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AN IMPROVEMENT OF THE REGION OF ACCESSIBILITY OF CHEBYSHEV'S METHOD FROM NEWTON'S METHOD

J. A. EZQUERRO AND M. A. HERNÁNDEZ

ABSTRACT. A simple modification of Chebyshev's method is presented, so that the region of accessibility is extended to the one of Newton's method.

1. INTRODUCTION

Many scientific and engineering problems can be brought in the form of a nonlinear equation,

$$F(z) = 0, \tag{1}$$

where F is a nonlinear operator defined on a nonempty open convex subset Ω of X with values in a Banach space Y. This problem is usually solved by iterative processes: from a starting point z_0, a sequence $\{z_n\}$ is constructed such that it converges to a solution of equation (1). The most used iterative processes are the well-known one-point iterations, that are defined as follows: $z_0 \in \Omega$, $z_{n+1} = G(z_n)$, $n \geq 0$. Three ideas are emphasized in these kinds of iterations: the convergence, the speed of convergence and the operational cost. In this paper, we are only centered in the analysis of the convergence. There are three types of convergence that can be considered: local, semilocal and global. The analysis of convergence presented here is focused only on the semilocal convergence, where two kinds of conditions are required: conditions on the starting point z_0 and conditions on the operator F that define equation (1). Our analysis is centered on the condition that the starting point of an iteration must satisfy, what is known as the region of accessibility of the iteration. This region of accessibility consists of every starting point such that the iteration converges to a solution of (1) from it. In particular, the problem that we can observe is that the regions of accessibility of iterations with higher speed of convergence is reducing, and consequently, it is more difficult to locate starting points such that the iterations converge from them.

The principal idea of this paper is to construct a modification of Chebyshev's method (order three, [1], [7]) that converges provided that Newton's method (order two, [8], [9]) does also. The basic idea is to use Newton's method until a certain finite step and then use this iterate as the starting point for Chebyshev's method, so that we can consider a larger domain of starting points to iterate. In Section 2, we present

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the construction of the modification of Chebyshev's method. In Section 3, we study the semilocal convergence of the new iterative method, the domains of existence and uniqueness of solutions, the R-order of convergence [11] and we compute the step in which the iterative method jumps from Newton's method to Chebyshev's method. Finally, in Section 4, we illustrate the above-mentioned with an example, where a nonlinear integral equation of mixed Hammerstein type is analyzed and solved.

2. Preliminaries and description of the method

In practice, we can see the problem mentioned above with the attraction basins (the set of all starting points such that an iteration converges to a solution of an equation, [10], [12]) of two of the best-known iterations to solve (1), Newton's method

$$\begin{cases} x_0 \in \Omega, \\ x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n \geq 0, \end{cases}$$

and Chebyshev's method

$$\begin{cases} z_0 \in \Omega, \\ y_n = z_n - [F'(z_n)]^{-1} F(z_n), \\ z_{n+1} = y_n + \frac{1}{2} L_F(z_n)(y_n - z_n), \quad n \geq 0. \end{cases}$$

Here $L_F(z)$ is the degree of logarithmic convexity [6], defined by

$$L_F(z) = [F'(z)]^{-1} F''(z) [F'(z)]^{-1} F(z), \quad z \in X,$$

provided that $[F'(z)]^{-1}$ exists at each step z_n. The operators F' and F'' denote the first and the second Fréchet derivatives of the operator F.

We now consider a complex equation $F(z) = 0$, where $F : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}$, and we are interested in identifying the attraction basin for two solutions z^* and z^{**} (see [12]). The main idea of this study is to apply, using computer experiments, Newton's and Chebyshev's methods for solving the equation $F(z) = \sin z - 1/3 = 0$, and show the fractal pictures that they generate to approximate $z^* = \arctan(1/2\sqrt{2}) = 0.33983\ldots$ and $z^{**} = \pi - \arctan(1/2\sqrt{2}) = 2.80176\ldots$. This also allows us to compare the regions of accessibility of both methods.

We take a rectangle $R \subseteq \mathbb{C}$ and iterations starting at "every" $z_0 \in R$. In practice, a grid of 512×512 points in R is considered and these points are chosen as z_0. We used the rectangle $[0, 3] \times [-2.5, 2.5]$, which contains the two zeros. The numerical methods starting at a point in the rectangle can converge to some of the zeros or, eventually, diverge.

In all of the cases, the tolerance 10^{-3} and the maximum of 25 iterations are used. If we have not obtained the desired tolerance with 25 iterations, we do not continue and we decide that the iterative method starting at z_0 does not converge to any zero.

In Figures 1 and 2, it is shown what happens when the two iterations are applied to approximate the zeros of the function $F(z) = \sin z - 1/3$ in the above mentioned rectangle. The strategy taken into account is the following. A colour is assigned to each attraction basin of a zero. The colour is made lighter or darker according to the number of iterations needed to reach the root with the fixed precision required. Finally, if the iteration does not converge, black is used. For more strategies, the reader can see [12] and the references appearing there. In particular, to obtain the pictures, two different colours have been assigned for the attraction basins of

