n_{l'}}} \beta|x|^r |v(x)|^\beta dx,$$

$\eta_1, \eta_2 \in [0,1]$. On the other hand, in the verification of Lemma 4.12, we have

$$(4.28) \qquad |\int_{\bar{\Omega}} |x|^r U^n(\gamma)(x)dx| \le 2^{\gamma-1}C^\gamma M\gamma\|v\|^{\gamma-1}\|v_n - v\|$$

for $\bar{\Omega} \subseteq \Omega$, $\gamma > 1$ for $m = 1,2$ and $1 < \gamma \le \frac{2m}{m-2}$ for $m \ge 3$, as n is large, where $U^n(\gamma) = |v_n|^\gamma - |v|^\gamma$ and $C > 0$ is a constant. Since $v_{n_{l'}} \to v$, (4.28) means, $\forall \epsilon > 0$,

$$|\int_{\hat{\Omega}} |x|^r |v_{n_{l'}}(x)|^\gamma dx - \int_{\hat{\Omega}} |x|^r |v(x)|^\gamma dx| \le \epsilon,$$

for $(\gamma, \hat{\Omega}) = (\alpha, \Omega^+_{n_{l'}}), (\beta, \Omega^0_{n_{l'}}), (\beta, \Omega^-_{n_{l'}})$, as $n_{l'}$ is large. By this inequality, $\forall \zeta > 0$,

$$(4.29) \qquad\qquad\qquad |G^{n_{l'}} - \bar{G}^{n_{l'}}| \le \zeta,$$

as $n_{l'}$ is large, where $\bar{G}^{n_{l'}} = \bar{G}^+_{n_{l'}} + \bar{G}^0_{n_{l'}} + \bar{G}^-_{n_{l'}} - (t + t_{n_{l'}})$ with

$$\bar{G}^+_{n_{l'}} = c\alpha(\eta_1 t + (1-\eta_1)t_{n_{l'}})^{\alpha-1} \int_{\Omega^+_{n_{l'}}} \alpha|x|^r |v_{n_{l'}}(x)|^\alpha dx,$$

$$\bar{G}^0_{n_{l'}} = c\beta(\eta_2 t + (1-\eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^0_{n_{l'}}} (\beta + k_{n_{l'}})|x|^r |v_{n_{l'}}(x)|^\beta dx,$$

and

$$\bar{G}^-_{n_{l'}} = c\beta(\eta_2 t + (1-\eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^-_{n_{l'}}} \beta|x|^r |v_{n_{l'}}(x)|^\beta dx.$$

Since $p(v_n) = t_n v_n$, $t_n > 0$, we have

$$t_n^2 - c\int_{\Omega^+_n} \alpha f_n(x)dx - c\int_{\Omega^0_n} (\beta + k_n)g_n(x)dx - c\int_{\Omega^-_n} \beta g_n(x)dx = 0,$$

where $f_n(x) = |x|^r |t_n v_n(x)|^\alpha$ and $g_n(x) = |x|^r |t_n v_n(x)|^\beta$. Thus, for large $n_{l'}$,

$$(4.30) \qquad \bar{G}^{n_{l'}} \ge (\beta - \frac{\beta-2}{4})t_{n_{l'}} - (2 + \frac{\beta-2}{4})t_{n_{l'}} = \frac{\beta-2}{2}t_{n_{l'}} \ge \frac{\beta-2}{4}t.$$

By (4.29) and (4.30), if we set $\zeta = \frac{\beta-2}{8}t$, then, for large $n_{l'}$,

$$(4.31) \qquad\qquad\qquad G^{n_{l'}} > \frac{\beta-2}{8}t.$$

By (4.27) and (4.31), for large $n_{l'}$,

$$\frac{|t_{n_{l'}} - t|}{s^{\beta_0}_{n_{l'}}} = \frac{t - t_{n_{l'}}}{s^{\beta_0}_{n_{l'}}} < \frac{8\bar{C}}{(\beta-2)t}.$$

This is a contradiction to (4.23). Hence, the conclusion is true. $\qquad\qquad \square$

Lemma 4.15. *Assume that $L = \{0\}$, $f(x, t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \le 1$, $v_n \to v$. Denote $p(v_n) = t_n v_n$, $t_n > 0$, $p(v) = tv$, $t > 0$ and $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. A_i^n, B_i^n, C_i^n, $i = 1, 2, 3$, are defined in Lemma 4.10. If $A_2^n \cup A_3^n \cup B_3^n \ne \emptyset$, $t < t_n$ and*

$$(4.32) \qquad \int_{\Omega_n} |t_n v_n(x)| dx \le D s_n^{\beta_0}$$

for $n = 1, 2, ...$, where $\Omega_n = A_2^n \cup A_3^n \cup B_3^n$, $v_n = \frac{1}{\sqrt{1+s_n^2}}(v - s_n d_n)$, $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$, $s_n \ge 0$ and $D > 0$ is a constant, then, there is a constant $L_h > 0$ such that

$$|t_n - t| \le L_h \|v_n - v\|^{\beta_0}.$$

Proof. Suppose that there is not a constant $L_h > 0$ such that

$$|t_n - t| \le L_h \|v_n - v\|^{\beta_0}.$$

Then, similar to Lemma 4.14, there is $\{t_{n_l}\}$ such that

$$(4.33) \qquad \frac{|t_{n_l} - t|}{s_{n_l}^{\beta_0}} \to +\infty.$$

By Lemma 4.10, there are the following two cases. In these two cases, $\{n_{l'}\}$ represents a subsequence of $\{n_l\}$.

The first case is there is $\{B_2^{n_{l'}}\}$ such that $B_2^{n_{l'}} \ne \emptyset$,

$$(4.34) \qquad \int_{B_2^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx \le \hat{D} s_{n_{l'}}^{\beta_0},$$

where $\hat{D} > 0$ is a constant.

The second case is there is $\{B_2^{n_{l'}}\}$ such that $B_2^{n_{l'}} = \emptyset$.

We only work on the first case and the second case can be discussed in a similar way.

