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Explicit Estimates for the Error Term in the Prime Number Theorem for Arithmetic Progressions

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Abstract. We give explicit numerical estimates for the Chebyshev functions $\psi(x; k, l)$ and $\theta(x; k, l)$ for certain nonexceptional moduli k. For values of ε and b, a constant c is tabulated such that $|\psi(x; k, l) - x/\varphi(k)| < \varepsilon x/\varphi(k)$, provided $(k, l) = 1$, $x \geqslant \exp(c \log^2 k)$, and $k \geqslant 10^b$. The methods are similar to those used by Rosser and Schoenfeld in the case $k = 1$, but are based on explicit estimates of $N(T, \chi)$ and an explicit zero-free region for Dirichlet L-functions.

1. Introduction. Let k and l be positive integers. The Chebyshev prime counting functions $\psi(x; k, l)$ and $\theta(x; k, l)$ are defined by

$$\theta(x; k, l) = \sum_{\substack{p \leqslant x \\ p \equiv l \,(\mathrm{mod}\, k)}} \log p, \qquad \psi(x; k, l) = \sum_{\substack{p^\alpha \leqslant x \\ p^\alpha \equiv l \,(\mathrm{mod}\, k)}} \log p,$$

where the sums extend over all primes p and prime powers p^α, respectively. The prime number theorem for arithmetic progressions is equivalent to the statement that

$$\psi(x; k, l) = x/\varphi(k) + o(x), \qquad x \to \infty,$$

if k and l are fixed relatively prime integers. An alternative statement is that for any positive ε there exists $x_0 = x_0(k, l, \varepsilon)$ such that

$$|\psi(x; k, l) - x/\varphi(k)| < \varepsilon x/\varphi(k), \qquad x \geqslant x_0.$$

The purpose of this paper is to give explicit numerical estimates for $x_0(k, l, \varepsilon)$ for some values of k and ε.

The case $k = 1$ or 2 has been investigated in a series of papers by J. B. Rosser and L. Schoenfeld. The methods used in this paper are similar to those used by Rosser and Schoenfeld, and we shall make frequent reference to their work.

The size of the error term in the prime number theorem depends on the location of zeros of the Riemann zeta function $\zeta(s)$. The estimates of $\psi(x; 1, 1)$ in [10] and [11] are based on the computation of 3,502,500 zeros of $\zeta(s)$ and a zero-free region for $\zeta(s)$ of the type originally proved by de la Vallée Poussin. A similar situation exists in the case of the prime number theorem for arithmetic progressions, where the size of $x_0(k, l, \varepsilon)$ depends on the location of zeros of Dirichlet L-functions formed with characters modulo k. In the case of a fixed modulus k we can make use

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of computational information concerning the zeros of L-functions modulo k in the same way that Rosser and Schoenfeld used information concerning the zeros of $\zeta(s)$. In the estimation of $x_0(k, l, \varepsilon)$ as k tends to infinity, we can no longer derive significant benefit from the mere computation of zeros, since it is no longer a finite computational problem to compute enough zeros. In this case we can base our estimates on the following explicit zero-free region.

Let $R = 9.645908801$ and $\mathfrak{L}_k(s) = \prod_{\chi \bmod k} L(s, \chi)$.

THEOREM 1.1. *There exists at most a single zero of $\mathfrak{L}_k(s)$ in the region $\{s = \sigma + it$: $\sigma \geqslant 1 - 1/[R \log \max\{k, k|t|, 10\}]\}$. The only possible zero in this region is a simple real zero arising from an L-function formed with a real nonprincipal character modulo k.*

If k is an integer for which there exists a real zero of $\mathfrak{L}_k(s)$ with $\beta > 1 - 1/(R \log k)$, then we shall refer to k as an exceptional modulus. A proof of Theorem 1.1 appears in [5], as well as a further result concerning exceptional moduli.

TABLE 1

b \ ε	1	.5	.2	.1	.05	.01	.005	.001	.0001	.00001
1	34.13	41.01	53.23	65.28	79.94	124.3	147.2	208.3	313.3	438.5
2	20.62	23.35	27.98	32.37	37.55	52.25	59.53	78.34	109.7	146.9
3	16.85	18.51	21.29	23.88	26.84	34.92	38.82	48.74	64.88	83.19
4	15.08	16.28	18.26	20.05	22.08	27.48	30.04	36.49	46.80	58.36
5	14.04	14.98	16.51	17.88	19.40	23.41	25.28	29.97	37.34	45.54
6	13.36	14.12	15.37	16.47	17.69	20.83	22.33	25.94	31.62	37.84
7	12.86	13.52	14.58	15.50	16.49	19.09	20.28	23.23	27.81	32.78
8	12.49	13.07	13.98	14.76	15.62	17.81	18.82	21.29	25.11	29.23
9	12.20	12.73	13.52	14.21	14.94	16.86	17.72	19.85	23.09	26.58
10	11.98	12.44	13.15	13.76	14.43	16.10	16.87	18.71	21.54	24.57
11	11.79	12.20	12.85	13.38	13.99	15.47	16.15	17.81	20.30	22.96
12	11.64	12.03	12.60	13.09	13.62	14.99	15.58	17.08	19.31	21.67
13	11.50	11.84	12.37	12.83	13.32	14.55	15.10	16.44	18.48	20.60
14	11.39	11.72	12.19	12.62	13.08	14.20	14.71	15.93	17.76	19.72
15	11.29	11.58	12.03	12.42	12.85	13.88	14.35	15.48	17.17	18.96
20	10.93	11.14	11.48	11.76	12.08	12.81	13.16	13.97	15.15	16.41
25	10.69	10.88	11.14	11.36	11.60	12.20	12.45	13.07	14.00	14.95
30	10.55	10.69	10.91	11.09	11.30	11.78	11.99	12.51	13.25	14.01
35	10.44	10.55	10.74	10.90	11.08	11.47	11.65	12.08	12.72	13.37
40	10.35	10.45	10.62	10.76	10.90	11.25	11.41	11.79	12.32	12.89
45	10.29	10.39	10.52	10.64	10.78	11.08	11.23	11.54	12.02	12.51
50	10.24	10.33	10.45	10.56	10.66	10.95	11.06	11.37	11.78	12.22
60	10.15	10.21	10.32	10.43	10.52	10.74	10.84	11.08	11.42	11.78
70	10.08	10.14	10.23	10.32	10.40	10.59	10.68	10.89	11.17	11.47
80	10.04	10.08	10.18	10.23	10.31	10.48	10.56	10.73	10.98	11.25
90	10.00	10.04	10.11	10.18	10.23	10.41	10.45	10.61	10.85	11.08
100	9.96	10.01	10.07	10.13	10.18	10.32	10.39	10.54	10.72	10.93

The main result of this paper is the following:

THEOREM 1.2. *Let k be a nonexceptional modulus, and let $(k, l) = 1$. For various values of ε and b, Table 1 gives values of c such that*

$$\left|\psi(x; k, l) - \frac{x}{\varphi(k)}\right| < \frac{\varepsilon x}{\varphi(k)} \quad \text{and} \quad \left|\theta(x; k, l) - \frac{x}{\varphi(k)}\right| < \frac{\varepsilon x}{\varphi(k)},$$

provided that $k \geqslant 10^b$ and $x \geqslant \exp(c \log^2 k)$.

For any given values of ε and b the methods of this paper will yield a value of c, but the methods are limited by the requirement that $c > R$. The methods could also be used to calculate an explicit constant A with the property that

$$\frac{\varphi(k)}{x}\left|\psi(x; k, l) - \frac{x}{\varphi(k)}\right| < Ak\sqrt{\frac{\log x}{R}}\exp\left(-\sqrt{\frac{\log x}{R}}\right),$$

provided $x > \exp(R \log^2 k)$ and k is not exceptional. In the interests of brevity this will be deferred to a later paper. Later papers will also deal with the case $k = 3$ and implications of the generalized Riemann hypothesis.