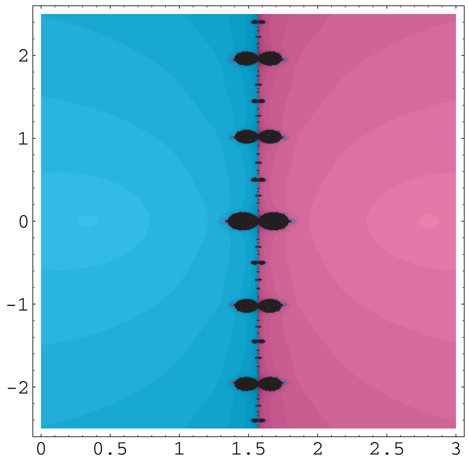

FIGURE 1. Newton's method

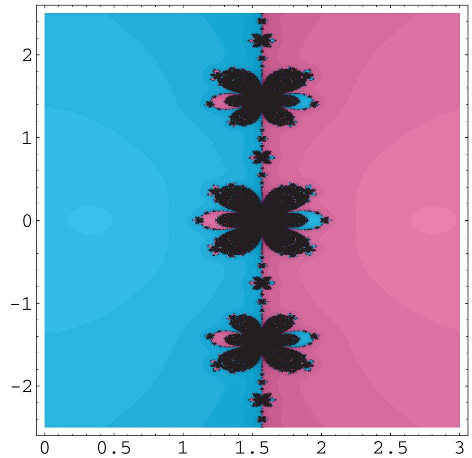

FIGURE 2. Chebyshev's method

the two zeros. Black is used to mark the points of the rectangle for which the corresponding iterations starting at them do not reach any root with tolerance 10^{-3} in a maximum of 25 iterations. The graphics shown here have been generated with Mathematica 5.1 ([13]).

We can see in Figures 1 and 2 the behaviour of Newton's and Chebyshev's methods. Note that Chebyshev's method is the most demanding with respect to the initial point; see the black areas. We can also observe lighter areas for Chebyshev's method as a consequence of the cubic convergence of the method.

On the other hand, in theory, the classical well-known Newton-Kantorovich conditions ([2], [3], [9]) are considered to establish the semilocal convergence of iterative processes, so that these conditions are more restrictive for iterations with a higher speed of convergence. For example, if we suppose that $[F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X and we also assume the following:

(C1) $\|[F'(x_0)]^{-1}\| \leq \beta$,
(C2) $\|[F'(x_0)]^{-1}F(x_0)\| \leq \eta$,
(C3) $\|F''(x)\| \leq M,\ x \in \Omega$,
(C4) $\|F''(x) - F''(y)\| \leq K\|x - y\|,\ x, y \in \Omega$,

we can then guarantee the semilocal convergence of Newton's method under conditions (C1)–(C3) (see [9]), and the semilocal convergence of third-order iterative processes under conditions (C1)–(C4); see [2], [3]. In addition, if we consider Newton's and Chebyshev's methods, we can then guarantee the semilocal results given below in Theorem 2.1.

Throughout this paper we denote $\overline{B(x,r)} = \{y \in X; \|y - x\| \leq r\}$ and $B(x,r) = \{y \in X; \|y - x\| < r\}$.

Theorem 2.1. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a twice Fréchet differentiable operator in an open convex domain Ω.*

- *We suppose that $[F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists for some $x_0 \in \Omega$ and (C1)–(C3) are satisfied. If $B(x_0, R_1) \subseteq \Omega$, where $R_1 = \frac{2(1-a)}{2-3a}\eta$ and $a = M\beta\eta$, and*

$$a = M\beta\eta < 1/2, \tag{2}$$

then equation (1) *has a solution* z^* *and Newton's process converges quadratically to this solution* (*see* [8]).

- *We suppose that* $[F'(z_0)]^{-1} \in \mathcal{L}(Y,X)$ *exists for some* $z_0 \in \Omega$ *and (C1)–(C4) are satisfied for* z_0. *If* $B(z_0,R_2)\subseteq\Omega$, *where* $R_2=\frac{6(\tilde{a}+2)(\tilde{a}^2+2\tilde{a}-2)}{24\tilde{a}+24\tilde{a}^2+3\tilde{a}^3+4(\tilde{b}-6)}\tilde{\eta}$, $\tilde{a} = M\tilde{\beta}\tilde{\eta}$, $\tilde{b} = K\tilde{\beta}\tilde{\eta}^2$ *with* $\|[F'(z_0)]^{-1}\| \leq \tilde{\beta}$ *and* $\|[F'(z_0)]^{-1}F(z_0)\| \leq \tilde{\eta}$, *and*

$$\tilde{a} = M\tilde{\beta}\tilde{\eta} < 1/2 \quad \textit{and} \quad \tilde{b} = K\tilde{\beta}\tilde{\eta}^2 < h(\tilde{a}), \tag{3}$$

where

$$h(t) = 3(t+2)(2t-1)(t+1+\sqrt{5})(t+1-\sqrt{5})/4, \tag{4}$$

then equation (1) *has a solution* z^* *and Chebyshev's process converges cubically to this solution* (*see* [7]).

To establish the semilocal convergence of Chebyshev's method, we can easily observe that conditions (3) are required for the starting point z_0, while, for Newton's method, condition (2) is only required for the starting point x_0. Therefore, the use of Chebyshev's method is more restrictive than the use of Newton's method. To illustrate this, the following scalar equation is considered. Let $F:(0,20)\to\mathbb{R}$ be the nonlinear scalar function $F(x) = x^3 - 2007$, and we determine the interval in which the starting point x_0 can move in order to obtain the convergence of Newton's and Chebyshev's methods to the solution $z^* = 12.6139\ldots$ of equation $F(x) = 0$ in $(0,20)$. Condition (2) is satisfied by Newton's method if the starting point is in $(11.2204\ldots, 20)$, and (3) is satisfied by Chebyshev's method if the starting point is in $(11.2280\ldots, 18.3238\ldots)$. In consequence, for this simple example it is clear that the region of accessibility for Newton's method is a little bigger than for Chebyshev's method.

If we now consider the complex equation $F(z) = \sin z - 1/3 = 0$, we can see in Figures 3 and 4, from conditions (2) and (3), the regions of accessibility of Newton's and Chebyshev's methods respectively. Observe the same as in the scalar example: the domain of starting points for Newton's method is a little bigger than for Chebyshev's method (see the size of the regions of convergence).

