A VERSION OF FABRY’S THEOREM FOR POWER SERIES
WITH REGULARLY VARYING COEFFICIENTS

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Abstract. For real power series whose non-zero coefficients satisfy $|a_m|^{1/m} \to 1$, we prove a stronger version of Fabry’s theorem relating the frequency of sign changes in the coefficients and analytic continuation of the sum of the power series.

For a set $\Lambda$ of non-negative integers, we consider the counting function $n(x, \Lambda) = \# \Lambda \cap [0, x]$.

We say that $\Lambda$ is measurable if the limit $\lim_{x \to +\infty} n(x, \Lambda)/x$ exists, and we call this limit the density of $\Lambda$.

Let $S = \{a_m\}$ be a sequence of real numbers. We say that a sign change occurs at the place $m$ if there exists $k < m$ such that $a_m a_k < 0$ while $a_j = 0$ for $k < j < m$.

**Theorem A.** Let $\Delta$ be a number in $[0, 1]$. The following two properties of a set $\Lambda$ of positive integers are equivalent:

(i) Every power series

$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$

of radius of convergence 1, with real coefficients and such that the changes of sign of $\{a_m\}$ occur only for $m \in \Lambda$, has a singularity on the arc

$$I_\Delta = \{e^{i\theta} : |\theta| \leq \pi \Delta\}.$$

(ii) For every $\Delta' > \Delta$ there exists a measurable set $\Lambda' \subset \mathbb{N}$ of density $\Delta'$ such that $\Lambda \subset \Lambda'$.

Implication (ii) $\longrightarrow$ (i) is a consequence of Fabry’s General Theorem [5, 3], as restated by Pólya. For the implication (i) $\longrightarrow$ (ii) see [9]. Fabry’s General Theorem takes into account not only the sign changes of coefficients but also the absolute values of coefficients. It has a rather complicated statement, and the sufficient condition of the existence of a singularity given by this theorem is not the best
possible. The best possible condition in Fabry’s General Theorem is unknown; see, for example, the discussion in [4].

Alan Sokal (private communication) asked what happens if we assume that the power series (1) satisfies the additional regularity condition:

\[ \lim_{m \in P, m \to \infty} |a_m|^{1/m} = 1, \]

where \( P = \{m : a_m \neq 0\} \). This condition holds for most interesting generating functions. The answer is somewhat surprising:

**Theorem 1.** Let \( \Delta \) be a number in \([0, 1]\). The following two properties of a set \( \Lambda \) of positive integers are equivalent:

a) Every power series (1) satisfying (2), with real coefficients and such that the changes of sign of the coefficients \( a_m \) occur only for \( m \in \Lambda \), has a singularity on the arc \( I_\Delta \).

b) All measurable subsets \( \Lambda' \subset \Lambda \) have densities at most \( \Delta \).

We recall that the minimum density

\[ D_2(\Lambda) = \lim_{r \to 0+} \liminf_{x \to +\infty} \frac{n((r + 1)x, \Lambda) - n(x, \Lambda)}{rx}, \]

can be alternatively defined as the sup of the limits

\[ \lim_{x \to -\infty} n(x, \Lambda')/x \]

over all measurable sets \( \Lambda' \subset \Lambda \).

Similarly the maximum density of \( \Lambda \) is

\[ D_2(\Lambda) = \lim_{r \to 0+} \limsup_{x \to \infty} \frac{n((r + 1)x, \Lambda) - n(x, \Lambda)}{rx}, \]

and it equals the inf of the limits (3) over all measurable sequences of non-negative integers \( \Lambda' \) containing \( \Lambda \).

For all these properties of minimum and maximum densities, see [12].

Thus condition (ii) is equivalent to \( D_2(\Lambda) \leq \Delta \), while condition b) is equivalent to \( D_2(\Lambda) \leq \Delta \).

Here is a gap version of Theorem 1:

**Theorem 2.** The following two properties of a set \( \Lambda \) of positive integers are equivalent:

A. Every power series

\[ \sum_{m \in \Lambda} a_m z^m \]

satisfying (2) has a singularity on \( I_\Delta \).

A’. Every power series (1) satisfying (2) has a singularity on every closed arc of length \( 2\pi \Delta \) of the unit circle.

B. \( D_2(\Lambda) \leq \Delta \).

The equivalence between A and A’ is immediate, as all assumptions of the statement A are invariant with respect to the change of the variable \( z \mapsto \lambda z, |\lambda| = 1 \).

**Proof of Theorem 1.** b) \( \rightarrow \) a).
Proving this by contradiction, we assume that $D_{\Lambda}(\Lambda) \leq \Delta$, and that there exists a function $f$ of the form (1) with the property (2) which has an analytic continuation to $I\Delta$, and such that the sign changes occur only for $m \in \Lambda$.

