REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the indexing system printed in Volume 22, Number 101, January 1968, page 212.

41[2, 2.05, 2.10, 2.20, 3, 4, 7, 8].—*Procedures ALGOL en Analyse Numérique,* Éditions du Centre National de la Recherche Scientifique, Paris, 1967, 324 pp., 24 cm. Price 35 F.

This book is the combined effort of several university numerical analysis teams working in cooperation with the French National Centre of Scientific Research. The book is essentially a collection of Algol procedures arranged by subject in seven chapters. At the beginning of each chapter is a brief description of each procedure in the chapter. Unfortunately, few reasons for preferring one particular algorithm to another are supplied. The actual description of each Algol procedure consists of a brief but clear account of the method, a listing of the actual Algol procedure, and a listing of a driver program showing the use of the procedure on a numerical example. This makes the use of these procedures an easy task. Evaluation of the numerical results of these examples is difficult since a description of the computers used (word length, arithmetic unit etc.) is supplied in only a few cases. There is no indication that the procedures have undergone any formal certification.

Chapter one, edited by C. Bonnemoy, is concerned with the solution of linear systems of equations. The more important procedures include Gaussian elimination (with and without partial pivoting), determinant evaluation, Cholesky decomposition, solution of tridiagonal systems, least squares solution of linear systems, and the computation of the pseudo-inverse of a square matrix.

The second chapter, edited by J. L. Rigal, deals with the problem of finding the eigenvalues and eigenvectors of matrices. A number of methods are presented for both the symmetric and unsymmetric cases. For the symmetric case, the methods of Jacobi, Givens, and Householder are given, as well as a number of methods for finding the eigenvalues of the tridiagonal systems that result from the latter two methods. Methods of Wilkinson and Householder for reducing a general matrix to lower Hessenberg form are given. Rutishauser's LR method is presented along with some variants of the ordinary power method.

Solution of nonlinear algebraic equations is the subject of chapter three, edited by J. L. Lagouanelle. The procedures given use the well-known methods of Lin, Bairstow, Muller, Newton, and Laguerre for polynomial equations, and an ordinary bisection algorithm for finding a zero of a general function of one variable. Two procedures for solving systems of nonlinear equations are provided, a special one for two equations in two unknowns, and a general procedure for n equations in n unknowns which uses Newton's method.

Chapter four, edited by P. Pouzet, deals with differential, integral, and integro-differential equations. The procedures given are primarily Runge-Kutta methods which differ in details depending on the type of equation being solved. One multistep method (Adams) and a mixed method are given.

Edited by J. Legras, chapter five is concerned with numerical quadrature. Here some standard procedures using Newton-Cotes formulas of degrees one, three, and five are presented for use with functions of one and two variables. Procedures utilizing Romberg integration for rectangular regions in one, two and three dimensions are given.

Approximation is the subject of chapter six, under P. J. Laurent's editorship. It consists of two parts, the first dealing with approximation using the infinity norm, and the second dealing with approximation in the least squares sense. In the first section two versions of Remez' algorithm are provided as well as several procedures for finding uniform approximations on discrete sets of points. Procedures for obtaining least squares approximations (a direct method and one using orthogonal polynomials) appear. A procedure for generating approximations using spline functions ends the chapter.

The last chapter, edited by P. L. Hennequin, is entitled "Probability and Special Functions" and, as the editor states, deals with algorithms which could not logically be placed elsewhere in the book. They include one related to the decomposition of the set of states in a stationary Markov process into classes, an optimized Runge-Kutta procedure, two random number generators, a procedure to find the upper limit of integration of a Gaussian distribution when the cumulative probability is known, and two algorithms related to Mathieu functions.

Although the book contains a large number of very useful procedures, the reviewer feels that it should be approached with caution by the inexperienced computer. A few of the algorithms presented are of questionable reliability, and some of the limitations of the algorithms are not stated. For example, the inversion of a matrix using Schmidt orthogonalization can lead to severe round-off error and it is not generally regarded as a satisfactory numerical procedure. (By contrast, the same basic procedure using Householder transformation matrices to carry out the factorization enjoys good numerical stability.) Repeated use of Hotelling's deflation with the power method is suggested for finding all the eigenvalues of a given matrix when the eigenvalues are known to be distinct. It is often satisfactory only for a few iterations.

Despite these reservations, the authors deserve a great deal of credit for gathering together such a comprehensive set of algorithms. The procedures in the book would enable one to attack the majority of standard numerical computation problems. There are a few misprints in the book and sources for most of the methods are supplied.

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42[2.10, 7].—David Galant, *Gauss Quadrature Rules for the Evaluation of* $2\pi^{-1/2}\int_0^\infty \exp(-x^2)f(x)dx$, 6 pages of tables, reproduced on the microfiche card attached to this issue.