Next, we are interested in observing the cubic and quadratic decreasing regions of Chebyshev's and Newton's methods respectively. These regions represent the regions of accessibility of both methods by means of the parameters a, $\tilde{a}$ and $\tilde{b}$ that are given from the starting point. If we observe the cubic decreasing region of Chebyshev's method (see [5]), Figure 5, where $\tilde{a}$ and $\tilde{b}$ are taken as coordinates, we see Chebyshev's method converges provided that the parameters $\tilde{a}$ an $\tilde{b}$ corresponding to the starting approximation is in the gray region, limited by the two coordinate axis and the line $\tilde{b} = h(\tilde{a})$, where h is defined in (4). But, for the quadratic decreasing region of Newton's method, it suffices that the parameter a of the starting approximation is in the vertical region limited only by the line $a = 1/2$.

Since the goal is to construct, from Chebyshev's method, an iterative method that converges when it starts from the same points as Newton's method, so that, from some approximation given by Newton's method, the new approximations to a solution of equation (1) are in the cubic decreasing region of Chebyshev's method,

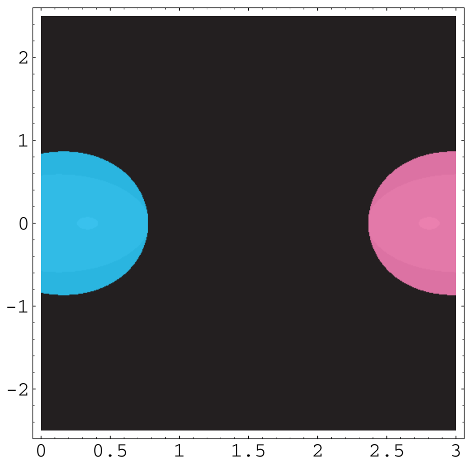

FIGURE 3. Newton's method

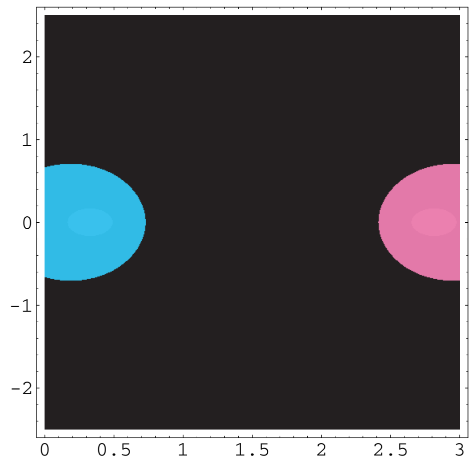

FIGURE 4. Chebyshev's method

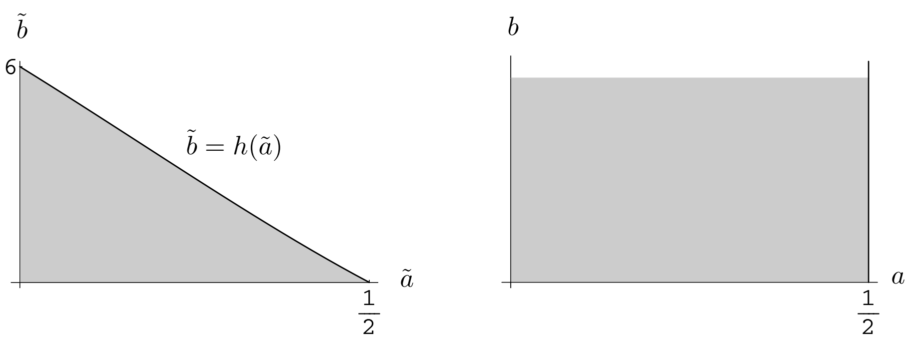

FIGURE 5. Cubic and quadratic decreasing regions of Chebyshev's and Newton's methods

we consider, in practice, the following algorithm:

$$
\begin{array}{l}
\texttt{input}\ x_0,\ maxiter,\ \tilde{a},\ \tilde{b},\ h \\
y \leftarrow F(x) \\
\texttt{output}\ 0, x, y \\
\texttt{for}\ k = 1, 2, \ldots, maxiter\ \texttt{do} \\
\quad \texttt{if}\ \tilde{b} \geq h(\tilde{a})\ \texttt{then} \\
\quad\quad x \leftarrow x - [F'(x)]^{-1} y \\
\quad\quad y \leftarrow F(x) \\
\quad \texttt{else} \\
\quad\quad z \leftarrow x - [F'(x)]^{-1} y \\
\quad\quad x \leftarrow z + \frac{1}{2}[F'(x)]^{-1} F''(x) [F'(x)]^{-1} y (z - x) \\
\quad\quad y \leftarrow F(x) \\
\quad \texttt{endif} \\
\quad \texttt{output}\ k, x, y \\
\texttt{end.}
\end{array}
\tag{5}
$$

If we take into account the previous example in the complex plane, $F(z) = \sin z - 1/3 = 0$, we can see in Figure 6 the attraction basin for the solution z^* and z^{**} when the last algorithm is used. Observe there that this algorithm is more demanding than Newton's method (see Figure 3), but less than Chebyshev's method (see Figure 4), with respect to the initial point (see the black areas).