2. Estimates of $N(T, \chi)$. Throughout this paper χ will be a Dirichlet character modulo k, and χ_1 modulo k_1 will be the primitive character which induces χ. We write χ_0 for the principal character, and in this case we take $k_1 = 1$ and $\chi_1 = 1$. Note that

$$L(s, \chi) = L(s, \chi_1)\prod_{p \mid k}(1 - \chi_1(p)p^{-s}). \tag{2.1}$$

Define $N(T, \chi) = N(T, \chi_1)$ as the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $0 < \beta < 1$ and $|\gamma| \leqslant T$. The main result of this section is the following.

THEOREM 2.1. *Let $T \geqslant 1$ and χ be a primitive nonprincipal character modulo k. If $0 < \eta \leqslant \frac{1}{2}$, then*

$$\left|N(T, \chi) - \frac{T}{\pi}\log\frac{kT}{2\pi e}\right| < C_1 \log kT + C_2, \tag{2.2}$$

where

$$C_1 = \frac{1 + 2\eta}{\pi \log 2}, \tag{2.3}$$

$$C_2 = .3058 - .268\eta + \frac{4\log\zeta(1 + \eta)}{\log 2} - \frac{2\log\zeta(2 + 2\eta)}{\log 2} + \frac{2}{\pi}\log\zeta\left(\frac{3}{2} + 2\eta\right). \tag{2.4}$$

Proof. The method of proof is essentially due to Backlund [2], with refinements due to Rosser [9] and the author. Assuming that $\pm T$ does not coincide with the ordinate of a zero, consider the rectangle R with vertices at $\sigma_1 - iT$, $\sigma_1 + iT$, $1 - \sigma_1 + iT$, and $1 - \sigma_1 - iT$, where $\sigma_1 > 1$. Then we have

$$N(T, \chi) = \frac{1}{2\pi}\Delta_R \arg \xi(s, \chi), \tag{2.5}$$

where

$$(2.6)\qquad \xi(s,\chi)=\left(\frac{k}{\pi}\right)^{(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi),\qquad a=(1-\chi(-1))/2.$$

Let $\mathcal{C}$ denote the portion of the contour in $\sigma \geqslant \frac{1}{2}$. From the functional equation of $\xi(s,\chi)$ and (2.6) we obtain

$$\begin{aligned}(2.7)\quad \Delta_R \arg \xi(s,\chi) &= 2\Delta_{\mathcal{C}} \arg \xi(s,\chi)\\ &= 2\left[\Delta_{\mathcal{C}}\arg\left(\frac{k}{\pi}\right)^{(s+a)/2}+\Delta_{\mathcal{C}}\arg\Gamma\left(\frac{s+a}{2}\right)+\Delta_{\mathcal{C}}\arg L(s,\chi)\right]\\ &= 2T\log\frac{k}{\pi}+4\operatorname{Im}\log\Gamma\left(\frac{1}{4}+\frac{a}{2}+i\frac{T}{2}\right)+2\Delta_{\mathcal{C}}\arg L(s,\chi).\end{aligned}$$

We shall apply Stirling's formula in the form

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\frac{1}{2}\log 2\pi+\frac{\theta}{6|z|},$$

where $|\theta|\leqslant 1$ and $|\arg z|\leqslant \pi/2$. This error term is well known and appears in Olver [6, p. 294]. Hence

$$\begin{aligned}(2.8)\quad \operatorname{Im}\log\Gamma\left(\frac{1}{4}+\frac{a}{2}+i\frac{T}{2}\right) &= \frac{T}{2}\log\frac{T}{2e}+\frac{T}{4}\log\left[1+\left(\frac{2a+1}{2T}\right)^2\right]\\ &\quad+\frac{2a-1}{4}\tan^{-1}\left[\frac{2T}{1+2a}\right]+\frac{\theta}{3\left|\frac{1}{2}+a+iT\right|}.\end{aligned}$$

Denote the last three terms by $f_1(T)$, $f_2(T)$ and $f_3(T)$. If $a=0$, then f_1 and f_2 are decreasing for $T\geqslant 1$, so that

$$\begin{aligned}|f_1+f_2+f_3| &\leqslant |f_1+f_2|+\frac{1}{3\sqrt{1.25}}\\ &\leqslant \max\{|f_1(1)+f_2(1)|,|f_1(\infty)+f_2(\infty)|\}+.2982<.6909.\end{aligned}$$

If $a=1$, then f_1 and f_2 are positive, f_1 is decreasing, and f_2 is increasing. The maximum value of f_1+f_2 occurs between $T=1.64$ and $T=1.65$, so that

$$|f_1+f_2+f_3|<f_1(1.64)+f_2(1.65)+\frac{1}{3\sqrt{3.25}}<.6425.$$

It follows from (2.5), (2.7) and (2.8) that

$$(2.9)\qquad N(T,\chi)=\frac{1}{\pi}\left[T\log\frac{kT}{2\pi e}+1.3818\theta+\Delta_{\mathcal{C}}\arg L(s,\chi)\right].$$

It remains to estimate $\Delta_{\mathcal{C}}\arg L(s,\chi)$. We divide $\mathcal{C}$ into 3 pieces $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ as follows:

$$\begin{aligned}&\mathcal{C}_1\colon \tfrac{1}{2}-iT \quad\text{to}\quad \sigma_1-iT,\\ &\mathcal{C}_2\colon \sigma_1-iT \quad\text{to}\quad \sigma_1+iT,\\ &\mathcal{C}_3\colon \sigma_1+iT \quad\text{to}\quad \tfrac{1}{2}+iT.\end{aligned}$$

We first estimate $\Delta_{\mathcal{C}_3}\arg L(s,\chi)$. In view of the fact that $L(\bar{s},\chi)=\overline{L(s,\bar{\chi})}$, an upper bound for the change in argument on $\mathcal{C}_3$ will also serve as an upper bound on

$\mathcal{C}_1$, provided the bound is valid for any primitive nonprincipal χ modulo k.

Let N be a positive integer, and define

$$f(s) = \tfrac{1}{2}\left[L(s + iT, \chi)^N + L(s - iT, \bar{\chi})^N\right].$$

Note that

$$f(\sigma) = \operatorname{Re} L(\sigma + iT, \chi)^N$$

if σ is real. Suppose $f(\sigma)$ has n real zeros in the interval $\frac{1}{2} \leqslant \sigma \leqslant \sigma_1$. These zeros partition the interval into $n + 1$ subintervals, and on each subinterval the quantity $\arg L(\sigma + iT, \chi)^N$ can change by at most π, since $\operatorname{Re} L(\sigma + iT, \chi)^N$ is nonzero on the interior of each subinterval. It follows that

$$|\Delta_{\mathcal{C}_3} \arg L(s, \chi)| = \frac{1}{N}|\Delta_{\mathcal{C}_3} \arg L(s, \chi)^N| \leqslant \frac{(n + 1)\pi}{N}. \tag{2.10}$$

We now estimate n from above. Let $0 < \eta \leqslant \frac{1}{2}$, and define $\sigma_1 = \frac{3}{2} + 2\eta$ and $\sigma_0 = 1 + \eta$. It follows from Jensen's theorem that

$$n \log 2 \leqslant \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} \log|f(\sigma_0 + (1 + 2\eta)e^{i\theta})|\, d\theta - \log|f(\sigma_0)|. \tag{2.11}$$

In order to estimate $|f(s)|$ we appeal to a result of Rademacher [8]. He proved that if $-\eta \leqslant \sigma \leqslant 1 + \eta$, then

$$|L(s, \chi)| \leqslant \left(\frac{k|s + 1|}{2\pi}\right)^{(1+\eta-\sigma)/2} \zeta(1 + \eta).$$

It follows that

$$\begin{aligned} &\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \log|f(\sigma_0 + (1 + 2\eta)e^{i\theta})|\, d\theta \\ &\quad \leqslant \frac{-N}{2\pi}\int_{\pi/2}^{3\pi/2} \frac{1}{2}(1 + 2\eta)\cos\theta \log\left(\frac{k\left(\sqrt{T^2 + (2 + \eta)^2} + 1 + 2\eta\right)}{2\pi}\right) d\theta \\ &\qquad + \frac{N}{2}\log\zeta(1 + \eta) \\ &\quad \leqslant \frac{N}{2\pi}(1 + 2\eta)\log(.74685\, kT) + \frac{N}{2}\log\zeta(1 + \eta), \end{aligned} \tag{2.12}$$

since $T \geqslant 1$ and $\eta \leqslant \frac{1}{2}$.