With $\{p_j(x)\}$ denoting the orthogonal polynomials associated with the weight function $\exp(-x^2)$ on $[0, \infty)$, the coefficients $\{b_j\}$ and $\{g_j\}$ in the recurrence relation $p_j(x) = (x - b_j)p_{j-1}(x) - g_j p_{j-2}(x)$ are given in Table I to 20S for $j = 1(1)20$.

Table II contains 20S values of the nodes $t_{j,n}$ and weights $w_{j,n}$ of the Gaussian quadrature rules $2\pi^{-1/2} \int_0^\infty \exp(-x^2)f(x)dx \doteq \sum_{j=1}^n w_{j,n}f(t_{j,n})$, $n = 1(1)20$. The recurrence coefficients were computed in 50S arithmetic from the moments of $\exp(-x^2)$ by means of the quotient-difference algorithm. The Gaussian nodes and weights were calculated in 30S arithmetic, using methods of Golub and Welsch (Gene H. Golub and John H. Welsch, "Calculation of Gauss quadrature rules," *Math. Comp.*, v. 23, 1969, pp. 221–230).

W. G.

43[2.20, 2.45, 5, 9, 11, 12, 13.20, 13.35, 13.50].—J. T. Schwartz, Editor, *Mathematical Aspects of Computer Science*, Vol. 19, Proc. Sympos. Appl. Math., Amer. Math. Soc., Providence, R. I., 1967, v + 224 pp. Price $6.80.

This volume contains research and expository papers on computer science and its mathematical facets. The eleven contributions will now be briefly treated in turn.

"A review of automatic theorem proving": J. A. Robinson surveys the methods and results and observes that under very general conditions a theorem-proving problem can be solved automatically if it can be solved at all. This expository paper, which is of particular interest to logicians and computer scientists, records that some theorem-proving problems considered unfeasible five years ago, can now be treated on a computer with relative ease although other problems involving set-theoretic notions have computationally inefficient solutions.

"Assigning meanings to programs": Robert W. Floyd offers a basis for formal definitions of the meanings of computer programs by defining programming languages in a way so that a rigorous standard is established for proofs about computer programs, including proofs of corrections, equivalence, and termination. This research paper is of interest principally to computer scientists who have followed the work of J. McCarthy, A. Perlis, and S. Gorn.

"Correctness of a compiler for arithmetic expressions": John McCarthy and James Painter give a proof of the correctness of a simple algorithm for compiling arithmetic expressions into machine language. Their ultimate goal is to make it possible to use a computer to check proofs that compilers are correct. This research paper is the first one in which the correctness of a compiler is proved.

"Context-free languages and Turing machine computations": J. Hartmanis derives a result that establishes a close tie between complements or intersections of context-free languages and Turing machine computations. In addition he gives some new results about the complements, intersections, and quotients of context-free languages. This expository and research paper is of interest mainly to computer scientists active in automata theory.

"Computer analysis of natural languages": Susumu Kuno surveys a portion of the field of computational linguistics. In fact, his survey is confined only to algebraic (i.e., nonstatistical) studies concerning the syntax and semantics of natural languages. The reviewer agrees with the author that such studies are more prevalent, but regards his statement that they are also more interesting as being a matter of personal taste. Most of this thorough expository paper is devoted to a detailed analysis of parsing algorithms for generative and transformational grammars of N. Chomsky.

"The use of computers in the theory of numbers": P. Swinnerton-Dyer presents a brief treatment of a topic in number theory in which the theory appears to be incomplete (i.e., there are yet undiscovered relations between the existing concepts) and in which a computer can be used to accumulate facts in the hope that a pattern will emerge. The particular problem is to find rational solutions of inhomogeneous cubic equations with rational coefficients and to study the set of its rational solutions.

"A machine calculation of a spectral sequence": M. E. Mahowald and M. D. MacLaren study Stiefel manifolds with a view to uncovering some internal structure by means of machine computation. After a brief description of the topology of the problem they discuss some of the details of the computations, which were performed on a CDC 3600.

"Numerical hydrodynamics of the atmosphere": C. E. Leith describes a numerical model for the long-term prediction of weather. The partial differential operators of the continuous model are approximated by finite-difference equations, which reduce the integration of the evolution equations to a numerical process. However, no computer calculations are presented.

"The calculation of zeros of polynomials and analytic functions": J. F. Traub studies a class of new methods for the calculation of zeros. Continuing his earlier work, he gives a simplified treatment of the case of a polynomial with distinct zeros and one zero of largest modulus. In other sections he treats the case of a zero of smallest modulus, the calculation of multiple zeros and equimodular dominant zeros of polynomials, and zeros of analytic functions. This research paper is of particular interest to numerical analysts.