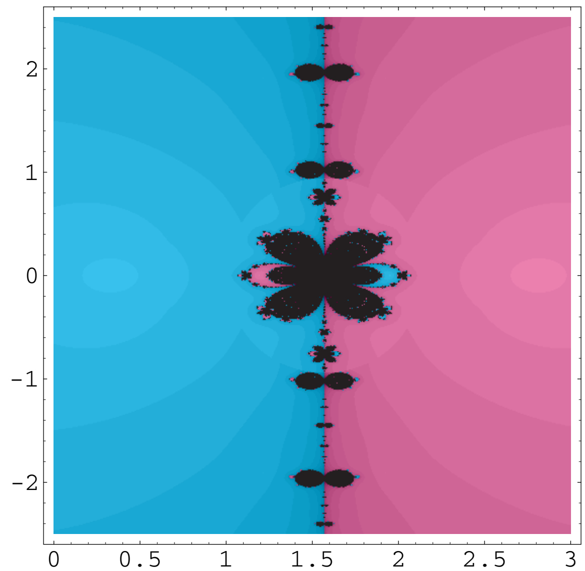

FIGURE 6. Algorithm (5)

Now, we define the last algorithm as follows:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} x_0 \in \Omega, \\ x_n = x_{n-1} - [F'(x_{n-1})]^{-1}F(x_{n-1}), \quad n = 1, 2, \ldots, N_0, \end{array}\right. \\ \left\{\begin{array}{l} z_0 = x_{N_0}, \\ y_{k-1} = z_{k-1} - [F'(z_{k-1})]^{-1}F(z_{k-1}), \\ z_k = y_{k-1} + \frac{1}{2}L_F(z_{k-1})(y_{k-1} - z_{k-1}), \quad k \geq 1, \end{array}\right. \end{array}\right. \tag{6}$$

where x_0 only satisfies (2) and $z_0 = x_{N_0}$ satisfies (3). We can then use Newton's method for a finite number of steps, N_0, provided that condition (2) is satisfied, until conditions given in (3) are satisfied for $z_0 = x_{N_0}$, and next faster Chebyshev's method takes over from Newton's method. The key of the problem is to guarantee that N_0 exists.

3. MAIN RESULTS

To establish a semilocal convergence result for iteration (6), certain conditions for the operator F and the initial approximation are required. Conclusions about the existence and uniqueness of solutions of equation (1) are also obtained, and the regions of existence and uniqueness of solutions from the theoretical significance of the method are provided.

A technique was developed to prove the semilocal convergence of Newton's and Chebyshev's sequences in [8] and [7] respectively, where it is constructed, from the initial scalar parameters, systems of recurrence relations where sequences of positive real numbers were involved. The convergence of the two sequences was then guaranteed from the fact that they were Cauchy sequences. We apply the same technique to prove the semilocal convergence of iteration (6).

3.1. **Semilocal convergence.** From now on, we suppose that the starting approximation x_0 satisfies (2), but not (3). In consequence, we want to prove that there exists $N_0 \in \mathbb{N}$ such that $z_0 = x_{N_0}$ satisfies (3).

From general conditions (C1)–(C3) for Newton's method, we can define the initial parameters a and b from

$$\begin{aligned} M\|[F'(x_0)]^{-1}\|\|[F'(x_0)]^{-1}F(x_0)\| \leq M\beta\eta = a,\\ K\|[F'(x_0)]^{-1}\|\|[F'(x_0)]^{-1}F(x_0)\|^2 \leq K\beta\eta^2 = b, \end{aligned}$$

and construct the following system of recurrence relations (see [8]):

$$\begin{aligned}
&\|[F'(x_n)]^{-1}\| \leq \tfrac{1}{1-a_{n-1}}\|[F'(x_{n-1})]^{-1}\|,\\
&\|x_{n+1}-x_n\| \leq \tfrac{a_{n-1}}{2(1-a_{n-1})}\|x_n - x_{n-1}\| \leq \left(\tfrac{a}{2(1-a)}\right)^n \|[F'(x_0)]^{-1}F(x_0)\|,\\
&M\|[F'(x_n)]^{-1}\|\|[F'(x_n)]^{-1}F(x_n)\|\\
&\qquad \leq f(a_{n-1})g(a_{n-1})M\|[F'(x_{n-1})]^{-1}\|\|[F'(x_{n-1})]^{-1}F(x_{n-1})\| \leq a_n,\\
&K\|[F'(x_n)]^{-1}\|\|[F'(x_n)]^{-1}F(x_n)\|^2\\
&\qquad \leq f(a_{n-1})g(a_{n-1})^2K\|[F'(x_{n-1})]^{-1}\|\|[F'(x_{n-1})]^{-1}F(x_{n-1})\|^2 \leq b_n,\\
&\|x_{n+1}-x_0\| \leq \tfrac{2}{2-af(a)}\left(1-\left(\tfrac{a}{2(1-a)}\right)^{n+1}\right)\|[F'(x_0)]^{-1}F(x_0)\|\\
&\qquad < \tfrac{2}{2-af(a)}\eta = R_1,
\end{aligned} \tag{7}$$

where

$$\begin{cases} a_0 = a, \quad a_{n+1} = a_n f(a_n) g(a_n), \quad n \geq 0,\\ b_0 = b, \quad b_{n+1} = b_n f(a_n) g(a_n)^2, \quad n \geq 0, \end{cases} \tag{8}$$

and

$$f(t) = \frac{1}{1-t} \qquad \text{and} \qquad g(t) = \frac{t}{2(1-t)}. \tag{9}$$

The real sequence $\{a_n\}$ guarantees the convergence of Newton's method. The key is now the strict decreasing of the sequence $\{a_n\}$, provided that (2) holds.

Notice that the sequence $\{b_n\}$ is not necessary to prove the semilocal convergence of Newton's method, but it is essential to locate a valid starting point for Chebyshev's method from Newton's sequence (see [8]). Observe also that the sequences $\{a_n\}$ and $\{b_n\}$ are strictly decreasing, provided that $a_0 < 1/2$, since

$$0 < a_n < \gamma_1^{2^n-1}a_0 \quad \text{and} \quad 0 < b_n < \gamma_1^{2(2^n-1)}b_0, \quad n \geq 0,$$

where $\gamma_1 = f(a_0)g(a_0) < 1$.