By (4.32),

$$|J_1^{n_{l'}}| = |c\beta \int_{A_3^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\beta dx - c\alpha \int_{A_3^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\alpha dx|$$

$$\le \quad c\beta M \int_{A_3^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx + c\alpha M \int_{A_3^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx \le c(\beta + \alpha) M D s_{n_{l'}}^{\beta_0},$$

$$|J_2^{n_{l'}}| = |ck \int_{B_3^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\beta dx| \le ckM \int_{B_3^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx \le ckM D s_{n_{l'}}^{\beta_0},$$

and

$$|J_3^{n_{l'}}| = |c(\beta + k_{n_{l'}}) \int_{A_2^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\beta dx - c\alpha \int_{A_2^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\alpha dx|$$

$$\le \quad c(\beta + k_{n_{l'}}) M \int_{A_2^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx + c\alpha M \int_{A_2^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx$$

$$\le \quad c(\beta + \alpha + k_{n_{l'}}) M D s_{n_{l'}}^{\beta_0},$$

where $M = \sup_{x \in \Omega} |x|^r$. These three inequalities mean that

$$(4.35) \qquad |H_2^{n_{l'}}| \le |J_1^{n_{l'}}| + |J_2^{n_{l'}}| + |J_2^{n_{l'}}| \le c(2\beta + 2\alpha + k_{n_{l'}} + k) M D s_{n_{l'}}^{\beta_0}.$$

By (4.34),

$$|H_3^{n_{l'}}| = |c(k_{n_{l'}} - k) \int_{B_2^{n_{l'}}} |x|^r |t_{n_{l'}} v_{n_{l'}}(x)|^\beta dx| \le c|k_{n_{l'}} - k|M \int_{B_2^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx,$$

i.e.,

$$(4.36) \qquad\qquad |H_3^{n_{l'}}| \le c|k - k_{n_{l'}}|M\hat{D}s_{n_{l'}}^{\beta_0}.$$

By (4.35), (4.36) and Lemma 4.13,

$$H_0^{n_{l'}} \le H_0^{n_{l'}} + H_1^{n_{l'}} = -B_{n_{l'}} - H_2^{n_{l'}} - H_3^{n_{l'}} \le |B_{n_{l'}}| + |H_2^{n_{l'}}| + |H_3^{n_{l'}}|$$

$$\le \bar{D}s_{n_{l'}} + c(2\alpha + 2\beta + k_{n_{l'}} + k)MDs_{n_{l'}}^{\beta_0} + c|k - k_{n_{l'}}|M\hat{D}s_{n_{l'}}^{\beta_0} \le \bar{C}s_{n_{l'}}^{\beta_0},$$

where $\bar{C}, \bar{D} > 0$ is a constant, i.e.,

$$(4.37) \qquad\qquad t_{n_{l'}} - t \le \frac{\bar{C}}{H^{n_{l'}}} s_{n_{l'}}^{\beta_0},$$

where $H^{n_{l'}} = H_{n_{l'}}^+ + H_{n_{l'}}^0 + H_{n_{l'}}^- - (t + t_{n_{l'}})$ with

$$H_{n_{l'}}^+ = c\alpha(\eta_1 t + (1 - \eta_1)t_{n_{l'}})^{\alpha-1} \int_{\Omega^+} \alpha|x|^r |v_{n_{l'}}(x)|\alpha dx,$$

$$H_{n_{l'}}^0 = c\beta(\eta_2 t + (1 - \eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^0} (\beta + k)|x|^r |v_{n_{l'}}(x)|^\beta dx,$$

and

$$H_{n_{l'}}^- = c\beta(\eta_2 t + (1 - \eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^-} \beta|x|^r |v_{n_{l'}}(x)|^\beta dx,$$

$\eta_1, \eta_2 \in [0, 1]$. Similar to Lemma 4.14, since $v_{n_{l'}} \to v$, by (4.28), we have, $\forall \epsilon > 0$,

$$|\int_{\hat{\Omega}} |x|^r |v_{n_{l'}}(x)|^\gamma dx - \int_{\hat{\Omega}} |x|^r |v(x)|^\gamma dx| \le \epsilon,$$

where $(\gamma, \hat{\Omega}) = (\alpha, \Omega^+), (\beta, \Omega^0), (\beta, \Omega^-)$, as $n_{l'}$ is large. By this inequality, $\forall \zeta > 0$,

$$(4.38) \qquad\qquad |H^{n_{l'}} - \bar{H}^{n_{l'}}| \le \zeta,$$

as $n_{l'}$ is large, where $\bar{H}^{n_{l'}} = \bar{H}_{n_{l'}}^+ + \bar{H}_{n_{l'}}^0 + \bar{H}_{n_{l'}}^- - (t + t_{n_{l'}})$.

$$\bar{H}_{n_{l'}}^+ = c\alpha(\eta_1 t + (1 - \eta_1)t_{n_{l'}})^{\alpha-1} \int_{\Omega^+} \alpha_1|x|^r |v(x)|^\alpha dx,$$

$$\bar{H}_{n_{l'}}^0 = c\beta(\eta_2 t + (1 - \eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^0} (\beta + k)|x|^r |v(x)|^\beta dx,$$

and

$$\bar{H}_{n_{l'}}^- = c\beta(\eta_2 t + (1 - \eta_2)t_{n_{l'}})^{\beta-1} \int_{\Omega^-} \beta|x|^r |v(x)|^\beta dx.$$

Since $p(v) = tv$, $t > 0$,

$$t^2 - c\int_{\Omega^+} \alpha|tv(x)|^\alpha dx - c\int_{\Omega^0} (\beta + k)|tv(x)|^\beta dx - c\int_{\Omega^-} \beta|tv(x)|^\beta dx = 0.$$

Then, for large $n_{l'}$,

$$(4.39) \qquad\qquad \bar{H}^{n_{l'}} \ge (\beta - \frac{\beta - 2}{4})t - (2 + \frac{\beta - 2}{4})t = \frac{\beta - 2}{2}t.$$