"Mathematical theory of automata": Michael O. Rabin surveys the major developments and trends in the theory of finite automata. The treatment is very complete since it covers the theories of nonprobabilistic finite-automata, probabilistic finite-automata, and finite tree-automata, whose foundations were largely established by the author. He appends a list of interesting problems for further research.

"Linearly unrecognizable patterns": Marvin Minsky and Seymour Papert study the classification of certain geometrical properties according to the type of computation necessary to determine whether a given figure has them. This lengthy research paper treats local vs. global geometric properties, series vs. parallel computation, and the theory of perceptrons.

In summary, this volume has bits of knowledge from many different branches of mathematics, with the computer or computation being the common thread. All authors are recognized in their field and their lists of references are generally excellent. Only a few minor typographical errors were noted, which the reader can easily detect and correct.

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44[4, 5, 6].—Francis B. Hildebrand, *Finite-Difference Equations and Simulations*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968, ix + 338 pp., 23 cm. Price $12.75.

This is an introductory text, in three chapters, dealing with the following topics: (i) Calculus of finite differences and difference equations, (ii) Numerical solution of ordinary differential equations, (iii) Numerical solution of partial differential equations. The term "simulations" in the title is to be understood in the restricted sense of simulating differential equations by finite-difference equations. An attempt has been made to incorporate recent advances in this field, particularly concerning the theory of error propagation and stability. Each chapter is followed by a short list of references and an extensive set of problems.

The text provides, at a modest level, a well-motivated introduction to the approximate solution of differential equations, and should serve well to prepare the student for a study of more specialized treatises on the subject.

W. G.

45[7, 9].—W. A. Beyer, N. Metropolis & J. R. Neergaard, *Square Roots of Integers 2 to 15 in Various Bases 2 to 10: 88062 Binary Digits or Equivalent*, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, December 1968. Plastic-bound computer printout, 277 pages, deposited in the UMT file.

The first ten tables here list $\sqrt{n}$ for $n = 2, 3, 5, 6, 7, 10, 11, 13, 14$, and 15 to 29354 octal digits.

The next five tables list $\sqrt{2}$ to the bases 3, 5, 6, 7, and 10 to the equivalent accuracies: 55296, 36864, 32768, 30720, and 24576 digits, respectively. The last ten tables give the corresponding data for $\sqrt{3}$ and $\sqrt{5}$. Thus, starting on page 237, we find

$$(\sqrt{5})_5 = 2.1042234\cdots.$$

The purpose of the authors to test the normality of these irrationals to different bases; their results and conclusions will appear elsewhere. For recent reviews on related matters, see [1], [2], [3] and the references cited there.

The three decimal numbers were compared with the slightly less accurate values in [2] in the vicinity of 22900D. No discrepancy was found. No details were given concerning programs or computer times, nor any explanation for the coincidence (?) that the number of digits to the base 6 turns out to be exactly 2^{15}.

D. S.

1. *Math. Comp.*, v. 21, 1967, pp. 258–259, UMT **17.**
2. *Math. Comp.*, v. 22, 1968, p. 234, UMT **22.**
3. *Math. Comp.*, v. 22, 1968, pp. 899–900, UMT **86.**

46[7, 9].—Daniel Shanks & John W. Wrench, Jr., *Calculation of e to 100,000 Decimals*, 1961. Computer printout deposited in the UMT file.

This calculation of e was performed seven years ago at the time that π was computed to the same accuracy [1]. In contrast to the latter computation, the programming for e had no special interest, inasmuch as it was based upon the obvious procedure of summing the reciprocals of successive factorials, and consequently it was dismissed in a footnote to [1].

Since a number of requests for copies of this approximation to e have been received, we accordingly deposit here two copies: the first, a full-size, 20-page, computer printout; the second, a photographic reduction thereof.

At the time of the computation, cumulative decimal-digit counts for $D = 10^3(10^3)10^5$ were tabulated, and nothing unexpected was observed. The final counts for $e - 2$ and $\pi - 3$ are as follows.

	0	1	2	3	4
e	9885	10264	9855	10035	10039
π	9999	10137	9908	10025	9971

	5	6	7	8	9
e	10034	10183	9875	9967	9863
π	10026	10029	10025	9978	9902

AUTHORS' SUMMARY

1. DANIEL SHANKS & JOHN W. WRENCH, JR., "Calculation of π to 100,000 decimals," *Math. Comp.*, v. 16, 1962, pp. 76–99.

47[7].—FREDERIC B. FULLER, *Tables for Continuously Iterating the Exponential and Logarithm*, ms. of 30 typewritten pages, 29 cm. Deposited in UMT file.

The theory of the continuous iteration of real functions of a real variable has been presented by a number of writers, including Bennett [1], Ward [2], and the present author [3].