If $\sigma \geqslant 1 + \eta$, then we use the trivial estimate

$$|f(s)| \leqslant \zeta(1 + \eta)^N,$$

and it follows from (2.11) and (2.12) that

$$n \log 2 \leqslant \frac{N(1 + 2\eta)}{2\pi}\log(.74685\, kT) + N\log\zeta(1 + \eta) - \log|f(\sigma_0)|. \tag{2.13}$$

Now write $L(\sigma_0 + iT, \chi) = re^{i\varphi}$. We choose a sequence of N's tending to infinity such that $N\varphi$ tends to 0 modulo 2π. It follows that

$$\lim_{N\to\infty} \frac{f(\sigma_0)}{|L(\sigma_0 + iT, \chi)|^N} = 1. \tag{2.14}$$

Note that for $\sigma > 1$ we have

$$|L(s,\chi)| = \prod_p |1 - \chi(p)p^{-s}|^{-1} \geqslant \prod_p \left(1 + \frac{1}{p^\sigma}\right)^{-1} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

Hence from (2.10), (2.13), and (2.14) we obtain

$$(2.15) \qquad |\Delta_{\mathcal{C}_1} \arg L(s,\chi)| \leqslant \frac{1+2\eta}{2\log 2}\log(.74685\,kT) + \frac{2\pi\log\zeta(1+\eta)}{\log 2} - \frac{\pi\log\zeta(2+2\eta)}{\log 2}.$$

Finally we estimate the change along $\mathcal{C}_2$. If $\sigma > 1$, then

$$|\arg L(s,\chi)| \leqslant |\log L(s,\chi)| \leqslant \log\zeta(\sigma).$$

Hence

$$|\Delta_{\mathcal{C}_2} \arg L(s,\chi)| \leqslant 2\log\zeta(\tfrac{3}{2} + 2\eta).$$

The result then follows from (2.9) and (2.15).

Theorem 2.1 may be stated as well for imprimitive or principal characters. Henceforth we shall abbreviate $N(T,\chi)$ as $N(T)$, and furthermore we use

$$F(T) = \frac{T}{\pi}\log\frac{kT}{2\pi e}, \qquad F_1(T) = \frac{T}{\pi}\log\frac{k_1T}{2\pi e},$$
$$R(T) = C_1\log kT + C_2, \qquad R_1(T) = C_1\log k_1T + C_2.$$

COROLLARY 2.2. *If $T \geqslant 1$ and C_1 and C_2 are as in Theorem 2.1, then*

$$(2.16) \qquad |N(T) - F_1(T)| < R_1(T).$$

Proof. If χ is nonprincipal this follow immediately from Theorem 2.1, since $N(T,\chi_1) = N(T,\chi)$. If $\chi = \chi_0$ is the principal character, then we appeal to a result of Rosser [9], who proved that (in our notation)

$$(2.17) \qquad \left|N(T,\chi_0) - \frac{T}{\pi}\log\frac{T}{2\pi e}\right| < \begin{cases} 3.75, & 0 < T \leqslant 280 \\ 5.75, & 0 < T \leqslant 1467 \\ .274\log T + .886\log\log T + 4.926, & 2 \leqslant T. \end{cases}$$

If $\sigma > 1$, note that

$$2\log\zeta(\sigma) - \log\zeta(2\sigma) = \sum_p \log\left(\frac{p^\sigma+1}{p^\sigma-1}\right)$$

is decreasing in σ. It follows that C_2 is decreasing in η, and

$$(2.18) \qquad C_2 > 5.365.$$

If $1 \leqslant T \leqslant 280$ or $280 \leqslant T \leqslant 1467$, the result follows immediately from (2.17) and (2.18). If $T \geqslant 1467$, then by (2.17) and (2.18) it suffices to prove that

$$(2.19) \qquad \left(\frac{1}{\pi\log 2} - .274\right)\log T - .886\log\log T + .439 > 0.$$

The left side of (2.19) is increasing in T for $T \geqslant 1467$, and is positive for $T = 1467$.

3. Bounds for $\psi(x;\, k, l)$. Let k and l be positive integers with $(k, l) = 1$. Our method of estimation for $\psi(x;\, k, l)$ is based on an "explicit formula" for certain integral averages of $\psi(x;\, k, l)$. This is the method used by Rosser [9] in the case $k = 1$, and reduces the problem to that of estimating certain sums involving zeros of Dirichlet L-functions.

Before we state the explicit formula we require some notation. If χ is a Dirichlet character modulo k, we use $z(\chi)$ to represent the set of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ with $\beta \geqslant 0$ and $\rho \neq 0$. Since χ_1 is the associated primitive character, $z(\chi_1)$ is the subset of $z(\chi)$ consisting of the zeros with $\beta > 0$. We use $b(\chi)$ for the constant term in the Laurent expansion of $L'(s, \chi)/L(s,\chi)$ about 0, $c(\chi)$ for the constant term in the expansion about -1, and $m(\chi)$ for the order of the zero of $L(s, \chi)$ at $s = 0$. Note that

$$0 \leqslant m(\chi) \leqslant \omega(k) \leqslant \frac{\log k}{\log 2}, \tag{3.1}$$

where $\omega(k)$ is the number of distinct prime factors of k. Unless otherwise indicated, a sum over χ is to be interpreted as a sum over all characters modulo k.

LEMMA 3.1. *Let* $\psi_1(x;\, k, l) = \int_1^x \psi(t;\, k, l)\, dt$, *where* $x > 1$. *Then*

$$\begin{aligned}\psi_1(x;\, k, l) = \frac{x^2}{2\varphi(k)} - \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{\rho \in z(\chi)} \frac{x^{\rho+1}}{\rho(\rho+1)} + g(x) \\ + d_1 x + d_2 x \log x + d_3 \log x + d_4,\end{aligned} \tag{3.2}$$

where

$$g(x) = -\frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{n=1}^{\infty} \frac{x^{-2n+1-a}}{2n(2n-1+2a)}, \tag{3.3}$$

$$a = (1 - \chi(-1))/2,$$

$$d_1 = \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l)[m(\chi) - b(\chi)], \tag{3.4}$$

$$d_2 = \frac{-1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) m(\chi), \tag{3.5}$$

$$d_3 = \frac{1}{\varphi(k)} \sum_{\chi(-1)=-1} \bar{\chi}(l), \tag{3.6}$$

$$d_4 = \frac{1}{\varphi(k)} \sum_{\chi(-1)=1} \bar{\chi}(l) \frac{L'}{L}(-1, \chi) + \frac{1}{\varphi(k)} \sum_{\chi(-1)=-1} \bar{\chi}(l)[c(\chi) + 1]. \tag{3.7}$$

Proof. A "smoothed" Perron inversion formula gives

$$\psi_1(x;\, k, l) = \frac{-1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \frac{1}{2\pi} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{L'}{L}(s, \chi)\, ds.$$

The remainder of the proof involves an application of the residue theorem to express the contour integral as a sum of residues. The details justifying this appear in Ingham [4, pp. 68–74], and Prachar [7, pp. 224–228].