By (4.38) and (4.39), if we set $\zeta = \frac{\beta-2}{4}t$, then, for large $n_{l'}$,

$$(4.40) \qquad\qquad H^{n_{l'}} > \frac{\beta - 2}{4}t.$$

By (4.37) and (4.40), for large $n_{l'}$,

$$\frac{|t_{n_{l'}} - t|}{s_{n_{l'}}^{\beta_0}} = \frac{t_{n_{l'}} - t}{s_{n_{l'}}^{\beta_0}} < \frac{4\bar{C}}{(\beta - 2)t}.$$

This is a contradiction to (4.33). Hence, the conclusion is correct. □

Lemma 4.16. *Assume that $L = \{0\}$, $f(x,t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$, p is continuous at $v \in S_{L^\perp}$ and $\{v_n\} \subset S_{L^\perp}$ satisfies $\|v_n - v\| \le 1$, $v_n \to v$. Denote $p(v_n) = t_n v_n$, $t_n > 0$, $p(v) = tv$, $t > 0$ and $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. A_i^n, B_i^n, C_i^n, $i = 1, 2, 3$, are defined in Lemma 4.10. If $A_2^n \cup A_3^n \cup B_3^n = \emptyset$ for $n = 1, 2, ...$, then, there is a constant $L_h > 0$ such that*

$$|t_n - t| \le L_h \|v_n - v\|^{\beta_0}.$$

Proof. By Lemma 2.12, denote $v_n = \frac{1}{\sqrt{1+s_n^2}}(v - s_n d_n)$, $\|d_n\| = 1$, $\langle d_n, v \rangle = 0$ and $s_n \ge 0$. Suppose that there is not a constant $L_h > 0$ such that

$$|t_n - t| \le L_h \|v_n - v\|^{\beta_0}.$$

Then, similar to Lemma 4.14, there is $\{t_{n_l}\}$ such that

$$(4.41) \qquad\qquad \frac{|t_{n_l} - t|}{s_{n_l}^{\beta_0}} \to +\infty.$$

Then, by Lemma 4.10 and Lemma 4.11, there are the following six cases. In these six cases, $\{n_{l'}\}$ represents a subsequence of $\{n_l\}$.

The first case is there are $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $t < t_{n_{l'}}$, $B_2^{n_{l'}} \ne \emptyset$ and

$$\int_{B_2^{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx \le D s_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The second case is there are $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $t < t_{n_{l'}}$ and $B_2^{n_{l'}} = \emptyset$.

The third case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$, $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} \ne \emptyset$, $t > t_{n_{l'}}$, $B_2^{n_{l'}} \ne \emptyset$ and

$$\int_{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}} |tv(x)| dx \le D s_{n_{l'}}^{\beta_0} \quad \text{and} \quad \int_{B_2^{n_{l'}}} |tv(x)| dx \le D s_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The fourth case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$, $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} \ne \emptyset$, $t > t_{n_{l'}}$, $B_2^{n_{l'}} = \emptyset$ and

$$\int_{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}} |tv(x)| dx \le D s_{n_{l'}}^{\beta_0}.$$

The fifth case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$, $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} = \emptyset$, $t > t_{n_{l'}}$, $B_2^{n_{l'}} \ne \emptyset$ and

$$\int_{B_2^{n_{l'}}} |tv(x)| dx \le D s_{n_{l'}}^{\beta_0},$$

where $D > 0$ is a constant.

The sixth case is there are $\{C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}}\}$, $\{t_{n_{l'}}\}$ and $\{B_2^{n_{l'}}\}$ such that $C_1^{n_{l'}} \cup C_2^{n_{l'}} \cup B_1^{n_{l'}} = \emptyset$, $t > t_{n_{l'}}$ and $B_2^{n_{l'}} = \emptyset$.

The first and second cases can be dealt with similarly to Lemma 4.15 and the third, fourth, fifth and sixth cases can be handled similarly to Lemma 4.14. $\square$

By Lemma 4.10–Lemma 4.16, we can proof the following theorem.

Theorem 4.17. *Assume that $L = \{0\}$, $f(x,t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$ and p is continuous at $v \in S_{L^\perp}$. Denote $p(v) = tv$, $t > 0$. Then, there is a constant $L_h > 0$ and a neighborhood $\mathcal{N}(v)$ of v on $S_{L^\perp}$ such that, $\forall u \in \mathcal{N}(v)$,*

$$\|p(u) - p(v)\| \le L_h \|u - v\|^{\beta_0},$$

where $\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$.

Proof. Suppose that the conclusion is not true. Then, there exists $\{v_n\} \subset S_{L^\perp}$ such that $\|v_n - v\| \le 1$, $v_n \to v$ and

$$\lim_{n \to \infty} \frac{\|p(v_n) - p(v)\|}{\|v_n - v\|^{\beta_0}} = +\infty.$$

On the other hand,

$$|t_n - t| = \|(t_n - t)v_n\| \ge \|(t_n - t)v_n + t(v_n - v)\| - \|t(v_n - v)\|$$
$$= \|t_n v_n - tv\| - t\|v_n - v\| = \|p(v_n) - p(v)\| - t\|v_n - v\|,$$

where $p(v_n) = t_n v_n$ and $t_n > 0$. Hence, we have

$$(4.42) \qquad \lim_{n \to \infty} \frac{|t_n - t|}{\|v_n - v\|^{\beta_0}} = +\infty.$$

It is obvious that there are two cases. The first is there is $\{A_2^{n_l} \cup A_3^{n_l} \cup B_3^{n_l}\}$ such that $A_2^{n_l} \cup A_3^{n_l} \cup B_3^{n_l} \ne \emptyset$ and the second is there is $\{A_2^{n_l} \cup A_3^{n_l} \cup B_3^{n_l}\}$ such that $A_2^{n_l} \cup A_3^{n_l} \cup B_3^{n_l} = \emptyset$.

