MATHEMATICS OF COMPUTATION
Volume 67, Number 221, January 1998, Pages 357–359
S 0025-5718(98)00924-7

STEINER SYSTEMS $S(5,6,v)$ WITH $v = 72$ AND 84

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Abstract. It is proved that there are precisely 4204 pairwise non-isomorphic Steiner systems $S(5,6,72)$ invariant under the group $\mathrm{PSL}_2(71)$ and which can be constructed using only short orbits.

It is further proved that there are precisely 38717 pairwise non-isomorphic Steiner systems $S(5,6,84)$ invariant under the group $\mathrm{PSL}_2(83)$ and which can be constructed using only short orbits.

1. Introduction

This is the third of a sequence of papers by the present authors on Steiner 5-designs. Steiner systems $S(5,6,v)$ are now known to exist for $v = 12, 24, 48, 72, 84$, 108 and 132. These are precisely the first seven admissible values of v which are 1 more than a prime $p \equiv 3 \pmod 4$ and this fact is utilized in constructing the systems from orbits of 6-sets under the group $\mathrm{PSL}_2(v-1)$. The next value of v satisfying these criteria is 168 and we continue to search for a Steiner system $S(5,6,168)$ invariant under $\mathrm{PSL}_2(167)$. With the exception of the system $S(5,6,12)$, for which the full automorphism group is the sporadic Mathieu group M_{12}, it is easy to show that the groups $\mathrm{PSL}_2(v-1)$ are the full automorphism groups of the systems $S(5,6,v)$. Together with the classical system $S(5,8,24)$ and the system $S(5,7,28)$ constructed by Denniston [1], the above systems together with their derived Steiner 4-designs are still the only known Steiner systems $S(t,k,v)$ with $t > 3$.

Enumeration results are also available. It is well known that up to isomorphism the system $S(5,6,12)$ is the unique system with these parameters [7]. There are precisely 3 pairwise non-isomorphic $S(5,6,24)$s invariant under $\mathrm{PSL}_2(23)$ [2]. In the first of this sequence of papers [3] the present authors proved that the number of pairwise non-isomorphic $S(5,6,48)$s invariant under $\mathrm{PSL}_2(47)$ is precisely 459. The underlying mathematical theory and technical computing details of all of this work are dealt with in the second of the papers [4] and so are not repeated here. Enumeration results of some very limited classes of systems $S(5,6,108)$ and $S(5,6,132)$ are also given in that paper. This leaves a gap for the values $v = 72$ and 84 where the only published systems are those given by Mills [5] and Denniston [1] respectively and 8 pairwise non-isomorphic $S(5,6,72)$s constructed recently by Schmalz [6]. It is the aim of this paper to fill this gap. Because of the size of the problem and numbers of orbits involved, unfortunately it is not possible to completely enumerate all systems $S(5,6,72)$ and $S(5,6,84)$ invariant under the groups $\mathrm{PSL}_2(71)$ and $\mathrm{PSL}_2(83)$ respectively. Therefore we have restricted our attention to an easily defined and important subclass; namely those systems which can be constructed

Received by the editor April 5, 1996.
1991 *Mathematics Subject Classification.* Primary 05B05.

only from short 6-set orbits under the appropriate group. This would appear to be the best that can be done at the present time. The veracity of the results has been checked by the first two authors and the third author working independently using different hardware and software and can be summarized as follows.

Theorems. *There are precisely* 4204 *pairwise non-isomorphic Steiner systems* $S(5,6,72)$ *invariant under the group* $\mathrm{PSL}_2(71)$ *and which can be constructed using only short 6-set orbits.*

There are precisely 38717 *pairwise non-isomorphic Steiner systems* $S(5,6,84)$ *invariant under the group* $\mathrm{PSL}_2(83)$ *and which can be constructed using only short* 6*-set orbits.*

2. Results

The results for $S(5,6,72)$ are given in Appendix 1 and for $S(5,6,84)$ in Appendix 2 (see supplement section at the end of this issue). We describe the interpretation of the tables given in the appendices with particular reference to the former. The information for $S(5,6,84)$ is given similarly. Call an orbit a full orbit if the number of blocks in it is g, the order of $\mathrm{PSL}_2(v-1)$, and a $1/n$ th orbit if its length is g/n.

Elementary calculation shows that there are precisely 78 full orbits and one one-fifth orbit of 5-sets under $\mathrm{PSL}_2(71)$. These are listed as #0 to #78 together with a pair of elements which with $\infty, 0, 1$ are a generating set for the orbit. Next all suitable (i.e. not containing a repeated 5-set orbit), short 6-set orbits together with a triple of elements which with $\infty, 0, 1$ are a generating set for the orbits are listed as #0 to #202. Also listed alongside are the 5-set orbits which each 6-set orbit contains. To construct a Steiner system $S(5,6,72)$ it is necessary to determine a collection of these 6-set orbits which collectively contain each 5-set orbit precisely once. Altogether there are 8408 distinct solutions but these are isomorphic in pairs under the mapping $z \mapsto -z$. The number of occurrences of the 6-set orbits in these solutions is given next. Thus the one-third 6-set orbit #187 does not occur in any solution! The next table is a statistical breakdown of the different possible "types" of the 4204 pairwise nonisomorphic short 6-set orbit solutions. N gives the total number of short orbits in the system and under $2, 3, 5$ and 6 are given respectively the numbers of half, third, fifth and sixth orbits. M gives the total number of the systems of this type. A selection of actual systems then follows. For each different type as identified in the previous table, the short 6-set orbit numbers for a solution of that type are given. For some selected types of solution, where the number of pairwise non-isomorphic solutions of that type is small, all solutions are listed.

Finally for information two further tables are included. The first gives the pairs of 5-set and short 6-set orbits which are isomorphic under the mapping $z \mapsto -z$. A missing orbit number indicates that the orbit is stabilized by the mapping. The second table shows which short 6-set orbits the 5-set orbits are contained in, and is the inverse table of the information given alongside the listing of the short 6-set orbits. This latter table is of particular use in constructing the Steiner systems.

3. Conclusion

The combined results of this and the other papers in the references leave little doubt that the conjecture that there exists a Steiner system $S(5,6,v)$ invariant under the group $\mathrm{PSL}_2(v-1)$ whenever v is admissible and 1 more than a prime $p \equiv 3 \pmod 4$ is safe. It would be a major advance to prove it. One approach is to study the enumeration results obtained for the cases $v = 24$ and 48 in previous papers [2], [3] and for $v = 72$ and 84 in this paper. We have spent a considerable amount of time doing this but unfortunately have been unable to detect any major clues even to a preliminary theory of these systems. The only slight hint to emerge has been the role which can be played by certain short 6-set orbits which we refer to as of type ABB'. The theory of these orbits is explained in detail in [4] where they are utilized to construct systems $S(5,6,108)$ and $S(5,6,132)$. However by themselves they are not powerful enough even with the aid of a computer to construct easily systems with larger v. A general theory awaits new ideas.

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