For $x > 1$, define

$$E(x) = \psi(x; k, l) - x/\varphi(k), \tag{3.8}$$

and for m a positive integer, $x + mh > 1$, define

$$E_m(x, h) = \int_0^h \cdots \int_0^h E(x + y_1 + \cdots + y_m)\, dy_1 \cdots dy_m. \tag{3.9}$$

Further let

$$f(x) = \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{n=1}^{\infty} \frac{x^{-2n+a}}{2n-a} + d_2 \log x + d_1 + d_2. \tag{3.10}$$

LEMMA 3.2. *If* $|h| < (x-1)/m$, *then*

$$\begin{aligned} E_m(x, h) &= \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{\rho \in z(\chi)} \frac{1}{\rho(\rho+1)\cdots(\rho+m)} \sum_{j=0}^{m} (-1)^{m+j+1} \binom{m}{j} (x + jh)^{\rho+m} \\ &\quad + \int_0^h \cdots \int_0^h f(x + y_1 + \cdots + y_m)\, dy_1 \cdots dy_m. \end{aligned}$$

Proof. We use induction on m. If $m = 1$, it follows from (3.8), (3.9), and Lemma 3.1 that

$$\begin{aligned} E_1(x, h) &= \int_0^h E(x + y)\, dy \\ &= \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{\rho \in z(\chi)} \frac{x^{\rho+1} - (x+h)^{\rho+1}}{\rho(\rho+1)} + d_1 h \\ &\quad + d_2(x+h)\log(x+h) - d_2 x \log x + d_3 \log(x+h) \\ &\quad - d_3 \log x + g(x+h) - g(x). \end{aligned}$$

The result then follows for $m = 1$ from (3.3), (3.4), (3.5), (3.6), and (3.10).

If $m > 1$, we have

$$\begin{aligned} E_m(x, h) &= \int_0^h E_{m-1}(x + y_m, h)\, dy_m \\ &= \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \sum_{\rho \in z(\chi)} \frac{1}{\rho(\rho+1)\cdots(\rho+m-1)} \\ &\quad \times \sum_{j=0}^{m-1} (-1)^{m+j} \binom{m-1}{j} \int_0^h (x + y_m + jh)^{\rho+m-1}\, dy_m \\ &\quad + \int_0^h \cdots \int_0^h f(x + y_1 + \cdots + y_m)\, dy_1 \cdots dy_m, \end{aligned}$$

the term-by-term integration being justified by the fact that $\sum_{\rho\in z(\chi)} 1/|\rho(\rho+1)|$ converges. The sum on j may be written as

$$\frac{1}{\rho+m}\sum_{j=0}^{m-1}(-1)^{m+j+1}\binom{m-1}{j}(x+jh)^{\rho+m}$$
$$+\frac{1}{\rho+m}\sum_{i=1}^{m}(-1)^{m+i+1}\binom{m-1}{i-1}(x+ih)^{\rho+m}$$
$$=\frac{1}{\rho+m}\sum_{j=0}^{m}(-1)^{m+j+1}\binom{m}{j}(x+jh)^{\rho+m},$$

and this completes the proof.

LEMMA 3.3. *If* $0 < h < (x-1)/m$, *then there exists a* z *such that* $0 \leqslant z \leqslant mh$ *and*

$$E(x+z) \leqslant \frac{E_m(x,h)}{h^m} + \frac{mh}{2\varphi(k)} - \frac{z}{\varphi(k)}.$$

If $0 < -h < (x-1)/m$, *then there exists a* z *such that* $mh \leqslant z \leqslant 0$ *and*

$$E(x+z) \geqslant \frac{E_m(x,h)}{h^m} + \frac{mh}{2\varphi(k)} - \frac{z}{\varphi(k)}.$$

Proof. Let $G(t) = E(x+t) + t/\varphi(k)$. If $h > 0$, then clearly there exists a z such that $0 \leqslant z \leqslant mh$ and

$$G(z) \leqslant \frac{1}{h^m}\int_0^h \cdots \int_0^h G(y_1 + \cdots + y_m)\, dy_1 \cdots dy_m,$$

and this proves the first part. If $h < 0$, then there exists a z such that $mh \leqslant z \leqslant 0$ and

$$G(z) \geqslant \frac{1}{(-h)^m}\int_h^0 \cdots \int_h^0 G(y_1 + \cdots + y_m)\, dy_1 \cdots dy_m.$$

LEMMA 3.4. *If* $0 < \delta < (x-1)/(mx)$, *then*

$$\frac{\varphi(k)E_m(x,-\delta x)}{(-\delta)^m x^{m+1}} - \frac{m\delta}{2} \leqslant \frac{\varphi(k)}{x}\psi(x;k,l) - 1 \leqslant \frac{\varphi(k)E_m(x,\delta x)}{\delta^m x^{m+1}} + \frac{m\delta}{2}.$$

Proof. In Lemma 3.3 we put $h = \delta x$, and it follows that there exists a $z > 0$ with

$$\psi(x+z;k,l) \leqslant \frac{x}{\varphi(k)} + \frac{E_m(x,\delta x)}{(\delta x)^m} + \frac{m\delta x}{2\varphi(k)},$$

but $\psi(x;k,l) \leqslant \psi(x+z;k,l)$, so that this proves the upper bound. The lower bound is proved with $h = -\delta x$.

This reduces the problem to the estimation of $|E_m(x, \pm\delta x)|$, for which we require a lemma.

LEMMA 3.5. *If* d_1 *and* d_2 *are defined by* (3.4) *and* (3.5), *and* k *is not exceptional, then*

$$|d_1 + d_2| \leqslant \frac{k}{4}\log k + C_3 \log^2 k + C_4 \log k + C_5,$$

where

$$C_3 = 11C_1 + 4, \tag{3.11}$$

$$C_4 = 11C_2 + 2C_1 - 8, \tag{3.12}$$

$$C_5 = C_1 + 2C_2 - 2. \tag{3.13}$$

Proof. From (3.4) and (3.5) we obtain

$$d_1 + d_2 = \frac{-1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) b(\chi);$$

hence

$$|d_1 + d_2| \leqslant \max_{\chi} |b(\chi)|. \tag{3.14}$$

If χ_0 is the principal character modulo k, then

$$\frac{L'}{L}(s, \chi_0) = \frac{\zeta'}{\zeta}(s) + \sum_{p \mid k} \frac{\log p}{p^s - 1},$$

and it follows that

$$b(\chi_0) = \log 2\pi - \frac{1}{2} \sum_{p \mid k} \log p.$$

Hence we have trivially

$$|b(\chi_0)| \leqslant \log 2\pi + \tfrac{1}{2} \log k \leqslant 4 \log^2 k,$$

and the result follows from (2.18).

If χ is nonprincipal, then from (2.1) we obtain

$$b(\chi) = b(\chi_1) - \frac{1}{2} \sum_{\substack{p \mid k \\ \chi_1(p) = 1}} \log p + \sum_{\substack{p \mid k \\ \chi_1(p) \neq 1}} \frac{\chi_1(p) \log p}{1 - \chi_1(p)}.$$

If $\chi_1(p) \neq 1$, note that

$$|1 - \chi_1(p)| \geqslant \left| 1 - \exp\left(\frac{2\pi i}{\varphi(k_1)} \right) \right| \geqslant \frac{4}{k};$$

hence

$$|b(\chi)| \leqslant |b(\chi_1)| + \sum_{p \mid k} \log p \max\left\{ \frac{1}{2}, \frac{k}{4} \right\} \leqslant |b(\chi_1)| + \frac{k}{4} \log k. \tag{3.15}$$

From Davenport [3, p. 85] we have

$$\frac{L'}{L}(s, \chi_1) = \frac{1}{2} \log \frac{\pi}{k_1} - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s + a}{2} \right) + B(\chi_1) + \sum_{\rho \in z(\chi_1)} \left(\frac{1}{s - \rho} + \frac{1}{\rho} \right).$$

If we subtract the same expression with s replaced by 2, we obtain

$$b(\chi_1) = \frac{L'}{L}(2, \chi_1) + a - \sum_{\rho \in z(\chi_1)} \frac{2}{\rho(2 - \rho)},$$

and it follows from (3.15) that

$$|b(\chi)| \leqslant \left| \frac{\zeta'}{\zeta}(2) \right| + 1 + \sum_{\rho \in z(\chi_1)} \frac{2}{|\rho(2 - \rho)|} + \frac{k}{4} \log k. \tag{3.16}$$