For the first case, by Lemma 4.10, there are two subcases. The first one is there is $\{A_2^{n_{l'}} \cup A_3^{n_{l'}} \cup B_3^{n_{l'}}\}$ such that $A_2^{n_{l'}} \cup A_3^{n_{l'}} \cup B_3^{n_{l'}} \ne \emptyset$ and $t > t_{n_{l'}}$ and the second one is there is $\{A_2^{n_{l'}} \cup A_3^{n_{l'}} \cup B_3^{n_{l'}}\}$ such that $A_2^{n_{l'}} \cup A_3^{n_{l'}} \cup B_3^{n_{l'}} \ne \emptyset$, $t < t_{n_{l'}}$ and

$$\int_{\Omega_{n_{l'}}} |t_{n_{l'}} v_{n_{l'}}(x)| dx \le C s_{n_{l'}}^{\beta_0},$$

where $\Omega_{n_{l'}} = A_2^{n_{l'}} \cup A_3^{n_{l'}} \cup B_3^{n_{l'}}$, $C > 0$ is a constant, $v_{n_{l'}} = \frac{1}{\sqrt{1 + s_{n_{l'}}^2}}(v - s_{n_{l'}} d_{n_{l'}})$, $\|d_{n_{l'}}\| = 1$, $\langle d_{n_{l'}}, v \rangle = 0$ and $s_{n_{l'}} \ge 0$, and $\{n_{l'}\}$ is a subsequence of $\{n_l\}$. By Lemma 4.14 and Lemma 4.15, there is a constant $\bar{L}_h > 0$ such that

$$|t_{n_{l'}} - t| \le \bar{L}_h \|v_{n_{l'}} - v\|^{\beta_0}.$$

This is a contradiction to (4.42).

For the second case, by Lemma 4.16, there is a constant $\hat{L}_h > 0$ such that

$$|t_{n_l} - t| \le \hat{L}_h \|v_{n_l} - v\|^{\beta_0}.$$

This is also a contradiction to (4.42). $\square$

By Theorem 4.17, a conclusion on super-linear property of peak selection to J (4.1) with $f(x,t)$ (4.2) can be verified.

Theorem 4.18. *Assume that $L = \{0\}$, $f(x, t)$ is in (4.2), p is a peak selection of J (4.1) defined on $S_{L^\perp}$, $p(w) = t_w w$, $t_w > 0$ for $w \in S_{L^\perp}$ and p is continuous at $v \in S_{L^\perp}$. Then, p has super-linear property at v.*

Proof. For every $\{v_n\} \subset S_{L^\perp}$ such that $\|v_n - v\| \leq 1$, $v_n \to v$, denote $p(v_n) = t_n v_n$, $t_n > 0$ and $p(v) = tv$, $t > 0$. By Theorem 4.17, there is a constant $L_h > 0$ such that, for large n,

$$(4.43) \qquad \qquad \|p(v_n) - p(v)\| \leq L_h \|v_n - v\|^{\beta_0},$$

$\beta_0 = \frac{1}{2} + \alpha_0$, $\alpha_0 \in (0, \frac{1}{6}]$. On the other hand,

$$\frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \langle \nabla u, \nabla w - \nabla u \rangle + \frac{1}{2}\int_\Omega |\nabla w - \nabla u|^2 dx$$

$$= -\langle \Delta u, w - u \rangle + \frac{1}{2}\|w - u\|^2.$$

Then, by Remark 2.9, $F(w; u, z) = \frac{1}{2}\|w - u\|^2$ is an upper-bound functional around u for $z \in \partial J(u)$.

Thus, by (4.43),

$$\lim_{n\to\infty} \frac{F(p(v_n); p(v), z)}{\|v_n - v\|} = \lim_{n\to\infty} \frac{\|p(v_n) - p(v)\|^2}{2\|v_n - v\|} = 0,$$

i.e., p has super-linear property at v. $\qquad \qquad \square$

Remark 4.19. To our numerical example, Lemma 4.1, Lemma 4.3, Lemma 4.5 and Theorem 4.18 guarantee that the peak selection $p(v) = tv$, where $t > 0$ and $J(tv) = \max_{s\geq0} J(sv)$, has super-linear property on $S_{L^\perp}$, where $L = \{0\}$.

Now, we start to discuss the details on numerical computation. To carry out the minimax algorithm, we need to find z_n^k to construct a descent direction. Thus, we have to solve the linear equation

$$(4.44) \qquad \begin{cases} \Delta z(x) = -\Delta p(v_n^k)(x) - \zeta_n^k(x), \ x \in \Omega, \\ z(x)|_{x\in\partial\Omega} = 0, \end{cases}$$

where $\zeta_n^k \in \partial G(p(v_n^k))$ and p is a peak selection to the variational functional J (4.1). By Theorem 1.3, it is still hard to find $\zeta_n^k \in \partial G(p(v_n^k))$ since the inclusion $\partial G(p(v_n^k))(x) \subseteq [\bar{f}(x, p(v_n^k)(x)), \hat{f}(x, p(v_n^k)(x))]$ offers little information on $\partial G(p(v_n^k))$. For nice $f(x, t)$, Theorem 1.5 gives us an equality

$$\partial G(u) = \{\zeta : \Omega \to \mathbb{R} | \zeta \text{ is measurable}, \ \zeta(x)$$
$$\in [f(x, u(x) - 0), f(x, u(x) + 0)] \ \forall x \in \Omega\}.$$

Indeed, we have the following simple lemma.

Lemma 4.20. *If $f(x, t)$ is a Baire-measurable function defined on $\Omega \times \mathbb{R}$, is non-decreasing in t and satisfies*

$$|f(x, t)| \leq C_1 + C_2|t|^\sigma,$$

where $0 < \sigma < \frac{m+2}{m-2}$ for $m \geq 3$, $\sigma > 0$ for $m = 1, 2$ and $C_1, C_2 > 0$ are two constants, then we have that, in $H_0^1(\Omega)$, for $v \in S_{L^\perp}$, $\zeta \in \partial G(p(v))$ and $(\Delta p(v) + \zeta) \perp [L, v]$ is equivalent to ζ is a solution to the convex optimization problem

$$(4.45) \qquad \min_{\eta\in\mathcal{D}} \mathcal{L}(\eta) = \sum_{i=1}^{n-1} \left(\int_\Omega (\Delta p(v) + \eta)u_i dx\right)^2 + \left(\int_\Omega (\Delta p(v) + \eta)v dx\right)^2,$$

where $\mathcal{D} = \{\eta : \Omega \to \mathbb{R} | \eta$ is measurable, $\eta(x) \in [f(x, p(v)(x) - 0), f(x, p(v)(x) + 0)] \ \forall x \in \Omega\}$ and p is a peak selection to the variational functional J (4.1) w.r.t. a finite dimensional subspace $L \subset H_0^1(\Omega)$.