The unique tables under review give 6D values of the continuously iterated function $F(x)$ and its inverse $G(x)$ for $x = 0(0.001)1$, with first differences, and for $x = 1(0.1)3$, without differences. Here $F(x)$ represents the exponential of zero iterated x times. Typical values for integral values of x are $F(0) = 0$, $F(1) = 1$, $F(2) = e$, and $F(3) = e^e$.

An introduction of five pages provides details of the procedures followed in the calculation of these tables. Appended notes explain how the tables can be extended in both directions with respect to the argument and include a discussion of the effect of the F operator on the number system of algebra.

It seems appropriate to mention here a similar study of Zavrotsky [4], which, however, led to radically different tables.

J. W. W.

1. A. A. BENNETT, "Note on an operation of the third grade," *Ann. of Math.*, v. 17, 1915–1916, pp. 74–75.
2. MORGAN WARD, "Note on the iteration of functions of one variable," *Bull. Amer. Math. Soc.*, v. 40, 1934, pp. 688–690.
3. MORGAN WARD & F. B. FULLER, "The continuous iteration of real functions," *Bull. Amer. Math. Soc.*, v. 42, 1936, pp. 393–396.
4. A. ZAVROTSKY, "Construccion de una escala continua de las operaciones aritmeticas," *Revista Ciencia e Ingeniería de la Facultad de Ingeniería de la Universidad de los Andes*, Mérida, Venezuela, December 1960, No. 7, pp. 38–53. (See *Math. Comp.*, v. 15, 1961, pp. 299–300, RMT **63**.)

48[8].—JOHN R. WOLBERG, *Prediction Analysis*, D. Van Nostrand Co., Princeton, N. J., 1967, xi + 291 pp., 24 cm. Price $10.75.

The author develops a general statistical approach, called prediction analysis, for designing experiments to estimate the structural parameters in functional relationships and regression surfaces by the general method of least squares. Structural parameters are the constants that link the true values of the independent and dependent variables. Prediction analysis considers such design problems as how to choose sample size and values of the independent variables to keep the predicted standard errors of the parameters less than a given upper bound. Treatment of these design problems in book form for scientists and engineers is long overdue. Mathematical prerequisites are elementary calculus and some exposure to matrix theory.

After a short introductory chapter defining the scope of prediction analysis, the statistical concepts essential for the remainder of the text are reviewed in chapter 2. Then, the method of least squares is developed in chapter 3 in sufficient generality to fit general nonlinear relationships to experimental data with measurement errors in the independent variables. Chapter 4 describes prediction analysis and its implementation on a computer. The author shows, for a proposed experiment, how to obtain the predicted variances for the least squares estimators of the structural parameters in a functional relationship when the measurement error variances and the form of the functional relationships are known and some knowledge is available about the structural parameters. Such predicted variances are the starting point from which an investigator would choose an experimental design. Examples to illustrate the usefulness of applying prediction analysis to the design of experiments to fit the polynomial, exponential, sine series and Gaussian function comprise chapters 5–8, respectively. Each chapter in the text concludes with a summary of the important topics, problems for solution and computer projects.

The author is only partially successful in carrying out his objective to provide a useful reference work for experimenters to design and/or analyze experiments to estimate parameters in functional relationships. Several reasons account for this assessment; only the most important difficulties will be outlined here. First, the iterative least squares procedure described in chapter 3, pages 39–45 to minimize the sum of squares in Equation 3.3.7 will not in general produce either consistent or minimum variance estimators of the structural parameters, contrary to the author's statements on pages 30, 31. The problem lies both with the quantity that is minimized and with the iterative procedure itself. Minimization of the defined sum of squares does not give the "best" estimators of the structural parameters when the measurement errors are not Gaussian. On the other hand, if the measurement errors have known variances and have a Gaussian distribution, minimization of the defined sum of squares will give the "best" estimators asymptotically; however, the iterative procedure will not yield such "best" estimators in general. In either situation when nonnegligible measurement error exists in the independent variables, the iterative procedure yields inconsistent estimators for at least some of the structural parameters in functional relationships nonlinear in the independent variables or the structural parameters. This difficulty with the iterative method means that the author's examples in chapters 5–8 need to be reworked to take the bias terms into account when they arise.

In addition to this first criticism, other important difficulties mar the text. Some remarks should have been made about minimization methods that supplement or are alternative to the iterative procedure described by the author. Next, if the distributions of the measurement errors are not Gaussian, then the estimated standard errors are not unbiased estimators of the population standard errors and the confidence intervals given in the text may produce misleading inferences. No discussion of closely related work on the design of experiments to estimate functional or regression relationships by Hoel, Kiefer and Wolfowitz and others is given or referenced. Finally, prediction analysis may be easily reformulated so that it is not necessarily based on the iterative least squares method and can make use of partial knowledge about the structural parameter and error variances in a formal way. Such a reformulation is possible through Bayesian decision theory.