It remains to estimate the sum on ρ. If $|\gamma| \leqslant 1$, we use the fact that

$$|\rho(2-\rho)| \geqslant \beta(2-\beta) > \frac{1}{5.5 \log k},$$

since k is not exceptional. It follows from (2.16) that

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| \leqslant 1}} \frac{2}{|\rho(2-\rho)|} < 11 \log k \, N(1) \tag{3.17}$$

$$< 11\left(\frac{1}{\pi} + C_1\right) \log^2 k + 11\left(C_2 - \frac{1}{\pi} \log 2\pi e\right) \log k.$$

If $|\gamma| > 1$, we use the estimate $|\rho(2-\rho)| \geqslant \gamma^2$, integrate by parts, and use (2.16) to obtain

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| > 1}} \frac{2}{|\rho(2-\rho)|} < 2\int_1^\infty \frac{dN(t)}{t^2} < 4\int_1^\infty \frac{N(t)}{t^3} dt$$

$$< \left(\frac{4}{\pi} + 2C_1\right) \log k - \frac{4}{\pi} \log 2\pi + C_1 + 2C_2.$$

The lemma then follows from (3.14), (3.16), (3.17), and the fact that $|\zeta'(2)/\zeta(2)| < .57$.

THEOREM 3.6. *If m is a positive integer, $x > 2$, $0 < \delta < (x-2)/(mx)$, $H \geqslant 1$, and $A_m(\delta) = \delta^{-m} \sum_{j=0}^m \binom{m}{j} (1 + j\delta)^{m+1}$, then*

$$\frac{\varphi(k)}{x}\left|\psi(x; k, l) - \frac{x}{\varphi(k)}\right| < \left(1 + \frac{m\delta}{2}\right) \sum_\chi \sum_{\substack{\rho \in z(\chi) \\ |\gamma| \leqslant H}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2}$$

$$+ A_m(\delta) \sum_\chi \sum_{\substack{\rho \in z(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|} + \varepsilon_1,$$

where

$$\varepsilon_1 = \frac{k}{x}\left[\frac{\log k \log x}{\log 2} + \frac{k}{4} \log k + C_3 \log^2 k + (C_4 + 1) \log k + C_5 + 1\right].$$

Proof. From Lemma 3.2 we obtain

$$|E_m(x, \pm\delta x)| \leqslant \frac{1}{\varphi(k)} \sum_\chi \sum_{\rho \in z(\chi)} \left| \frac{\sum_{j=0}^m \binom{m}{j} (-1)^{m+j+1} (x \pm j\delta x)^{m+\rho}}{\rho(\rho+1)\cdots(\rho+m)} \right| \tag{3.18}$$

$$+ \left| \int_0^{\pm\delta x} \cdots \int_0^{\pm\delta x} f(x + y_1 \cdots + y_m)\, dy_1 \cdots dy_m \right|.$$

For the zeros with $|\gamma| > H$ the summand is bounded in absolute value by

$$\frac{x^{\beta+m}}{|\rho(\rho+1)\cdots(\rho+m)|} \sum_{j=0}^m \binom{m}{j} (1 + j\delta)^{m+1}. \tag{3.19}$$

For the zeros with $|\gamma| \leqslant H$, we write the summand inside the absolute values as

$$\frac{x^{\rho+m}}{\rho} \int_0^{\pm\delta} \cdots \int_0^{\pm\delta} (1 + y_1 + \cdots + y_m)^\rho \, dy_1 \cdots dy_m.$$

The integrand satisfies

$$|(1 + y_1 + \cdots + y_m)^\rho| \leqslant 1 + \sum_{j=1}^{m} |y_j|,$$

so that the absolute value of the summand does not exceed

$$\frac{x^{\beta+m}}{|\rho|}\int_0^\delta \cdots \int_0^\delta (1 + y_1 + \cdots + y_m)\, dy_1 \cdots dy_m = \frac{x^{\beta+m}}{|\rho|}\delta^m\left(1 + \frac{m\delta}{2}\right). \tag{3.20}$$

If $y > 1$, then from (3.10) we obtain

$$|f(y)| \leqslant \frac{1}{2}\sum_{n=1}^{\infty} \frac{y^{-n}}{n} + |d_2|\log y + |d_1 + d_2|,$$

since $a = 0$ for half of the characters and $a = 1$ for the rest. Hence for $|y_i| \leqslant \delta x$ and $0 < \delta < (x - 2)/(mx)$ we have

$$|f(x + y_1 + \cdots + y_m)| < \tfrac{1}{2}\log 2 + |d_2|\log 2x + |d_1 + d_2|.$$

From (3.1) and Lemma 3.5 we obtain

$$\int_0^{\pm\delta x} \cdots \int_0^{\pm\delta x} f(x + y_1 + \cdots + y_m)\, dy_1 \cdots dy_m \tag{3.21}$$
$$< (\delta x)^m\left[\frac{1}{2}\log 2 + \frac{\log k}{\log 2}\log 2x + \frac{k}{4}\log k + C_3\log^2 k + C_4\log k + C_5\right].$$

The result then follows from Lemma 3.4, (3.18), (3.19), (3.20), and (3.21).

In order to simplify the statements of results, we shall use the notation $L = \log k$ and $H = k^\alpha$. As in Rosser and Schoenfeld [10], we use

$$K_n(z, y) = \frac{1}{2}\int_y^\infty u^{n-1}\exp\left[-\frac{z}{2}\left(u + \frac{1}{u}\right)\right] du$$

and also

$$\varphi_n(t) = \frac{x^{-1/R\log kt}}{t^{n+1}}.$$

LEMMA 3.7. *If k is not an exceptional modulus, $k \geqslant 10$, $x \geqslant \exp(\lambda RL^2)$, and $\lambda \geqslant (1 + \alpha)^2$, then*

$$\sum_{\substack{\rho\in z(\chi)\\ |\gamma|\leqslant H}} \frac{x^{\beta-1}}{|\rho|} < \varepsilon_2 + \varepsilon_3 + \varepsilon_4,$$

where

$$\varepsilon_2 = \frac{1}{2}x^{-1/2}\left\{\frac{1 + 4\alpha + \alpha^2}{2\pi}L^2 + \frac{2 + \alpha}{\pi}L + \frac{R(H)}{H} + 2R(1) + C_1\right\} \tag{3.22}$$
$$+ x^{-1}\{kL + \alpha L^2\},$$

$$\varepsilon_3 = \varphi_0(H)R(H), \tag{3.23}$$

and

$$\varepsilon_4 = \frac{\lambda L^2}{2\pi}\left\{\lambda L^2\left[\Gamma\left(-2, \frac{\lambda}{1+\alpha}L\right) - \Gamma(-2, \lambda L)\right]\right. \tag{3.24}$$
$$\left. - \log 2\pi\left[\Gamma\left(-1, \frac{\lambda}{1+\alpha}L\right) - \Gamma(-1, \lambda L)\right]\right\}.$$

Proof. Consider first the contribution from the zeros with $\beta = 0$. These zeros arise from the factors $1 - \chi_1(p)p^{-s}$ in (2.1). Let $N_p(T)$ be the number of zeros of $1 - \chi_1(p)p^{-s}$ in the region $|s| \leqslant T$. An elementary argument yields the estimate

$$N_p(T) \leqslant \frac{T \log p}{\pi} + 2. \tag{3.25}$$

Furthermore for each p there exists at most a single zero $\rho \neq 0$ within $\pi/\log p$ of the origin, and for this zero we have $|\rho| > 2\pi/k_1 \log p$. It follows from (3.25) that

$$\sum_{\substack{\rho \in z(\chi) \\ \beta = 0 \\ |\gamma| \leqslant 1}} \frac{1}{|\rho|} < \sum_{p \mid k} \left[\frac{k_1 \log p}{2\pi} + \frac{\log p}{\pi}\left(\frac{\log p}{\pi} + 2\right)\right] < \frac{kL}{2\pi} + \frac{L^2}{\pi^2} + \frac{2L}{\pi}. \tag{3.26}$$