Proof. If ζ satisfies $\zeta \in \partial G(p(v))$ and $(\Delta p(v) + \zeta) \perp [L, v]$, then, by Theorem 1.5, $\zeta \in \mathcal{D}$ and $\mathcal{L}(\zeta) = 0$, i.e., ζ is a solution to the convex optimization problem (4.45). If ζ is a solution to the convex optimization problem (4.45), then, $\zeta \in \mathcal{D}$, i.e., $\zeta \in \partial G(p(v))$ by Theorem 1.5. On the other hand, Lemma 2.2 says $\min_{\eta \in \mathcal{D}} \mathcal{L}(\eta) = 0$, i.e., $(\Delta p(v) + \zeta) \perp [L, v]$. Hence, ζ satisfies $\zeta \in \partial G(p(v))$ and $(\Delta p(v) + \zeta) \perp [L, v]$. $\square$

To $L = \{0\}$, we have a conclusion as follows.

Lemma 4.21. *Assume that $f(x, t) = g(x)h(t)$ is a Baire-measurable function defined on $\Omega \times \mathbb{R}$, $0 \leq g(x) \leq C$ in Ω for constant C, and $h(t)$ is nondecreasing in t and satisfies*

$$|h(t)| \leq C_1 + C_2|t|^\sigma,$$

for $t \in \mathbb{R}$, where $0 < \sigma < \frac{m+2}{m-2}$ for $m \geq 3$, $\sigma > 0$ for $m = 1, 2$ and $C_1, C_2 > 0$ are two constants. If $h(t)$ is discontinuous at $t_1, ..., t_k$ and continuous elsewhere, then, for $v \in H_0^1(\Omega)$ with $\|v\| = 1$, there are $d_i \in [h(t_i - 0), h(t_i + 0)]$, $i = 1, ..., k$, such that $\zeta \in \partial G(p(v))$ and $(\Delta p(v) + \zeta) \perp v$, where

$$(4.46) \qquad \zeta(x) = \begin{cases} g(x)h(p(v)(x)), & \text{if } p(v)(x) \notin \{t_1, ..., t_k\}, \\ g(x)d_i, & \text{if } p(v)(x) = t_i, \ i = 1, ..., k, \end{cases}$$

p is a peak selection to the variational functional J (4.1) and $p(v) \neq 0$.

Proof. Denote $p(v) = k(v)v$, $k(v) \in \mathbb{R}$. Without loss of generality, we assume $k(v) > 0$. By Lemma 2.2, there is a $z \in \partial G(p(v))$ such that $(\Delta p(v) + z) \perp v$. Thus, by Theorem 1.5,

$$\zeta_0^+(x) = \begin{cases} g(x)h(p(v)(x)), & \text{if } p(v)(x) \notin \{t_1, ..., t_k\}, \\ g(x)h(t_i + 0), & \text{if } p(v)(x) = t_i \geq 0, \\ g(x)h(t_i - 0), & \text{if } p(v)(x) = t_i < 0 \end{cases}$$

and

$$\zeta_0^-(x) = \begin{cases} g(x)h(p(v)(x)), & \text{if } p(v)(x) \notin \{t_1, ..., t_k\}, \\ g(x)h(t_i - 0), & \text{if } p(v)(x) = t_i \geq 0, \\ g(x)h(t_i + 0), & \text{if } p(v)(x) = t_i < 0, \end{cases}$$

satisfy $\zeta_0^+, \zeta_0^- \in \partial J(p(v))$,

$$\int_\Omega (\Delta p(v) + \zeta_0^+)v dx \geq \int_\Omega (\Delta p(v) + z)v dx = 0$$

and

$$\int_\Omega (\Delta p(v) + \zeta_0^-)v dx \leq \int_\Omega (\Delta p(v) + z)v dx = 0.$$

Then, we can construct $\{\zeta_i^+\}$ and $\{\zeta_i^-\}$ such that $\zeta_i^+, \zeta_i^- \in \partial J(p(v))$,

$$\int_\Omega (\Delta p(v) + \zeta_i^+)v dx \geq 0 \quad \text{and} \quad \int_\Omega (\Delta p(v) + \zeta_i^-)v dx \leq 0,$$

where

$$\begin{cases} \zeta_i^- = \zeta_{i-1}^-, \\ \zeta_i^+ = \frac{1}{2}(\zeta_{i-1}^- + \zeta_{i-1}^+), \end{cases}$$

if $\int_\Omega (\Delta p(v) + \frac{1}{2}(\zeta_{i-1}^- + \zeta_{i-1}^+))v dx \geq 0$, or

$$\begin{cases} \zeta_i^- = \frac{1}{2}(\zeta_{i-1}^- + \zeta_{i-1}^+), \\ \zeta_i^+ = \zeta_{i-1}^+, \end{cases}$$

if $\int_\Omega (\Delta p(v) + \frac{1}{2}(\zeta_{i-1}^- + \zeta_{i-1}^+))v dx \leq 0$, $i = 1, 2, ...$ During this process, if $\int_\Omega (\Delta p(v) + \zeta_i^+)v dx = 0$ or $\int_\Omega (\Delta p(v) + \zeta_i^-)v dx = 0$, then $\zeta = \zeta_i^+$ or $\zeta = \zeta_i^-$. Otherwise,

$$\zeta(x) = \lim_{i \to \infty} \zeta_i^+(x) = \lim_{i \to \infty} \zeta_i^-(x)$$

is in (4.46) and satisfies $\zeta \in \partial G(p(v))$ and $(\Delta p(v) + \zeta) \perp v$.   □

Remark 4.22. The proof gives us a simple iterative way to numerically capture a $\zeta \in \partial G(p(v))$ such that $(\Delta p(v) + \zeta) \perp v$.