To sum up, this text is a contribution to the statistical design and analysis of experiments to estimate the structural parameters in functional relationship and regression surfaces of known form. The detailed techniques developed here are reasonable to use in choosing the number of observations to take on the dependent variable and in estimating the structural parameters when the values of the independent variables are chosen in advance, the independent variables are measured without error, the variance of the measurement error for the dependent variable is known, some knowledge is available about the structural parameters and the probability distribution of the measurement error is "approximately" Gaussian. When one or more of these assumptions is not true, use of the iterative least squares method needs to be justified in each particular application.

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49[9].—M. Lal & P. Gillard, *Table of Euler's Phi Function, n < 10⁵*, Memorial University of Newfoundland, St. John's, Newfoundland, October 1968, 200 pp., paperbound. Deposited in the UMT file.

The number-theoretic function $\phi(n)$ is listed for $n = 0(1)99999$, 500 values per page. If

$$n = \prod_i p_i^{a_i},$$

then

$$\phi(n) = n \prod_i \frac{p_i - 1}{p_i}, \tag{1}$$

and the function was computed here by (1).

Earlier well-known tables of $\phi(n)$ were by J. J. Sylvester [1] (to $n = 10^3$) and J. W. L. Glaisher [2] (to $n = 10^4$). Both of these earlier authors seemed primarily interested not in $\phi(n)$, as such, but rather in $\sum \phi(n)$ and in the inverse of $\phi(n)$. In the present case, the interest seems to be in finding solutions of

$$\phi(n) = \phi(n + 1)$$

and similar functional equations.

D. H. Lehmer points out [3] that computation of a table of $\phi(n)$ usually has some indirect purpose inasmuch as any desired individual value of $\phi(n)$ can be rather easily obtained. See [3] for further discussion.

D. S.

1. J. J. Sylvester, "On the number of fractions contained in any Farey series . . . ," *Philos. Mag.*, v. 15, 1883, pp. 251–257.
2. J. W. L. Glaisher, *Number-Divisor Tables*, British Association Mathematical Tables, v. 8, Cambridge, 1940, Table I.
3. D. H. Lehmer, *Guide to Tables in the Theory of Numbers*, National Acad. of Sciences, Washington, D. C., 1941, pp. 6–7.

50[9].—Morris Newman, *Table of the Class Number $h(-p)$ for p Prime, $p \equiv 3 \pmod 4$, $101987 \leqq p \leqq 166807$*, National Bureau of Standards, 1969, 49 pages of computer output deposited in the UMT file.

This is an extension of Ordman's tables [1] previously deposited and reviewed. Those tables were computed because the undersigned wished to examine all cases of $h(-p) = 25$; this extension to $p = 166807$ was computed because (you guessed it) he wished to examine all cases of $h(-p) = 27$.

Unlike Ordman's tables, all $p = 4n + 3$ are listed consecutively here; those of the forms $8n + 3$ and $8n + 7$ are not listed separately.

We may now extend the table in our previous review of the first and last examples of a given odd class number:

h	$8n + 3$		$8n + 7$	
27	3299	103387	983	11383
29	2939	166147	887	8863
31	3251	133387	719	13687

For $p = 8n + 7$ our table here could be much extended, but not for $p = 8n + 3$, since there are known $p = 8n + 3 > 166807$ with $h(-p) = 33$.

D. S.

1. Edward T. Ordman, *Tables of the Class Number for Negative Prime Discriminants*, UMT 29, *Math. Comp.*, v. 23, 1969, p. 458.

51[9].—A. E. Western & J. C. P. Miller, *Indices and Primitive Roots*, Royal Society Mathematical Tables, Vol. 9, University Press, Cambridge, 1968, liv + 385 pp., 29 cm. Price $18.50.

To describe fully what is in this volume would be a long task; we therefore abbreviate somewhat. Let P be prime and let

$$P - 1 = \prod_i q_i^{\alpha_i} \tag{1}$$

be the factorization of $P - 1$ into prime-powers. If ξ is the smallest positive exponent such that

$$y^\xi \equiv 1 \pmod P$$

and $\nu = (P - 1)/\xi$, then ν is the *residue-index* of y (mod P). If $\nu = 1$, y is a *primitive root* of P. Let

$$(2) \qquad g, -g', h$$

be, respectively, the least positive, least negative, and least positive prime primitive roots of P. For any primitive root G and

$$y \equiv G^m \qquad (\text{mod } P)$$

we say m is the *index* of y to the base G (mod P) and write it

$$(3) \qquad m = \text{ind}_G\,(y)\,.$$

Let p_n be the nth prime: $p_1 = 2$, $p_2 = 3$, $\cdots$.