In the same way (see [7]), from general conditions (C1)–(C4) for Chebyshev's method and z_0, we can define the initial parameters $\tilde{a}$ and $\tilde{b}$ from

$$M\|[F'(z_0)]^{-1}\|\|[F'(z_0)]^{-1}F(z_0)\| \le M\tilde{\beta}\tilde{\eta} = \tilde{a},$$
$$K\|[F'(z_0)]^{-1}\|\|[F'(z_0)]^{-1}F(z_0)\|^2 \le K\tilde{\beta}\tilde{\eta}^2 = \tilde{b},$$

and construct the system of recurrence relations:

(10)
$$\begin{aligned}
&\|[F'(z_n)]^{-1}\| \le \tilde{f}(\tilde{a}_{n-1})\|[F'(z_{n-1})]^{-1}\|,\\
&\|[F'(z_n)]^{-1}F(z_n)\| \le \tilde{f}(\tilde{a}_{n-1})\tilde{g}(\tilde{a}_{n-1},\tilde{b}_{n-1})\|[F'(z_{n-1})]^{-1}F(z_{n-1})\|\\
&\qquad \le (\tilde{f}(\tilde{a})\tilde{g}(\tilde{a},\tilde{b}))^n\|[F'(z_0)]^{-1}F(z_0)\|,\\
&M\|[F'(z_n)]^{-1}\|\|[F'(z_n)]^{-1}F(z_n)\|\\
&\qquad \le \tilde{f}(\tilde{a}_{n-1})^2\tilde{g}(\tilde{a}_{n-1},\tilde{b}_{n-1})M\|[F'(z_{n-1})]^{-1}\|\|[F'(z_{n-1})]^{-1}F(z_{n-1}) \le \tilde{a}_n,\\
&K\|[F'(z_n)]^{-1}\|\|[F'(z_n)]^{-1}F(z_n)\|^2\\
&\qquad \le \tilde{f}(\tilde{a}_{n-1})^3\tilde{g}(\tilde{a}_{n-1},\tilde{b}_{n-1})^2K\|[F'(z_{n-1})]^{-1}\|\|[F'(z_{n-1})]^{-1}F(z_{n-1})^2 \le \tilde{b}_n,\\
&\|z_{n+1}-z_n\| \le (1+\tilde{a}_n/2)\|[F'(z_n)]^{-1}F(z_n)\|,\\
&\|z_{n+1}-z_0\| \le (1+\tilde{a}/2)\,\tfrac{1-(\tilde{f}(\tilde{a})\tilde{g}(\tilde{a},\tilde{b}))^{n+1}}{1-\tilde{f}(\tilde{a})\tilde{g}(\tilde{a},\tilde{b})}\|[F'(z_0)]^{-1}F(z_0)\|\\
&\qquad < \tfrac{2+\tilde{a}}{2(1-\tilde{f}(\tilde{a})\tilde{g}(\tilde{a},\tilde{b}))}\tilde{\eta} = R_2,
\end{aligned}$$

so that the convergence of Chebyshev's method is guaranteed from the strict decreasing of the real sequences

$$\begin{cases} \tilde{a}_0 = \tilde{a}, \quad \tilde{a}_{n+1} = \tilde{a}_n\tilde{f}(\tilde{a}_n)^2\tilde{g}(\tilde{a}_n,\tilde{b}_n), \quad n \ge 0,\\ \tilde{b}_0 = \tilde{b}, \quad \tilde{b}_{n+1} = \tilde{b}_n\tilde{f}(\tilde{a}_n)^3\tilde{g}(\tilde{a}_n,\tilde{b}_n)^2, \quad n \ge 0, \end{cases}$$

where

$$\tilde{f}(t) = \frac{2}{2-t-t^2} \qquad \text{and} \qquad \tilde{g}(t,u) = \frac{t^2}{2}\left(1+\frac{t}{4}\right)+\frac{u}{6}, \tag{11}$$

which is satisfied if (3) holds (see [7]).

As we have mentioned above, the idea is to apply Chebyshev's method to approximate a solution z^* of (1) from an iteration z_0 such that $z_0 = x_{N_0}$, where $x_0, x_1, \ldots, x_{N_0}$ are approximated by Newton's method from $x_0 \in \Omega$. We can do this. Indeed, since h is a decreasing function and $\{a_n\}$ is a strictly decreasing sequence, the sequence $\{h(a_n)\}$ is strictly increasing and, as $b_0 \ge h(a_0)$ and $\{b_n\}$ is strictly decreasing to zero, we can say that there exists $N_0 \in \mathbb{N}$ such that

$$b_{N_0} < h(a_0) = h(a), \tag{12}$$

for $h(a_0) > 0$ given. On the other hand, since $\{h(a_n)\}$ is strictly increasing, we have that $b_{N_0} < h(a_{N_0})$, and consequently, for x_{N_0}, it follows that

$$M\|[F'(x_{N_0})]^{-1}\|\|[F'(x_{N_0})]^{-1}F(x_{N_0})\| \le a_{N_0} < 1/2,$$
$$K\|[F'(x_{N_0})]^{-1}\|\|[F'(x_{N_0})]^{-1}F(x_{N_0})\|^2 \le b_{N_0} < h(a_{N_0}).$$

In consequence, we can choose $z_0 = x_{N_0}$ and apply Chebyshev's method starting at this point to guarantee the convergence of method (6).

Provided that sequence (6) is well defined, the convergence reduces to show that (6) is a Cauchy sequence. In the next result the semilocal convergence of (6) is provided and it is also used to draw conclusions about the existence of a solution and the domain in which it is located.