From (3.25) we obtain

$$\sum_{\substack{\rho \in z(\chi) \\ \beta = 0 \\ 1 < |\gamma| \leqslant H}} \frac{1}{|\rho|} = \sum_{\substack{p \mid k \\ p \nmid k_1}} \int_1^H \frac{dN_p(t)}{t}$$

$$\leqslant \sum_{p \mid k} \left[\frac{N_p(H)}{H} + \int_1^H \frac{N_p(t)}{t^2}\, dt \right] < \frac{1}{\pi}(1 + \log H)L.$$

It follows from (3.26) that

$$\sum_{\substack{\rho \in z(\chi) \\ \beta = 0 \\ |\gamma| \leqslant H}} \frac{1}{|\rho|} < \frac{k}{2\pi} L + \frac{L^2}{\pi^2} + \frac{3L}{\pi} + \frac{L}{\pi} \log H < kL + L \log H, \tag{3.27}$$

since $k \geqslant 10$.

The zeros of $L(s, \chi)$ with $\beta > 0$ are symmetrically located with respect to the line $\sigma = \frac{1}{2}$. Hence

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| \leqslant 1}} \frac{x^{\beta - 1}}{|\rho|} \leqslant \sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| \leqslant 1}} \frac{1}{2}\left(\frac{x^{\beta-1}}{\beta} + \frac{x^{-\beta}}{1 - \beta}\right).$$

By Theorem 1.1 and the fact that k is not exceptional we have $\beta > (\log x)^{-1}$, so that x^β/β is an increasing function of β. It follows that

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| \leqslant 1}} \frac{x^{\beta-1}}{|\rho|} < \frac{1}{2} N(1)\left(\frac{x^{-1/RL}}{1 - 1/RL} + \frac{x^{-1/2}}{\frac{1}{2}}\right) \tag{3.28}$$

$$= \frac{1}{2} N(1) \varphi_0(1) \frac{RL}{RL - 1} + N(1) x^{-1/2}.$$

For the zeros with $1 < |\gamma| \leqslant H$ Theorem 1.1 yields

$$\sum_{\substack{\rho \in z(\chi_1) \\ 1 < |\gamma| \leqslant H}} \frac{x^{\beta-1}}{|\rho|} < \frac{1}{2} \int_1^H \varphi_0(t)\, dN(t) + \frac{1}{2} \int_1^H \frac{x^{-1/2}}{t}\, dN(t). \tag{3.29}$$

In the second integral we integrate by parts and apply (2.16) to obtain

$$(3.30)\quad \int_1^H \frac{dN(t)}{t} < \frac{N(H)}{H} - N(1) + \frac{F(H)\log H}{H} - \frac{1}{2\pi}\log^2 H - \frac{R(H)}{H} + R(1) + C_1\left(1 - \frac{1}{H}\right) < \frac{N(H)}{H} - N(1) + \frac{1}{2\pi}\log^2 H + F(1)\log H + R(1) + C_1.$$

Integration by parts yields

$$(3.31)\quad \frac{1}{2}\int_1^H \varphi_0(t)\,dN(t) = \frac{1}{2}\int_1^H \varphi_0(t)\,dF_1(t) + \frac{1}{2}\int_1^H \varphi_0(t)\,d[N(t) - F_1(t)] = \frac{1}{2}\int_1^H \varphi_0(t)\,dF_1(t) + \frac{1}{2}\varphi_0(H)[N(H) - F_1(H)] - \frac{1}{2}\varphi_0(1)[N(1) - F_1(1)] - \frac{1}{2}\int_1^H [N(t) - F(t)]\varphi_0'(t)\,dt.$$

The condition $\lambda \geqslant (1+\alpha)^2$ implies that $\varphi_0'(t) > 0$ for $1 \leqslant t < H$, and (2.16) yields

$$-\frac{1}{2}\int_1^H [N(t) - F_1(t)]\varphi_0'(t)\,dt < \frac{1}{2}\int_1^H R_1(t)\varphi_0'(t)\,dt < \frac{1}{2}R(H)\int_1^H \varphi_0'(t)\,dt = \frac{1}{2}R(H)[\varphi_0(H) - \varphi_0(1)].$$

It follows from (3.28), (3,29), (3.30), and (3.31) that

$$(3.32)\quad \sum_{\substack{\rho\in z(\chi_1)\\ |\gamma|\leqslant H}} \frac{x^{\beta-1}}{|\rho|} < \frac{1}{2}x^{-1/2}\left\{\frac{N(H)}{H} + N(1) + \frac{1}{2\pi}\log^2 H + F(1)\log H + R(1) + C_1\right\} + \frac{1}{2}\int_1^H \varphi_0(t)\,dF(t) + \varphi_0(H)R(H) + \frac{1}{2}\varphi_0(1)\left[\frac{N(1)}{RL-1} + F(1) - R(H)\right].$$

From (2.16) and the trivial estimate $F(t) < (t\log kt)/\pi$ we obtain

$$(3.33)\quad \frac{N(H)}{H} + N(1) + \frac{1}{2\pi}\log^2 H + F(1)\log H + R(1) + C_1 < \frac{1+4\alpha+\alpha^2}{2\pi}L^2 + \frac{2+\alpha}{\pi}L + \frac{R(H)}{H} + 2R(1) + C_1.$$

Furthermore we have

$$\frac{N(1)}{RL-1} + F(1) - R(H) < F(1)\frac{RL}{RL-1} + \frac{R(1)}{RL-1} - R(1)$$
$$< \frac{RL^2}{\pi(RL-1)} - C_1\frac{RL-2}{RL-1}L < 0,$$

since $L \geqslant \log 10$ and $C_1 > 1/\pi \log 2$. The lemma then follows from (3.27), (3.32), (3.33) and the fact that

$$\frac{1}{2}\int_1^H \varphi_0(t)\, dF(t) = \varepsilon_4. \tag{3.34}$$

LEMMA 3.8. *If* $k \geqslant 10$, $x \geqslant \exp(\lambda RL^2)$, *and* $\lambda \leqslant (m+1)(1+\alpha)^2$, *then*

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|} < \varepsilon_5 + \varepsilon_6 + \varepsilon_7,$$

where

$$\varepsilon_5 = \frac{x^{-1/2}}{2H^{m+1}}\left\{\frac{H}{\pi m}(1+\alpha)L + 2R(H) + \frac{C_1}{m+1}\right\} + \frac{4L}{xH^m}, \tag{3.35}$$

$$\begin{aligned}\varepsilon_6 = k^m L\Bigg\{&\frac{\lambda L}{\pi m}K_2\left(2\sqrt{\lambda m}\,L, (1+\alpha)\sqrt{\frac{m}{\lambda}}\right) \\ &- \sqrt{\frac{\lambda}{m}}\,\frac{\log 2\pi}{\pi}K_1\left(2\sqrt{\lambda m}\,L, (1+\alpha)\sqrt{\frac{m}{\lambda}}\right) \\ &+ C_1 k\sqrt{\frac{\lambda}{m+1}}\,K_1\left(2\sqrt{(m+1)\lambda}\,L, (1+\alpha)\sqrt{\frac{m+1}{\lambda}}\right)\Bigg\}\end{aligned} \tag{3.36}$$

and

$$\varepsilon_7 = R(H)\varphi_m(H). \tag{3.37}$$

Proof. From (3.25) and integration by parts we obtain

$$\begin{aligned}\sum_{\substack{\rho \in z(\chi) \\ \beta = 0 \\ |\gamma| > H}} \frac{1}{|\gamma|^{m+1}} &= \sum_{\substack{p \mid k \\ p \nmid k_1}} \int_H^\infty \frac{dN_p(t)}{t^{m+1}} \\ &\leqslant \sum_{p \mid k} (m+1)\int_H^\infty \frac{N_p(t)}{t^{m+2}}\,dt \leqslant \sum_{p \mid k}\left[\frac{m+1}{\pi m H^m}\log p + \frac{2}{H^{m+1}}\right] \\ &\leqslant \frac{L}{H^m}\left[\frac{m+1}{\pi m} + \frac{2}{H\log 2}\right] \leqslant \frac{4L}{H^m}.\end{aligned} \tag{3.38}$$