Choose $G = g$ unless $g > g'$, in which case choose $G = -g'$. The main Table 1 (216 pages) lists for each odd prime $P \leqq 50021$, the factorization (1), the data (2), and, for $y = p_1, \cdots, p_{12} = 37$, their indices (3), and residue-indices, ν. The arguments $y = 6$, 10, and 12 are also given.

Table 2 continues with only $P \equiv 1$ (mod 24) to $P < 10^5$. Table 3 continues with only $P \equiv 1, 49$ (mod 120) to $P < \frac{1}{4}\cdot 10^6$. The arguments 6, 10, and 12 are dropped here. Table 4 continues with only $P \equiv 1$ (mod 120) to $P < 10^6$. The arguments 6, 10, and 12 are replaced by $y = p_{13}, p_{14}, p_{15} = 47$. The brief Table 5 gives any further $y = p_i$ up to $h = p_n$ for any previous P such that $n > 12$ or 15, respectively.

The 45-page Introduction is very interesting. It takes up many topics, including the long, intricate history of these tables, going back at least 50 or 60 years. The ingenious, but involved techniques for computing g and the other data were developed by Cunningham, Woodall, and Western, and, of course, preceded modern high-speed machines. We forego a description of these methods ourselves, but recommend that the reader study this part of the Introduction.

Tables 6 and 7 are related to this method, and the first is of interest in its own right. One defines A_n numbers (they are much used in the foregoing techniques) as those divisible by no prime $> p_n$. Table 6 gives the number of such $A_n \leqq N$ for $n = 1(1)51$—the last representing $p_{51} = 233$ — and from $N = 10^3$ to 10^8 by varying increments.

The uses of the main tables are discussed, together with examples. In spite of the limited number of arguments y listed, usually (3) can be found fairly quickly for any y. If y is an A_{12} (or A_{15}) number, one has its index merely by addition. Consider ind 97 (mod 4933). Since

$$97\cdot 51 \equiv 14 \qquad (\text{mod } 4933)\,,$$

one has ind 97 = ind 2 + ind 7 − ind 3 − ind 17. That is, one expresses y in terms of A_n numbers. To compute y from ind y may be longer, and a desk computer is desirable.

Other minor tables of interest in their own right, and related to these methods, are given in the Introduction. Thus, $P_a(n)$ is the smallest prime P such that every $p_i \leqq p_n$ has a ν divisible by a. Tables are given for a up to 17, and varying n up to 16. In connection with $a = 5$, a long discussion of quintic residues is given by Western.

Various conjectures are examined. Artin's conjecture, as modified by the

Lehmers, Heilbronn, and Hooley, is discussed at some length, and Table 8 gives data for $P < 50000$. Let $G(x)$ denote max $g(P)$ for $P \leqq x$. One has $G(760321) = 73$, and it is conjectured that $G(x) = O\{(\ln x)^3\}$.

Reviewer's comments: The whole book is interesting, and the tables are useful. Owing to their long history, the present tables exhibit a conflict between Cunningham's original purpose, the study of g and ν, and the later purpose of printing a useful table of indices. No explanation is given for the favoritism towards certain arithmetic progressions for P in Tables 2–4. For the second purpose alone, *all* $P < 10^5$ would perhaps be preferable. Again, the varying arguments y in Tables 1–4 is not explained. The *indices* for $y = 6$, 10, and 12 are obviously redundant, but Cunningham was interested in their values of ν, e.g., that for $y = 10$ relates to the decimal expansion of P^{-1}. For the second purpose alone, a uniform $y = p_i$, $i = 1(1)15$ would perhaps be preferable. Thus, besides their interest and utility, the tables are replete with charming vestigial structures, as befits the national character.

Perhaps there are legal aspects: Besides leaving his unpublished extensions of [1] to the London Mathematical Society, Cunningham also left a legacy to pay for their completion and publication. That is a similar situation to that of Mansell's *Logarithms* in the immediately preceding volume of the Royal Society Mathematical Tables, and inverts the situation in American universities, where the slogan is "Publish or Perish."

With the use of a modern computer, one would probably select a more direct program, and not use Cunningham's intricate method. Presumably, [2] was so computed. Of course, that is no criticism of the present tables.

The conjecture for $G(x)$ seems to the reviewer unduly conservative: the data shown would justify the stronger

$$G(x) = O\{(\ln x)^2\}$$

or

$$(G(x))^{1/2} = O(\ln x) .$$

If $g(x)$ is the largest gap between successive primes, the reviewer [3] has conjectured

$$(g(x))^{1/2} \sim \ln x .$$

When one reflects upon the meaning of $G(x)$, this may be less of a coincidence than appears at first.