First, we rewrite (6) in the following form:

$$w_n = \begin{cases} x_n, & \text{if} \quad n \leq N_0, \\ z_{n-N_0}, & \text{if} \quad n > N_0. \end{cases}$$

Theorem 3.1. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a twice Fréchet differentiable operator on a non-empty open convex domain Ω. Let $x_0 \in \Omega$ and assume all conditions (C1)–(C4) hold. If* (2) *and $B(x_0, R_1 + R_2) \subseteq \Omega$ are satisfied, then sequence* (6)*, defined by $\{w_n\}$ and starting from w_0, converges to a solution z^* of* (1)*, and the solution z^* and the iterates w_n belong to $\overline{B(x_0, R_1 + R_2)}$.*

Proof. First, from the ideas previously indicated, we consider that N_0 exists. We then observe that $w_0, w_1, \ldots, w_{N_0} \in \Omega$, since they are iterates generated by Newton's method, and consequently, $\|w_i - x_0\| \leq R_1 < R_1 + R_2$, for $i = 1, 2, \ldots, N_0$. Therefore, $w_i \in B(x_0, R_1) \subset B(x_0, R_1 + R_2) \subseteq \Omega$, for $i = 1, 2, \ldots, N_0$.

Next, if we choose $w_{N_0} = z_0 = x_{N_0}$, the iterates w_i, for $i > N_0$, are given by Chebyshev's method and $\|w_i - w_{N_0}\| < R_2$, for $i > N_0$. Then $\|w_i - x_0\| \leq \|w_i - w_{N_0}\| + \|w_{N_0} - x_0\| < R_1 + R_2$ and $w_i \in B(x_0, R_1 + R_2) \subseteq \Omega$, for $i > N_0$. In consequence, $w_n \in \Omega$, for all $n \in \mathbb{N}$, and $\{w_n\}$ is well defined.

Finally, we prove that $\{w_n\}$ is a Cauchy sequence in Ω. To do this, it suffices to prove it for $\{w_n\}_{n \geq N_0}$, which is the sequence generated by Chebyshev's method (see [7]). Hence, there exists $z^* \in \overline{B(x_0, R_2)} \subset \overline{B(x_0, R_1 + R_2)}$ such that $z^* = \lim_n w_n$. Moreover, it is immedite to follow that $F(z^*) = 0$ (see [7]). □

In addition, we establish the uniqueness of the solution z^* of equation (1) in the next theorem.

Theorem 3.2. *Let us suppose conditions (C1)–(C4) hold. The solution z^* of equation* (1) *is unique in the region $B\left(x_0, \frac{2}{M\beta} - (R_1 + R_2)\right) \cap \Omega$ provided that $R_1 + R_2 < \frac{2}{M\beta}$.*

Proof. We assume y^* is another solution of (1) in $B\left(x_0, \frac{2}{M\beta} - (R_1 + R_2)\right) \cap \Omega$. Then, from

$$\int_0^1 F'(z^* + t(y^* - z^*))\, dt\, (y^* - z^*) = F(y^*) - F(z^*) = 0,$$

we have to prove that the operator $T = [F'(x_0)]^{-1} \int_0^1 F'(z^* + t(y^* - z^*))\, dt$ is invertible to obtain $z^* = y^*$. By the Banach lemma, we have to prove $\|I - T\| < 1$. Indeed,

$$\begin{aligned} \|I - T\| &\leq \|[F'(x_0)]^{-1}\| \int_0^1 \|F'(z^* + t(y^* - z^*)) - F'(x_0)\|\, dt \\ &\leq M\beta \int_0^1 \|z^* + t(y^* - z^*) - x_0\|\, dt \\ &\leq M\beta \int_0^1 \left((1-t)\|z^* - x_0\| + t\|y^* - x_0\|\right)\, dt \\ &< \frac{M\beta}{2} \left(R_1 + R_2 + \frac{2}{M\beta} - (R_1 + R_2)\right) = 1. \end{aligned}$$

This completes the proof. □

Remark. We observe that the domain of the existence of solution $B(x_0, R_1 + R_2)$ obtained for iteration (6) is bigger than the domains of existence obtained for Newton's and Chebyshev's methods, $B(x_0, R_1)$ and $B(x_0, R_2)$ respectively. We also observe that the domain of uniqueness of solution $B\left(x_0, \frac{2}{M\beta} - (R_1 + R_2)\right)$ is subject to the condition $R_1 + R_2 < \frac{2}{M\beta}$, as we can see in Theorem 3.2. But we underline that the domain of starting points for iteration (6) is extended with regard to Chebyshev's method, so that we can obtain domains of existence and uniqueness of solutions that cannot be obtained with Chebyshev's method, as we can see in the example presented in Section 4.

3.2. **On the R-order of convergence.** It is well known that Newton's method is of R-order at least two, under conditions (C1)–(C3) and Chebyshev's method is of R-order at least three, under conditions (C1)–(C4). Therefore, iteration (6) has R-order of convergence at least two until iteration N_0 and R-order of convergence at least three from iteration $N_0 + 1$.

3.3. **In practice.** Observe that we can apply algorithm (6) provided that the conditions of Theorem 3.1 are satisfied, since there always exists N_0. However, the algorithm can be improved if the value of N_0 can be estimated a priori, since the verification of $\tilde{b} < h(\tilde{a})$ is saved in every step. In relation to this, we fix the value of N_0 in the following result.

Theorem 3.3. *Under the general hypotheses of the last theorem, we suppose that* (2) *holds, but* (3) *does not hold for $x_0 \in \Omega$ satisfying (C1) and (C2). Set $z_0 = x_{N_0}$ with $N_0 = 1 + \left[\frac{\ln(h(a)/b)}{\ln(f(a)g(a)^2)}\right]$, where f and g are defined in* (9), *h is given in* (4) *and $[t]$ denotes the integer part of the real number t. Then, z_0 satisfies condition* (3).