For the zeros with $\beta > 0$ we use Theorem 1.1 and the symmetric location of the zeros with respect to the line $\sigma = \frac{1}{2}$ to obtain

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|} < \frac{1}{2}\int_H^\infty \varphi_m(t)\,dN(t) + \frac{1}{2}\int_H^\infty \frac{x^{-1/2}}{t^{m+1}}\,dN(t). \tag{3.39}$$

If we integrate by parts in the second integral and apply (2.16), we find that

$$\begin{aligned}\int_H^\infty \frac{dN(t)}{t^{m+1}} &< \frac{-N(H)}{H^{m+1}} \\ &\quad + \frac{1}{H^{m+1}}\left\{\frac{m+1}{m}F_1(H) + \frac{m+1}{\pi m^2}H + R_1(H) + \frac{C_1}{m+1}\right\} \\ &< \frac{1}{H^{m+1}}\left\{\frac{1}{m}F(H) + \frac{m+1}{\pi m^2}H + 2R(H) + \frac{C_1}{m+1}\right\} \\ &< \frac{1}{H^{m+1}}\left\{\frac{H}{\pi m}\log kH + 2R(H) + \frac{C_1}{m+1}\right\}.\end{aligned} \tag{3.40}$$

For the first integral we proceed as in (3.31) to obtain

$$\begin{aligned}\frac{1}{2}\int_H^\infty \varphi_m(t)\,dN(t) &= \frac{1}{2}\int_H^\infty \varphi_m(t)\,dF_1(t) + \frac{1}{2}\varphi_m(H)[F_1(H) - N(H)] \\ &\quad + \frac{1}{2}\int_H^\infty [F_1(t) - N(t)]\varphi_m'(t)\,dt.\end{aligned}$$

The condition $\lambda \leqslant (m+1)(1+\alpha)^2$ implies that $\varphi_m'(t) < 0$ for $t > H$, so we apply (2.16) and integrate by parts again to obtain

$$\begin{aligned}\frac{1}{2}\int_H^\infty \varphi_m(t)\,dN(t) &< \frac{1}{2}\int_H^\infty \varphi_m(t)\,dF(t) + \varphi_m(H)R(H) \\ &\quad + \frac{C_1}{2}\int_H^\infty \varphi_{m+1}(t)\,dt = \varepsilon_6 + \varepsilon_7.\end{aligned}$$

The lemma then follows from (3.38), (3.39), and (3.40).

THEOREM 3.9. *Let k be a nonexceptional modulus, $(k, l) = 1$, $k \geqslant k_0 \geqslant 10$, m be a positive integer, $0 < \delta < (x-2)/(mx)$, and $x \geqslant \exp(\lambda R L^2)$. Let $(1+\alpha)^2 \leqslant \lambda \leqslant (m+1)(1+\alpha)^2$ and*

$$L > \max\left\{\frac{2+2\alpha}{\lambda - 1 - \alpha} + \frac{\log 2\pi}{1+\alpha}, \frac{2}{\lambda - 1} + \log 2\pi\right\}. \tag{3.41}$$

If $\lambda > m(1+\alpha)^2$, then let

$$L > \frac{2}{2\sqrt{m\lambda} - m - 1} + \frac{\log 2\pi}{1+\alpha}. \tag{3.42}$$

Then

$$\frac{\varphi(k)}{x}\left|\psi(x;k,l)-\frac{x}{\varphi(k)}\right| < \left(1+\frac{m\delta}{2}\right)k_0[\varepsilon_2(k_0)+\varepsilon_3(k_0)+\varepsilon_4(k_0)]+\frac{m\delta}{2} +\varepsilon_1(k_0)+A_m(\delta)k_0[\varepsilon_5(k_0)+\varepsilon_6(k_0)+\varepsilon_7(k_0)].$$

Proof. We may assume that $x=\exp(\lambda RL^2)$, since our upper bound from Theorem 3.6 is decreasing in x. By Theorem 3.6 and Lemmas 3.7 and 3.8 it suffices to prove that (for fixed λ, η, m, and α) $\varepsilon_1(k)$ and $k\varepsilon_i(k)$, $i=2,\dots,7$, are decreasing in k. Of these, the functions ε_1, $k\varepsilon_2$, and $k\varepsilon_5$ are easily shown to be decreasing in L.

It follows from (3.41) that $L\exp[L(1-\lambda/(1+\alpha)-\alpha)]$ is decreasing in L, and this suffices to prove that $k\varepsilon_3$ and $k\varepsilon_7$ are decreasing in L.

From (3.24) we obtain

$$2\pi k\varepsilon_4(k)=\int_1^{1+\alpha} f(L,u)\,du,$$

where

$$f(L,u)=(L^2u-L\log 2\pi)\exp\left[\left(1-\frac{\lambda}{u}\right)L\right].$$

Note that for $1\leqslant u\leqslant 1+\alpha$ we have

$$\begin{aligned}(3.43)\quad L^{-1}\exp\left[\left(\frac{\lambda}{u}-1\right)L\right]\frac{d}{dL}f(L,u) &< (L+2)u+\frac{\lambda\log 2\pi}{u}-\log 2\pi-\lambda L\\ &< \max\left\{(L+2)(1+\alpha)+\frac{\lambda\log 2\pi}{1+\alpha},\ L+2+\lambda\log 2\pi\right\}\\ &\quad -\log 2\pi-\lambda L<0,\end{aligned}$$

by (3.41). Hence $k\varepsilon_4$ is decreasing in L.

From (3.36) we obtain

$$(3.44)\qquad 2k\varepsilon_6(k)=\frac{1}{\pi}\int_{1+\alpha}^{\infty} g_1(L,u)\,du+C_1\int_{1+\alpha}^{\infty} g_2(L,u)\,du,$$

where

$$g_1(L,u)=(L^2u-L\log 2\pi)\exp\left[L\left(m+1-mu-\frac{\lambda}{u}\right)\right],$$

$$g_2(L,u)=L\exp\left\{L\left[m+2-(m+1)u-\frac{\lambda}{u}\right]\right\}.$$

The first integrand satisfies

$$\frac{d}{dL}g_1(L,u)<L\left\{(Lu-\log 2\pi)\left(m+1-mu-\frac{\lambda}{u}\right)+2u\right\} \times\exp\left[L\left(m+1-mu-\frac{\lambda}{u}\right)\right]<0,$$

provided

$$L > \frac{2}{\lambda/u + mu - m - 1} + \frac{\log 2\pi}{1 + \alpha}, \qquad u \geqslant 1 + \alpha.$$

If $1 + \alpha < \sqrt{\lambda/m}$, this condition is met by (3.42) and if $1 + \alpha \geqslant \sqrt{\lambda/m}$, then it follows from (3.41).

The second integral in (3.44) can similarly be shown to be decreasing provided $L > (\lambda/(1 + \alpha) + (m + 1)\alpha - 1)^{-1}$.

4. Computations. In this section we describe the methods used in the preparation of Table 1. Note that Theorem 1.2 gives estimates for $\theta(x; k, l)$ as well as $\psi(x; k, l)$. By a result of Schoenfeld [11], we have

$$\begin{aligned} (4.1) \qquad 0 &\leqslant \psi(x; k, l) - \theta(x; k, l) \leqslant \psi(x; 1, 1) - \theta(x; 1, 1) \\ &< 1.001093x^{1/2} + 3x^{1/3}. \end{aligned}$$

Hence we obtain the estimate

$$\left|\theta(x; k, l) - \frac{x}{\varphi(k)}\right| < 1.001093x^{1/2} + 3x^{1/3} + \left|\psi(x; k, l) - \frac{x}{\varphi(k)}\right|,$$

and the extra terms are negligible for the range of x under consideration.