Two errata in the Introduction are these: On p. xlv, for $P - 1 = 2 \cdot 59$, read $P - 1 = 2^2 \cdot 3^3$, and, in the line below, for "P exceeding 41" read "P exceeding 109." More serious, in Section 31, p. xlvi is the implication that the tables always use $G = g$. Sometimes, in Table 1, $G = -g'$, as is indicated above. Another pertinent reference that should have been listed is [4].

D. S.

1. Lt. Col. Allan J. C. Cunningham, H. J. Woodall & T. G. Creak, *Haupt-Exponents, Residue-Indices, Primitive Roots, and Standard Congruences*, F. Hodgson, London, 1922.

2. J. C. P. Miller, *Table of Primitive Roots*, *Math. Comp.*, v. 17, 1963, pp. 88–89, UMT **2**.

3. Daniel Shanks, "On maximal gaps between successive primes," *Math. Comp.*, v. 18, 1964, pp. 646–651.

4. C. A. Nichol, John L. Selfridge & Lowry McKee, *A Table of Indices and Power Residues for all Primes and Prime Powers below 2000*, W. W. Norton, New York, 1962. (See *Math. Comp.*, v. 17, 1963, pp. 463–464, RMT **72**.)

52[9].—Edgar Karst, *The First 2500 Reciprocals and their Partial Sums of all Twin Primes (p, p + 2) between (3, 5) and (102761, 102763)*, Department of Mathematics, University of Arizona, Tucson, Arizona, January 1969. Ms. of 271 computer sheets deposited in the UMT file.

Herein are tabulated to 20D the reciprocals of the first 2500 twin primes, together with the cumulative sums, calculated on an IBM 1130 system. This table is preceded by a listing of the computer program employed in its construction. An appended table of two pages lists the first member of each of the prime pairs considered.

In his introductory remarks the author notes the accuracy of the counts of twin primes published by Glaisher [1] and by Hardy & Littlewood [2] and confirms the errors in Sutton [3] as announced by Sexton [4]. However, he fails to refer to extensive counts of twin primes by Lehmer [5] and by Gruenberger & Armerding [6].

The author also remarks upon the slow convergence of the series of the reciprocals of the twin primes; for example, his table reveals that the sum of the reciprocals of the first 2500 such primes is 1.6733 . . . , whereas Fröberg [7] has calculated the sum to 4D of all such reciprocals to be 1.7019 (herein referred to as Brun's constant).

Regrettably, the appearance of these tables is marred by the occasional suppression of zeros in the computer-printed output.

J. W. W.

1. J. W. L. Glaisher, "An enumeration of prime-pairs," *Messenger of Math.*, v. 8, 1878, pp. 28–33.
2. G. H. Hardy & J. E. Littlewood, "Partitio numerorum III: On the expression of a number as a sum of primes," *Acta Math.*, v. 44, 1923, pp. 1–70.
3. C. S. Sutton, "An investigation of the average distribution of twin prime numbers," *J. Math. and Phys.*, v. 16, 1937, pp. 1–42.
4. C. R. Sexton, "Counts of twin primes less than 100000," *MTAC*, v. 8, 1954, pp. 47–49, Note **158.**
5. D. H. Lehmer, "Tables concerning the distribution of primes up to 37 millions," 1957, ms. deposited in UMT file. (See *MTAC*, v. 13, 1959, pp. 56–57, RMT **3.**)
6. F. Gruenberger & G. Armerding, *Statistics on the First Six Million Prime Numbers*, Rand Corporation, Santa Monica, California, 1961. (See *Math. Comp.*, v. 19, 1965, pp. 503–505, RMT **73.**)
7. Carl-Erik Fröberg, "On the sum of inverses of primes and of twin primes," *Nordisk Mat. Tidskr. Informations-Behandling*, v. 1, 1961, pp. 15–20.

53[10, 13.35].—Michael A. Arbib, Editor, *Algebraic Theory of Machines, Languages, and Semigroups*, Academic Press, New York, 1968, xvi + 359 pp., 23 cm. Price $16.00.

This book is a collection of papers or chapters in the general areas of finite state machines, context-free languages, and finite semigroups. Some of the contributions are appropriately called "chapters" as they are written in a consistent notation and provide a basis for later contributions. Other contributions are accurately described as "papers" as they are written in their own notation and are independent of the other contributions.

A major portion of this book is devoted to topics relating to the Krohn-Rhodes decomposition theorem for finite semigroups and machines. This portion includes three chapters on finite semigroups supplying needed background for two algebraic proofs of the decomposition theorem and applications to semigroup complexity. Although self-contained, these chapters are heavy on notation and will be difficult reading for people without a strong mathematical orientation. This main development is supplemented by a number of self-contained and more expository papers which approach the semigroups from a variety of more machine-oriented points of view. This includes Zeiger's independent proof of the decomposition theorem.