Without loss of generality we assume that $a_0 = 1$ and $\Delta < 1$.

**Lemma 1.** For a function $f$ as in (1) to have an immediate analytic continuation from the unit disc to the arc $I\Delta$ it is necessary and sufficient that there exists an entire function $F$ of exponential type with the properties

\begin{align}
  a_m &= (-1)^m F(m), \quad \text{for all } m \geq 0, \\
  \limsup_{t \to \infty} \frac{\log |F(te^{i\theta})|}{t} &\leq \pi b |\sin \theta|, \quad |\theta| < \alpha,
\end{align}

with some $b < 1 - \Delta$ and some $\alpha \in (0, \pi)$.

This result can be found in [1]; see also [2, 4].

Consider the sequence of subharmonic functions

\begin{equation}
  u_m(z) = \frac{1}{m} \log |F(mz)|, \quad m = 1, 2, 3, \ldots.
\end{equation}

This sequence is uniformly bounded from above on every compact subset of the plane, because $F$ is of exponential type. Moreover, $u_m(0) = 0$ because of our assumption that $a_0 = F(0) = 1$. The Compactness Principle [8, Th. 4.1.9] implies that from every sequence of integers $m$ one can choose a subsequence such that the limit $u = \lim u_m$ exists. This limit is a subharmonic function in the plane that satisfies, in view of (6),

\begin{equation}
  u(re^{i\theta}) \leq \pi br |\sin \theta|, \quad |\theta| < \alpha,
\end{equation}

with some $b$ satisfying $0 < b < 1 - \Delta$.

We use the following result of Pólya [11, footnote 18, p. 703]:

**Lemma 2.** Let $f$ be a power series (1) of radius of convergence 1. Let $\{a_{mk}\}$ be a subsequence of coefficients with the property

\[ \lim_{k \to \infty} |a_{mk}|^{1/m_k} = 1, \]

and assume that for some $r > 0$ the number of non-zero coefficients $a_j$ on the interval $m_k \leq j \leq (1 + r)m_k$ is $o(m_k r)$ as $k \to \infty$. Then $f$ has no analytic continuation to any point of the unit circle.

Lemma 2 also follows from the results of [1] or [4].

Now we show that (2) implies the following:

**Lemma 3.** Every limit function has the property $u(x) = 0$ for $x \geq 0$.

Proof of Lemma 3. Let $U = \{x : x \geq 0, u(x) < 0\}$. This set is open because $u$ is upper semi-continuous. Take any closed interval $J = [c, d] \subset U$. Then $u(x) \leq -\epsilon$, $x \in J$, with some $\epsilon > 0$. Let $\{m_k\}$ be the sequence of integers such that $u_{m_k} \to u$. Then from the definition of $u_m$ we see that

\[ \log |F(m_kx)| \leq -m_k \epsilon/2 \quad \text{for } x \in J \]

and for all large $k$. Together with (3) and (2) this implies that $a_j = 0$ for all $j \in m_kJ$. Let $a_{m_k}'$ be the last non-zero coefficient before $cm_k$. Applying Lemma 2
to the sequence \( \{m'_k\} \) we conclude that \( f \) has no analytic continuation from the unit disc. This is a contradiction, which proves Lemma 3. \( \square \)

Now we use the following general fact:

**Grishin’s Lemma.** Let \( u \leq v \) be two subharmonic functions, and let \( \mu \) and \( \nu \) be their respective Riesz measures. Let \( E \) be a Borel set such that \( u(z) = v(z) > -\infty \) for \( z \in E \). Then the restrictions of the Riesz measures on \( E \) satisfy

\[
\mu|_E \leq \nu|_E.
\]

The references are [13] [7] [6].

In view of Lemma 2, we can apply Grishin’s Lemma to \( u \) and \( v(z) = \pi b|\text{Im } z| \) and \( E = [0, \infty) \subset \mathbb{R} \). We obtain that the Riesz measure \( d\mu \) of any limit function \( u \) of the sequence \( \{u_k\} \) satisfies

\[
d\mu|_{[0,\infty)} \leq b \, dx.
\]

Now we go back to our coefficients and function \( F \). By our assumption, the sign changes occur on a sequence \( \Lambda \) whose minimum density is at most \( \Delta \). Choose a number \( a \) such that \( b < a < 1 - \Delta \). By the first definition of the minimum density, there exist \( r > 0 \) and a sequence \( x_k \to \infty \) such that

\[
n((1 + r)x_k, \Lambda) - n(x_k, \Lambda) \leq (1 - a)r x_k.
\]

**Lemma 4.** Let \( (a_0, a_1, \ldots, a_N) \) be a sequence of real numbers, and let \( f \) be a real analytic function on the closed interval \([0, N]\), such that \( f(n) = (-1)^n a_n \). Then the number of zeros of \( f \) on \([0, N]\), counting multiplicities, is at least \( N \) minus the number of sign changes of the sequence \( \{a_n\} \).