Proof. We take into account that the above-mentioned ideas carried out where we guarantee that algorithm (6) is well defined, since there always exists $N_0 \in \mathbb{N}$ so that Chebyshev's method can be applied starting at $z_0 = x_{N_0}$. Remember that condition (12) must be satisfied. On the other hand,

$$b_{N_0} = b_{N_0-1} f(a_{N_0-1}) g(a_{N_0-1})^2 = \cdots = b_0 \prod_{j=0}^{N_0-1} f(a_j) g(a_j)^2,$$

and, since the sequence $\{a_n\}$ is decreasing and f and g are increasing functions in $(0, 1/2)$, we can write $b_{N_0} < b\left(f(a)g(a)^2\right)^{N_0}$. If $b\left(f(a)g(a)^2\right)^{N_0} < h(a)$ is then required, from (12), it follows that x_{N_0} is a starting point for Chebyshev's method. Then, if condition

$$N_0 \log\left(f(a)g(a)^2\right) + \log b < \log h(a)$$

is satisfied, or

$$N_0 > \frac{\log h(a) - \log b}{\log\left(f(a)g(a)^2\right)},$$

the proof is now complete. □

Example 1. We again consider the scalar equation $F(x) = x^3 - 2007 = 0$ and choose $x_0 = 19.9$ as the starting approximation. From the domains of starting points obtained previously, observe that $x_0 = 19.9$ satisfies (2), but not (3), so that we can apply Newton's method and method (6) to approximate the solution $z^* = 12.6139\ldots$, but we cannot do Chebyshev's method. We then start with Newton's method and apply the last theorem to obtain $N_0 = 5$, so that in the fifth iteration we can already apply Chebyshev's method to approximate the solution of the equation in $(0, 20)$. In Table 1, the estimated errors $|x_n - z^*|$ and $|w_n - z^*|$ for Newton's method and method (6), respectively, are given if the starting point is $x_0 = 19.9$; 150 significant digits are used and they are smaller than 10^{-150}. Observe that the method proposed in this paper improves then the approximations given by Newton's method and the speed of convergence is also improved.

TABLE 1. Errors for Newton's method and method (6)

i	$\|x_n - z^*\|$	$\|w_n - z^*\|$
0	$7.28610\ldots$	$7.28610\ldots$
1	$2.34212\ldots$	$2.34212\ldots$
2	$3.47632\ldots \times 10^{-1}$	$3.47632\ldots \times 10^{-1}$
3	$9.24029\ldots \times 10^{-3}$	$9.24029\ldots \times 10^{-3}$
4	$6.76235\ldots \times 10^{-6}$	$6.76235\ldots \times 10^{-6}$
5	$3.62532\ldots \times 10^{-12}$	$3.62532\ldots \times 10^{-12}$
6	$1.04194\ldots \times 10^{-24}$	$4.99103\ldots \times 10^{-37}$
7	$8.60675\ldots \times 10^{-50}$	$1.30233\ldots \times 10^{-111}$
8	$5.87259\ldots \times 10^{-100}$	

Example 2. On the other hand, if we now take into account the complex equation $F(z) = \sin z - 1/3 = 0$, in Figure 7 we can see the region of accessibility of method (6). Observe that the domain of starting points is the same as for Newton's method, but the colour intensity is different, lighter or darker, according to the number of iterations needed to reach the roots. There are lighter areas for method (6) as a consequence of the faster speed of convergence.

4. EXAMPLE

We next illustrate how iteration (6) can be used to solve a non-linear integral equation of mixed Hammerstein type

$$x(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t)\ell_i(t, x(t))\, dt = y(s), \quad s \in [a,b],$$

where $-\infty < a < b < \infty$, y, ℓ_i and G_i $(i = 1, 2, \ldots, m)$ are known functions and x is a solution to determine. The analysis and computation of the mixed Hammerstein equation is justified by the dynamic model of a chemical reactor (see [4]).

In this paper, we consider the following non-linear integral equation of mixed Hammerstein type

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)\, x(t)^3\, dt, \quad s \in [0,1], \tag{13}$$

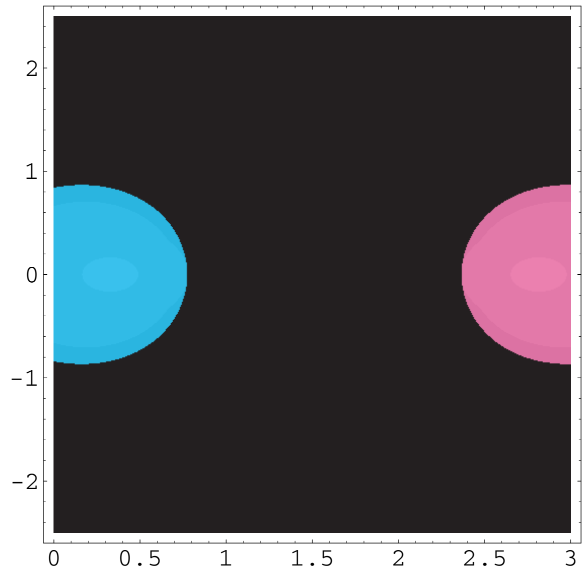

FIGURE 7. Method (6)

where $x \in C[0,1]$, $t \in [0,1]$, and the kernel G is $G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$

First, note that a solution z^* of (13) in $C[0,1]$ with the max-norm must satisfy

$$\|z^*\| - \|z^*\|^3/24 - 1 \le 0,$$

i.e., $\|z^*\| \le r_1 = 1.0479\ldots$ and $\|z^*\| \ge r_2 = 4.2902\ldots$, where r_1 and r_2 are the positive roots of the real equation $t - t^3/24 - 1 = 0$. Consequently, if we look for a solution such that $\|z^*\| < r_1$, we can consider $\Omega = B(0,\rho) \subseteq C[0,1]$, with $\rho \in (r_1, r_2)$, as a non-empty open convex domain. We then take, for example, $\rho = 2.21$.

Second, we discretize (13) to transform it into a finite dimensional problem. This procedure consists of approximating the integral appearing in (13) by a numerical quadrature formula. To obtain a numerical solution, we use the Gauss-Legendre formula to approximate an integral

$$\int_0^1 v(t)\,dt \simeq \sum_{i=1}^{m} \varpi_i v(t_i),$$

where the nodes t_i and the weights ϖ_i are determined; in particular, see Table 2 for $m = 8$.