The final two papers show two approaches to context-free and other languages: a grammatical approach and a power series approach.

The somewhat specialized subject matter of the book and its diversity of notation, level, and style make it an unlikely choice as a textbook, but it could prove valuable as a reference book and most people with interest in automata theory are likely to find some of the material of interest. Its strongest virtue is in its diversity of approaches and viewpoint.

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54[12].—D. W. Barron, *Recursive Techniques in Programming*, American Elsevier Publishing Co., Inc., New York, 1968, 64 pp., 22 cm. Price $5.25.

This monograph deals with the use of recursive techniques in programming. It considers very briefly and sketchily the ideas of recursion, the mechanisms for implementation and the formal relationship between recursion and iteration. It also contains some examples and applications. It is the sort of material that should be covered in a lecture or so in an introductory course in computer science.

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55[12].—Peter Wegner, *Programming Languages, Information Structures and Machine Organization*, McGraw-Hill Book Co., New York, 1968, xx + 401 pp., 23 cm. Price $10.95.

This book is very much a mixed bag. On the one hand, it is a useful collection of information on many of the most modish topics in computer science today; on the other hand, it suffers from a general lack of organization, occasional narrowness of viewpoint, and sometimes obscure explanations. The author states in his introduction that he plans ". . . to classify programming techniques and to develop a framework for the characterization of programming languages, programs, and computations. In the present text such a framework is developed, starting from the notions that a program with its data constitutes an *information structure*, and that a computation results in a sequence of information structures generated from an *initial representation* by the execution of a sequence of *instructions*." In fact, it appears that these notions are too abstract to be applied in any meaningful way to the subject matter of this book.

The first chapter of the book is on machine language and machine organization. The author treats the instruction set of the IBM 7094 as a paradigm for computer instruction sets, but unfortunately does not point out that he is doing so. The result leaves the impression to the naive reader that all machines are constructed in more or less this way, when in fact current machine architecture is far more diverse than this book would lead one to believe. The material on virtual address spaces, paging, and segmentation is not quite so narrow in its view, but it is plagued by repetitions of the same material in slightly altered form, with no cross-referencing among these repetitions. For instance, the term *activation record* is defined three different times in the first chapter (and several more times in succeeding chapters) as though the earlier definition had never existed. Also, the author defines access modes to information in a virtual memory, and later defines the same access modes for information in segments, but no relationship is given between these two sets of definitions.

The second chapter is a discussion of assembly programs, and is somewhat better than the first chapter. However, this chapter also suffers from a certain vagueness and lack of cohesiveness.

The third chapter is on macro generators and the lambda calculus. The discussions of Strachey's general-purpose macro-generator and of Mooers' TRAC system are well written, and there is some effort made to relate the two. The material on the lambda calculus is unfortunately more obscure than it needs to be, since the author fails to introduce appropriate abbreviative devices and thus is saddled with awkward notations for much of the exposition. However, there is much useful material here, particularly the section on Landin's SECD machine for evaluating lambda-expressions.

The fourth chapter is on procedure-oriented languages, and is probably the best chapter in the book. In this chapter, the author discusses ALGOL and several strategies for its implementation. He also describes a number of aspects of PL/I, with emphasis on those parts of the language, such as controlled storage allocation and structure definition, that are not in ALGOL. The chapter concludes with a discussion of simulation languages. There are also apprendices on syntactic specification and analysis and on the syntax of ALGOL 60.

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56[13.15, 13.20].—JULIAN D. COLE, *Perturbation Methods in Applied Mathematics*, Blaisdell Publishing Co., Waltham, Mass., 1968, vii + 260 pp., 23 cm. Price $9.50.

The literature of applied mathematics contains many ordinary or partial differential equations solved by various kinds of asymptotic expansions and connection procedures. It is difficult for a graduate student or research worker, not active in the field, to find a good reference from which to start learning the techniques. An earlier book devoted to this aim, *Perturbation Methods in Fluid Mechanics* by Milton Van Dyke, Academic Press, New York, 1964, fulfills the goal for those interested in fluid dynamics, but would be difficult reading for others. The book under review is similar in approach and style to Van Dyke's book, but the newer one covers a distinctly broader range of problems and is readily accessible to a general reader. While the style is somewhat too formal for this reviewer's taste, the book does represent an excellent source to which the student or researcher may turn.

A sampling of the subjects discussed includes the van der Pol oscillator, shock structure, some adiabatic invariants, the Mathieu equation, WKB method, numerous fluid dynamic boundary layer problems, slender body theory, the piston problem of gas dynamics, and brief excursions into elasticity, shallow water theory and magneto-hydrodynamics.

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