Estimates for the incomplete gamma function and incomplete Bessel functions may be found in [11] and [10]. Upper bounds for $K_\nu(z, x)$ are provided by Lemma 4, Lemma 5, (2.30), and (2.31) of [10]. In addition, if $x < 1$, we can use Lemma 3 combined with Lemma 4 and the asymptotic expansions of $K_\nu(z)$ (9.7.2 of [1]). A lower bound for $K_1(z, x)$ is provided by Lemma 4 or (2.22) and (2.33) of [10], resulting in the estimate

$$\begin{aligned} K_1(z, x) > \frac{e^{-z}}{2z}\Bigg\{&\left[1 + \frac{\sqrt{2}}{8}y\left(3 - y^2 - \frac{3}{2z}\right)\right]e^{-zy^2} \\ &+ \left(z + \frac{3}{8} - \frac{3}{16z}\right)\sqrt{2}\int_y^\infty e^{-zw^2}\,dw\Bigg\}. \end{aligned}$$

If $x < 1$, another method for bounding $K_1(z, x)$ from below is to use (2.10) of [10] and 9.7.2 of [1]. Other methods for estimating $K_\nu(z, x)$ are available in [10] and [12], but in the interests of simplifying the computations these were not used in the preparation of Table 1.

The choices of the parameters m, η, α, and δ are completely at our disposal. We used $m = 2$ since it seemed to give the best results. Tables 2 and 3 give the values of η and α used in the preparation of Table 1. The best values of α turn out to be only slightly less than $\sqrt{\lambda} - 1$, and the choice $\alpha = \sqrt{\lambda} - 1$ would lead to results that are nearly as good. The major effect of η is to control the size of ε_3 and ε_7. For this reason, and the fact that the best α is near $\sqrt{\lambda} - 1$, we chose η to minimize $R(k^{\sqrt{\lambda}-1})$.

This leaves only δ to be chosen. For $m = 2$, the optimal δ is approximately that which minimizes

$$\delta(1 + w_1 + 10w_2) + (4\delta^{-2} + 12\delta^{-1} + 18)w_2,$$

where

$$w_1 = k\varepsilon_2 + k\varepsilon_3 + k\varepsilon_4, \qquad w_2 = k\varepsilon_5 + k\varepsilon_6 + k\varepsilon_7.$$

We can then find δ by elementary calculus. If $1 + w_1 < 102w_2$, a minimum exists at the positive real root of $\delta^3 - a\delta - 2a/3$, where $a = 12w_2/(1 + w_1 + 10w_2)$. This leads to the choice

$$\delta = D + a/(3D),$$

where

$$D = \left[\frac{a}{3}\left(1 + \sqrt{1 - \frac{a}{3}}\right)\right]^{1/3}.$$

All computations were performed on the CDC Cyber Computer at Michigan State University, using double precision Fortran (approximately 28 significant decimal digits). We have listed in Table 1 only values of c for which we were able to find appropriate values of η and α, but Theorem 3.9 may actually yield slightly smaller values of c.

TABLE 2

η

b \ ε	1	.5	.2	.1	.05	.01	.005	.001	.0001	.00001
1	.500	.500	.500	.500	.500	.500	.500	.436	.369	.321
2	.500	.500	.500	.500	.497	.435	.413	.369	.321	.285
3	.495	.476	.451	.430	.410	.369	.352	.321	.285	.255
4	.411	.398	.379	.365	.351	.320	.308	.285	.255	.233
5	.352	.342	.329	.318	.307	.284	.275	.255	.233	.213
6	.308	.301	.291	.283	.274	.255	.249	.233	.213	.198
7	.275	.269	.261	.254	.248	.233	.226	.213	.198	.183
8	.249	.244	.237	.232	.226	.213	.208	.198	.183	.171
9	.227	.223	.217	.212	.208	.197	.193	.183	.171	.162
10	.208	.206	.201	.197	.193	.183	.179	.171	.162	.152
11	.194	.191	.186	.183	.179	.171	.168	.162	.152	.144
12	.180	.177	.174	.171	.168	.161	.159	.152	.144	.136
13	.168	.166	.164	.161	.158	.152	.149	.144	.136	.129
14	.159	.157	.154	.152	.149	.144	.141	.136	.129	.124
15	.150	.148	.145	.143	.141	.136	.134	.129	.124	.119
20	.117	.116	.115	.113	.112	.109	.108	.105	.101	.097
25	.096	.095	.094	.093	.092	.090	.089	.087	.085	.083
30	.082	.082	.081	.080	.080	.078	.077	.076	.074	.072
35	.071	.071	.070	.070	.069	.068	.068	.067	.065	.063
40	.063	.063	.062	.062	.061	.061	.060	.059	.058	.057
45	.056	.056	.056	.055	.055	.054	.054	.053	.052	.051
50	.051	.051	.050	.050	.050	.049	.049	.048	.048	.047
60	.043	.043	.042	.042	.042	.042	.041	.041	.041	.041
70	.038	.038	.038	.037	.037	.037	.037	.036	.036	.036
80	.033	.033	.033	.033	.033	.033	.033	.032	.032	.032
90	.030	.030	.030	.030	.029	.029	.029	.029	.029	.028
100	.027	.027	.027	.027	.027	.026	.026	.026	.026	.026

TABLE 3

α

b \ c	1	.5	.2	.1	.05	.01	.005	.001	.0001	.00001
1	.879	1.06	1.345	1.599	1.873	2.564	2.879	3.584	4.584	5.584
2	.461	.552	.700	.829	.969	1.296	1.451	1.791	2.281	2.786
3	.319	.383	.484	.564	.647	.851	.953	1.187	1.524	1.858
4	.246	.294	.369	.434	.480	.644	.717	.892	1.143	1.394
5	.202	.242	.298	.338	.391	.522	.579	.715	.915	1.114
6	.170	.207	.253	.286	.331	.431	.488	.599	.762	.932
7	.149	.177	.214	.233	.287	.380	.412	.516	.653	.794
8	.134	.153	.177	.207	.238	.323	.358	.445	.574	.694
9	.123	.129	.160	.186	.213	.281	.332	.406	.512	.617
10	.106	.118	.146	.169	.195	.254	.281	.367	.463	.557
11	.098	.116	.134	.154	.178	.232	.264	.335	.421	.506
12	.085	.101	.124	.143	.164	.215	.236	.288	.388	.466
13	.085	.096	.115	.133	.153	.199	.219	.275	.360	.432
14	.075	.089	.108	.125	.143	.186	.204	.249	.326	.402
15	.070	.083	.101	.117	.134	.174	.191	.232	.302	.375
20	.048	.064	.067	.091	.089	.133	.147	.177	.228	.285
25	.045	.046	.055	.063	.072	.108	.119	.143	.178	.228
30	.033	.039	.047	.054	.061	.091	.099	.120	.150	.185
35	.030	.033	.041	.047	.053	.072	.082	.104	.129	.154
40	.026	.030	.036	.042	.047	.063	.076	.091	.113	.135
45	.024	.027	.033	.037	.042	.062	.058	.081	.101	.120
50	.022	.025	.030	.034	.038	.048	.058	.074	.091	.109
60	.018	.021	.025	.029	.033	.041	.045	.062	.077	.091
70	.016	.018	.021	.025	.028	.035	.039	.046	.066	.078
80	.015	.016	.020	.021	.024	.031	.034	.047	.057	.069
90	.013	.015	.017	.020	.021	.028	.032	.042	.052	.062
100	.012	.013	.015	.018	.020	.025	.027	.033	.047	.056

The conditions (3.41) and (3.42) fail to hold for several entries of Table 1, and this required a check of all values of k up to a point where (3.41) and (3.42) were in effect.

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