**Proof.** Consider first an interval \((k, n)\) such that \( a_k a_n \neq 0 \) but \( a_j = 0 \) for \( k < j < n \). We claim that \( f \) has at least

\[
n - k - \#(\text{sign changes in the pair } (a_k, a_n))
\]

zeros on the open interval \((k, n)\). Indeed, the number of zeros of \( f \) on this interval is at least \( n - k - 1 \) in any case. This proves the claim if there is a sign change in the pair \((a_k, a_n)\). If there is no sign change, that is, if \( a_k a_n > 0 \), then \( f(n) f(k) = (-1)^{n-k} \). So the number of zeros of \( f \) on the interval \((n, k)\) is of the same parity as \( n - k \).

But \( f \) has at least \( n - k - 1 \) zeros on this interval; thus the total number of zeros is at least \( n - k \). This proves our claim.

Now let \( a_k \) be the first and \( a_n \) the last non-zero term of our sequence. As the interval \((k, n)\) is a disjoint union of the intervals to which the above claim applies, we conclude that the number of zeros of \( f \) on \((k, n)\) is at least \( n - k \) minus the number of sign changes of our sequence. On the rest of the interval \([0, N]\) our function has at least \( N - n + k \) zeros, so the total number of zeros is at least \( N \) minus the number of sign changes. \( \square \)

Let \( u \) be a limit function of the subsequence \( \{u_{m_k}\} \) with \( m_k = [x_k] \). By Lemma 4, the function \( F \) has at least \( ar x_k - 2 \) zeros on each interval \([x_k, (1 + r)x_k]\), which implies that the Riesz measure \( \mu \) of \( u \) satisfies

\[
\mu([1, 1 + r]) \geq ar.
\]

This contradicts (9) and thus proves the implication b) \( \rightarrow a \).
Proof of Theorem 2, $B \rightarrow A$. This is essentially the same as the previous proof. Proving by contradiction, we assume that $B$ holds but that there exists a function $f$ of the form (4) with the property (2) which has an analytic continuation to $I_\Delta$. Applying Lemma 1, we obtain an entire function $F$ with properties (5) and (6). Then we consider subharmonic functions $u_m$ and the limit functions $u$ of this sequence. Using Lemmas 2, 3 and Grishin’s lemma, we obtain the inequality (9) for the Riesz measure $\mu$ of $u$, exactly as in the proof of Theorem 1.

Now we notice that condition $B$ of Theorem 2 means that the entire function $F$ has zeros at some sequence of integers of maximum density at least $1 - \Delta$. Denoting by $n(x)$ the number of zeros of $F$ on $[0, x]$ and choosing a number $a \in (b, 1 - \Delta)$, we obtain that there exist $r > 0$ and a sequence of integers $m_k \rightarrow \infty$ such that

$$n((1 + r)m_k) - n(m_k) \geq arm_k.$$  

This implies that for the limit function $u$ of the sequence $u_{m_k}$, the Riesz measure $\mu$ satisfies $\mu([1, 1 + r]) \geq ar$, which contradicts (9). This contradiction proves implication $B \rightarrow A$ in Theorem 2.

Proof of implications $a) \rightarrow b)$ of Theorem 1 and $A \rightarrow B$ of Theorem 2. Suppose that a set $\Lambda$ of positive integers does not satisfy $b)$, $B$. We will construct power series $f$ of the form (4) with real coefficients which has an immediate analytic continuation from the unit disc to the arc $I_\Delta$. This will simultaneously prove the implications $a) \rightarrow b)$ of Theorem 1 and $A \rightarrow B$ of Theorem 2, as the number of sign changes of any sequence does not exceed the number of its non-zero terms.

Let $\Lambda' \subset \Lambda$ be a measurable set of density $\Delta' > \Delta$. Let $S$ be the complement of $\Lambda'$ in the set of positive integers. Then $S$ is also measurable and has density $1 - \Delta'$.

Consider the infinite product

$$F(z) = \prod_{t \in S} \left(1 - \frac{2z^2}{t^2}\right).$$

This is an entire function of exponential type with indicator $\pi(1 - \Delta')|\sin \theta|$, and furthermore,

$$\log |F(z)| \geq \pi(1 - \Delta')|\text{Im } z| + o(|z|),$$

as $z \rightarrow \infty$ outside the set $\{z : \text{dist}(z, S) \leq 1/4\}$. (See [10] Ch. II, Thm. 5 for this result.) Now we use the sufficiency part of Lemma 1 and define the coefficients of our power series by $a_m = (-1)^m F(m)$. Then we have all the needed properties; in particular $a)$ follows from $[10]$. \hfill \Box

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