If we denote the approximation of $x(t_j)$ by x_j ($j = 1, 2, \ldots, 8$), (13) is now equivalent to the following non-linear system of equations

$$x_j = 1 + \frac{1}{3}\sum_{k=1}^{8} \alpha_{jk}\, x_k^3, \quad j = 1, 2, \ldots, 8, \tag{14}$$

TABLE 2. Nodes and weights for the Gauss-Legendre formula

i	t_i	ϖ_i	i	t_i	ϖ_i
1	0.019855	0.050614	5	0.591717	0.181342
2	0.101667	0.111191	6	0.762766	0.156853
3	0.237234	0.156853	7	0.898333	0.111191
4	0.408283	0.181342	8	0.980145	0.050614

where

$$\alpha_{jk} = \begin{cases} \varpi_k t_k(1-t_j) & \text{if } k \leq j, \\ \varpi_k t_j(1-t_k) & \text{if } k < j. \end{cases}$$

System (14) can now be written in the form $\mathbf{x} = \mathbf{1} + \frac{1}{3}A\mathbf{x}^3$, or

$$F : B(0,\rho) \subseteq \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \quad F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - \frac{1}{3}A\mathbf{x}^3 = 0,$$

where

$$\mathbf{x} = (x_1, x_2, \dots, x_8)^T,\ \mathbf{1} = (1,1,\dots,1)^T,\ A = (\alpha_{jk})_{j,k=1}^8,\ \mathbf{x}^3 = (x_1^3, x_2^3, \dots, x_8^3)^T.$$

For this F, we have

$$F'(\mathbf{x})(\mathbf{u}) = \begin{pmatrix} 1-\alpha_{11}x_1^2 & -\alpha_{12}x_2^2 & \cdots & -\alpha_{18}x_8^2 \\ -\alpha_{21}x_1^2 & 1-\alpha_{22}x_2^2 & \cdots & -\alpha_{28}x_8^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{81}x_1^2 & -\alpha_{82}x_2^2 & \cdots & 1-\alpha_{28}x_8^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{pmatrix},$$

$$F''(\mathbf{x})(\mathbf{u},\mathbf{v}) = -2 \begin{pmatrix} \sum_{i=1}^8 \alpha_{1i}x_iu_iv_i \\ \sum_{i=1}^8 \alpha_{2i}x_iu_iv_i \\ \vdots \\ \sum_{i=1}^8 \alpha_{8i}x_iu_iv_i \end{pmatrix},$$

where $\mathbf{u} = (u_1, u_2, \dots, u_8)^T$ and $\mathbf{v} = (v_1, v_2, \dots, v_8)^T$.

If we choose $\mathbf{x_0} = (1.1, 1.1, \dots, 1.1)^T$ and the max-norm, then

$$M = 4.42, \quad \beta = 1.1715\dots, \quad \eta = 0.0964\dots, \quad a_0 = M\beta\eta = 0.4996\dots < 1/2,$$
$$K = 2, \quad b_0 = K\beta\eta^2 = 0.0217\dots > 0.0075\dots = h(a_0).$$

Observe that we can apply Newton's method to solve (14), but we cannot use Chebyshev's method because the second condition of (3) is not satisfied. However, by Theorem 3.3, we can use Chebyshev's method after the second approximation given by Newton's method, since $N_0 = 2$, and obtain the numerical solution $\mathbf{z}^* = (z_1^*, z_2^*, \dots, z_8^*)^T$, which is shown in Table 3, after two more approximations. If only Newton's method is applied, six iterations are needed to reach the solution $\mathbf{z}^*$.

Moreover, the existence of the solution is guaranteed in the ball $\overline{B(\mathbf{x_0}, 0.1922\dots)}$ by Theorem 3.1, and the unicity in $B(\mathbf{x_0}, 0.1939\dots)$ by Theorem 3.2.

Finally, we interpolate the points of Table 3 and taking into account that the solution of (13) satisfies $x(0) = x(1) = 1$, an approximation $\hat{\mathbf{z}}$ of the numerical

TABLE 3. Numerical solution $\mathbf{z}^*$ of (14)

i	z_i^*	i	z_i^*
1	1.0036371689137973...	5	1.0463047141982516...
2	1.0172285203889426...	6	1.0344614429104666...
3	1.0344614429104666...	7	1.0172285203889426...
4	1.0463047141982516...	8	1.0036371689137973...

solution $\mathbf{z}^*$ is obtained (see Figure 8). Notice that the interpolated approximation $\hat{\mathbf{z}}$ lies within the existence domain of the solutions obtained above.

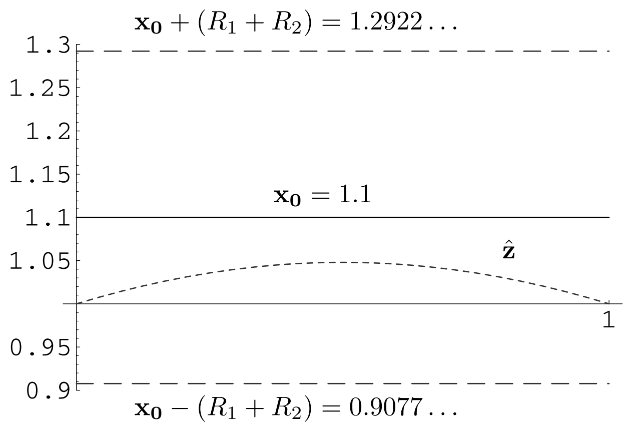

FIGURE 8. Approximated solution $\hat{\mathbf{z}}$ of equation (13)

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University of La Rioja, Department of Mathematics and Computation, C/ Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: jezquer@unirioja.es

University of La Rioja, Department of Mathematics and Computation, C/ Luis de Ulloa s/n, 26004 Logroño, Spain
E-mail address: mahernan@unirioja.es