It is easy to check that $u = 0$ is a minimal point of J (4.1) if f is in (4.2). Among saddle points, people usually pay more attention to Mountain Pass type saddle points. In this paper, we concentrate on computing them. Of course, more saddle points can be captured if symmetry is used. Thus, in our numerical computation, Lemma 4.21 is used for computing ζ_n^k and then the Poisson equation (4.44) is solved to get z_n^k in the algorithm; even or odd symmetry about the origin is used for capturing symmetric saddle points as $\Omega = (-1, 1)$ and even or odd symmetry about x_1-axis, x_2-axis or the diagonal lines $x_2 = -x_1$ is used to capture symmetric saddle points as $\Omega = (-1, 1) \times (-1, 1)$, $\{(x_1, x_2) \in \mathbb{R} | x_1^2 + x_2^2 < 1\}$. To solve the Poisson equation (4.44), the finite element method is employed. Over 5×10^4 elements on $\Omega \subset \mathbb{R}$ and 5×10^5 triangle elements on $\Omega \subset \mathbb{R}^2$ are used. The profiles of approximations for saddle points are listed in Figures 1–10. The corresponding values of J (4.1) are listed in captions. Symmetry of saddle point is also pointed out in captions. Then $\min\{\|\zeta\| \mid \zeta \in \partial J(u)\}$ of every approximation u is less than 10^{-3}. Hence, these approximations are good approximations of saddle points.

We set $\alpha = 8$, $\beta = 6$ and $\Omega = (-1, 1)$ in Figures 1 and 2, $c = 1$ in Figure 1 and $c = 24.0625$ in Figure 2. For the first and the second in Figures 1 and 2, $r = 0$ and in the third and the fourth, $r = 4$. Odd symmetry about the origin is used for capturing the second and even symmetry about the origin is used for calculating the fourth in these two figures. We set $\alpha = 8$, $\beta = 6$ to Figures 3–10, $c = 1$ to Figures 3–6, $c = \frac{14}{11}$ to Figures 7 and 8 and $c = 60$ to Figures 9 and 10. For Figures 3, 5, 7, and 8, $r = 0$ and to Figures 4, 6, 9, and 10, $r = 4$. For the domain, $\Omega = (-1, 1) \times (-1, 1)$ to Figures 3, 4, 7, and 9 and $\Omega = \{(x_1, x_2) \in \mathbb{R} | x_1^2 + x_2^2 < 1\}$ to Figures 5, 6, 8, and 10. For capturing the third in Figures 3, 6, 7, and 10 and the second in Figures 5, 8, odd symmetry for x_2-axis is used and for computing the second in Figures 4, 6, 9, and 10, even symmetry for x_2-axis is used. For computing the second in Figures 3 and 7 and the third in Figures 4 and 9, odd and even symmetry for the line $x_2 = -x_1$ are used respectively. Odd symmetry for x_1-axis and x_2-axis are used for the fourth in Figures 3 and 7 and the third in Figures 5 and 8 and even symmetry for x_1-axis and x_2-axis are used for the fourth in Figures 4, 6, 9, and 10. $v_1^1 = \frac{v}{\|v\|}$ was used as initial point for calculating the

first in Figure 7, where

$$v(x) = \begin{cases} 2(x_2 + 1), & x_2 \in [-1, -\frac{1}{2}], x_1 + x_2 \leq 0, x_1 - x_2 \geq 0, \\ 2(1 - x_2), & x_2 \in [\frac{1}{2}, 1], x_1 + x_2 \geq 0, x_1 - x_2 \leq 0, \\ 2(x_1 + 1), & x_1 \in [-1, -\frac{1}{2}], x_1 + x_2 \leq 0, x_1 - x_2 \leq 0, \\ 2(1 - x_1), & x_1 \in [\frac{1}{2}, 1], x_1 + x_2 \geq 0, x_1 - x_2 \geq 0, \\ 1, & (x_1, x_2) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]. \end{cases}$$

It is easy to check that

$$\int\int_\Omega v^6(x)dx = \frac{37}{28} \text{ and } \int\int_\Omega |\nabla v(x)|^2 dx = 12,$$

where $\Omega = (-1, 1) \times (-1, 1)$. Denote $\phi(t) = J(tv_1^1)$ for J (4.1). Then, we have

$$\partial\phi(t) = \{\langle z, v_1^1 \rangle | z \in \partial J(tv_1^1)\}.$$

For the first in Figure 7, by (4.4),

$$\partial\phi(t) = \{-\int\int_\Omega (\Delta u + \zeta)v_1^1 dx | \zeta \text{ is measurable, } \zeta(x) \in f_-^+(x, u(x)) \forall x \in \Omega\},$$

where

$$f_-^+(x, u(x)) = [f(x, u(x) - 0), f(x, u(x) + 0)] = \begin{cases} \{8c|u(x)|^6 u(x)\}, & \text{if } |u(x)| > 1, \\ \{6c|u(x)|^4 u(x)\}, & \text{if } |u(x)| < 1, \\ [6c, 8c], & \text{if } u(x) = 1, \\ [-8c, -6c], & \text{if } u(x) = -1, \end{cases}$$

$u = tv_1^1$ and $c = \frac{14}{11}$. The equality,

$$\int\int_\Omega |\nabla v|^2 dx - c(6\int\int_\Omega v^6 dx + \frac{3}{2}) = 0,$$

means $0 \in \partial\phi(\|v\|)$. As t is less than $\|v\|$ and very close to $\|v\|$,

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{1}{t}(\int\int_\Omega |\nabla(tv_1^1)|^2 dx - 6c\int\int_\Omega (tv_1^1)^6 dx) \\ &> \frac{1}{t}(\int\int_\Omega |\nabla v|^2 dx - c(6\int\int_\Omega v^6 dx + \frac{3}{2})) = 0 \end{aligned}$$

and as t is larger than $\|v\|$ and very close to $\|v\|$,

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{1}{t}(\int\int_\Omega |\nabla(tv_1^1)|^2 dx - c(6\int\int_{\Omega_0} (tv_1^1)^6 dx + 8\int\int_{\Omega_1} (tv_1^1)^8 dx)) \\ &< \frac{1}{t}(\int\int_\Omega |\nabla(tv_1^1)|^2 dx - c(6\int\int_\Omega (tv_1^1)^6 dx + 2\int\int_{\Omega_1} (tv_1^1)^8 dx)) \\ &< \frac{1}{t}(\int\int_\Omega |\nabla v|^2 dx - c(6\int\int_\Omega v^6 dx + \frac{3}{2})) = 0, \end{aligned}$$

where $\Omega_0 = \{x \in \Omega | tv_1^1(x) \leq 1\}$ and $\Omega_1 = \{x \in \Omega | tv_1^1(x) > 1\}$. Thus, $p(v_1^1) = v$. The point $p(v_1^1)$ is a nonsmooth point of J.
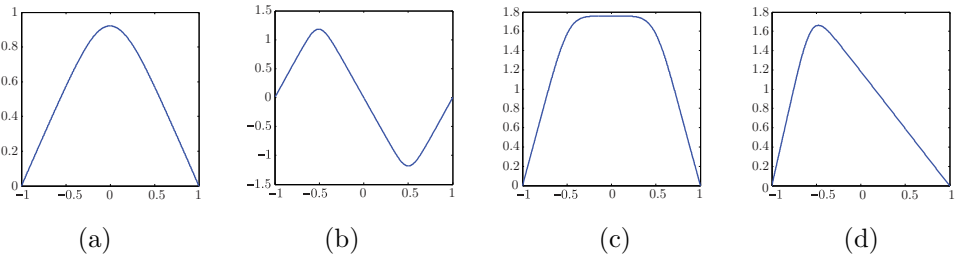
FIGURE 1. $c=1$, $r=0$ for (a) and (b), $r=4$ for (c) and (d). (a) $J=0.6342$, positive saddle point, (b) $J=4.4818$, saddle point odd symmetric about the origin, (c) $J=2.9981$, positive saddle point, (d) $J=3.9094$, saddle point even symmetric about the origin.
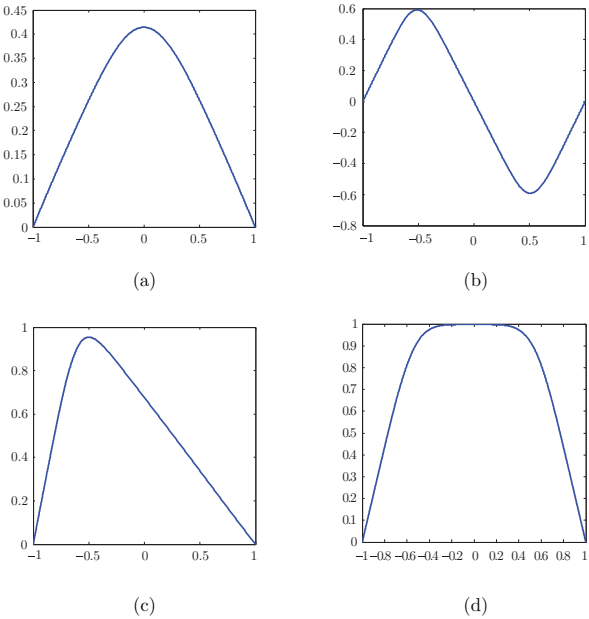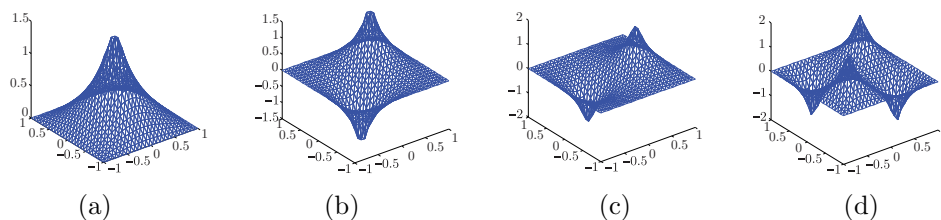

FIGURE 2. $c=24.0625$, $r=0$ for (a) and (b), $r=4$ for (c) and (d). (a) $J=0.1296$, positive saddle point, (b) $J=1.0331$, saddle point odd symmetric about the origin, (c) $J=0.9372$, positive saddle point, (d) $J=1.2156$, saddle point odd symmetric about the origin.
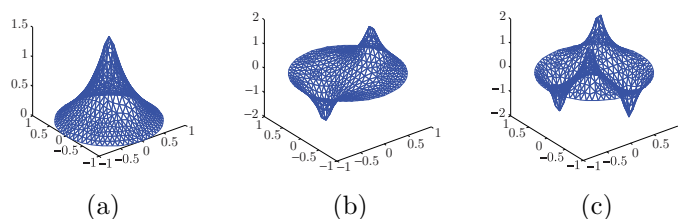
FIGURE 3. $c = 1$, $r = 0$. (a) $J = 1.5746$, positive saddle point, (b) $J = 4.3662$, saddle point odd symmetric for $x_2 = -x_1$, (c) $J = 4.5596$, saddle point odd symmetric for $x_1 = 0$ (d) $J = 10.1556$, saddle point odd symmetric for $x_1 = 0$ and $x_2 = 0$.
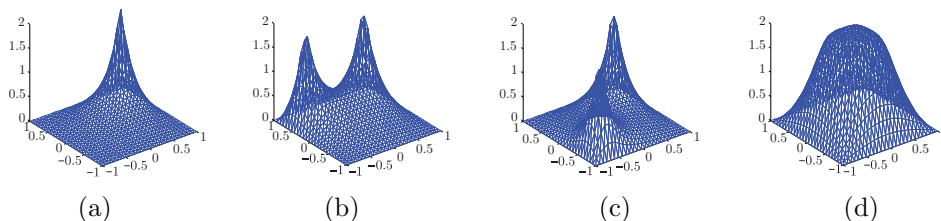
FIGURE 4. $c = 1$, $r = 4$. (a) $J = 3.0167$, positive saddle point, (b) $J = 5.8567$, saddle point even symmetric for $x_1 = 0$, (c) $J = 5.9545$, saddle point odd symmetric for $x_2 = -x_1$, (d) $J = 9.2199$, saddle point even symmetric for $x_1 = 0$ and $x_2 = 0$.

FIGURE 5. $c = 1$, $r = 0$. (a) $J = 1.6614$, positive saddle point, (b) $J = 4.7202$, saddle point odd symmetric for $x_1 = 0$, (c) $J = 11.2632$, saddle point odd symmetric for $x_1 = 0$ and $x_2 = 0$.
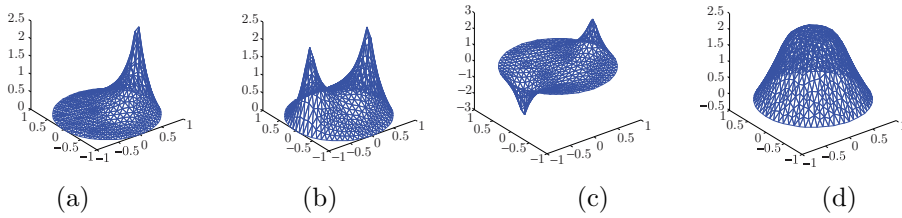
FIGURE 6. $c = 1$, $r = 4$. (a) $J = 4.2512$, positive saddle point, (b) $J = 8.1264$, saddle point even symmetric for $x_1 = 0$, (c) $J = 8.8090$, saddle point odd symmetric for $x_1 = 0$, (d) $J = 10.1558$, saddle point even symmetric for $x_1 = 0$ and $x_2 = 0$.
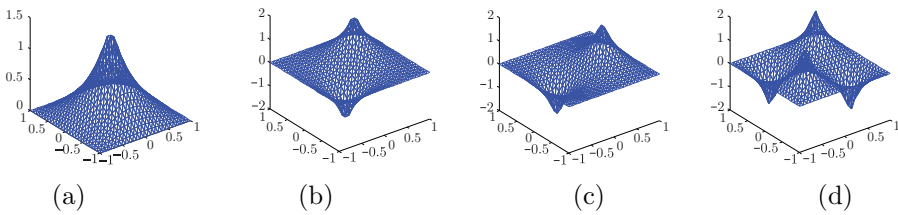

FIGURE 7. $c = \frac{14}{11}$, $r = 0$. (a) $J = 1.4439$, positive saddle point, (b) $J = 4.0388$, saddle point odd symmetric for $x_2 = -x_1$, (c) $J = 4.1995$, saddle point odd symmetric for $x_1 = 0$ (d) $J = 9.3593$, saddle point odd symmetric for $x_1 = 0$ and $x_2 = 0$.
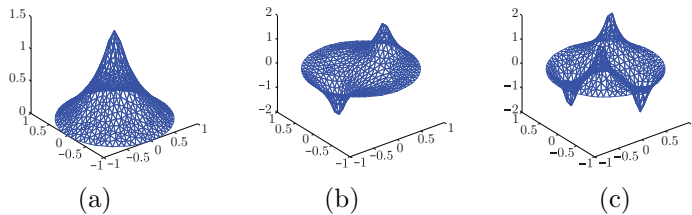

FIGURE 8. $c = \frac{14}{11}$, $r = 0$. (a) $J = 1.5250$, positive saddle point, (b) $J = 4.3483$, saddle point odd symmetric for $x_1 = 0$ (c) $J = 10.3838$, saddle point odd symmetric for $x_1 = 0$ and $x_2 = 0$.
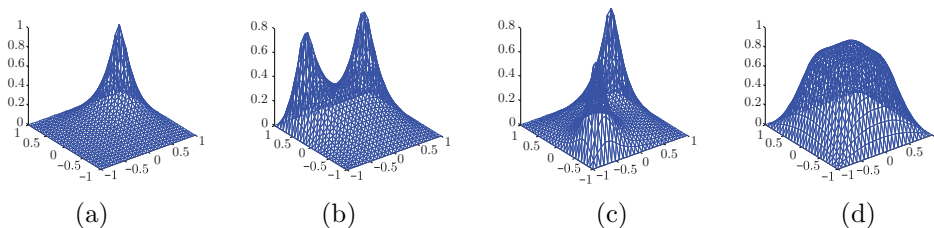
(a)   (b)   (c)   (d)

FIGURE 9. $c = 60$, $r = 4$. (a) $J = 0.6098$, positive saddle point, (b) $J = 1.1663$, saddle point even symmetric for $x_1 = 0$ (c) $J = 1.1958$, saddle point even symmetric for $x_2 = -x_1$, (d) $J = 1.8055$, saddle point even symmetric for $x_1 = 0$ and $x_2 = 0$.
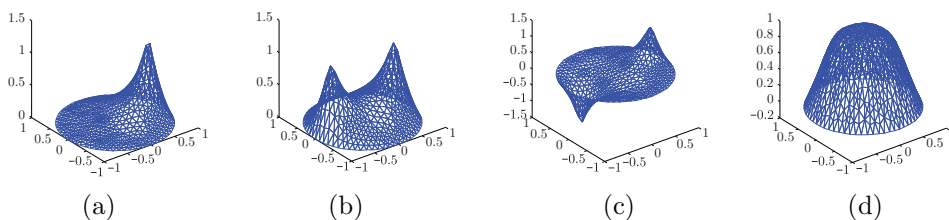

(a)   (b)   (c)   (d)

FIGURE 10. $c = 60$, $r = 4$. (a) $J = 1.0044$, positive saddle point, (b) $J = 1.8906$, saddle point even symmetric for $x_1 = 0$ (c) $J = 2.1015$, saddle point odd symmetric for $x_1 = 0$ (d) $J = 2.2920$, saddle point even symmetric for $x_1 = 0$ and $x_2 = 0$.

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