GLOBAL EXISTENCE OF SOLUTIONS TO A HYPERBOLIC-PARABOLIC SYSTEM

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ABSTRACT. In this paper, we investigate the global existence of solutions to a hyperbolic-parabolic model of chemotaxis arising in the theory of reinforced random walks. To get $L^2$-estimates of solutions, we construct a nonnegative convex entropy of the corresponding hyperbolic system. The higher energy estimates are obtained by the energy method and a priori assumptions.

1. INTRODUCTION AND MAIN RESULT

In this paper, we consider the following system:

$$\begin{cases}
    u_t - u_x = 0, \\
    v_t - (uv)_x = v_{xx},
\end{cases}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geq 0,$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) > 0, \quad x \in [0,1].$$

Here the compatibility conditions on $u_0, v_0$ assume that $(u_0(0), v_0(0)) = (0,0)$, which will be used in Section 3.

Motivated by biological considerations and numerical computations carried out by Othmer and Stevens in [6] and Levine and Sleeman in [3], the system (1.1) comes from:

$$\begin{cases}
    \frac{\partial p}{\partial t} = D \frac{\partial}{\partial x} \left( p \frac{\partial}{\partial x} \left( \ln \frac{p}{\Phi(w)} \right) \right), \quad x \in (0,l), \quad t > 0, \\
    \frac{\partial w}{\partial t} = R(p,w),
\end{cases}$$

where $p(x,t)$ is the particle density of a particular species, $w(x,t)$ is the concentration of the “active agent”, and $D$ and $B$ are positive constants.
In fact, as in [7], let
\( \Phi(w) = w^{-\alpha}, \quad R(p, w) = \lambda pw - \mu w, \)
where \( \alpha \) and \( \lambda, \mu \) are positive constants.

Then the system (1.4) is transformed into the following form:
\begin{align}
\begin{cases}
p_t &= Dp_{xx} + D\alpha \left( \frac{pw}{w} \right)_x, \\
w_t &= \lambda pw - \mu w.
\end{cases}
\end{align}

Furthermore, set
\begin{equation}
q = (\ln w)_x = \frac{w_x}{w}.
\end{equation}

Then the system (1.6) can be rewritten as:
\begin{align}
\begin{cases}
p_t &= Dp_{xx} + D\alpha (pq)_x, \\
q_t &= \lambda p_x.
\end{cases}
\end{align}

Let
\begin{equation}
\tau = At, \quad \xi = lx, \quad p = Bv, \quad q = c_1 u,
\end{equation}
where \( A, l \) and \( c_1 \) are positive constants to be determined below.

Then the system (1.8) becomes
\begin{align}
\begin{cases}
u_\tau &= \frac{\lambda B}{Ac_1} v_\xi, \\
v_\tau &= \frac{Dl^2}{A} v_\xi + \frac{Dalc_1}{A} (uv)_\xi.
\end{cases}
\end{align}

Choosing
\begin{align}
\begin{cases}
\frac{\lambda B}{Ac_1} = 1, \\
\frac{Dl^2}{A} = 1, \\
\frac{Dalc_1}{A} = 1,
\end{cases}
\end{align}
i.e.,
\begin{equation}
A = B\alpha \lambda > 0, \quad l = \sqrt{\frac{B\alpha \lambda}{D}} > 0, \quad c_1 = \sqrt{\frac{B\lambda}{\alpha D}} > 0,
\end{equation}
then it is easy to verify that \( u, v \) satisfy
\begin{align}
\begin{cases}
u_\tau &= v_\xi, \\
v_\tau &= v_\xi + (uv)_\xi.
\end{cases}
\end{align}

If we replace \((\xi, \tau)\) by \((x, t)\), then (1.12) can be rewritten as (1.1).

The system (1.4) describes the model of chemotaxis in biology. Othmer and Stevens [6] have developed a number of mathematical models of chemotaxis to illustrate aggregation leading (numerically) to nonconstant steady-states, blow-up resulting in the formation of singularities and collapse or the formation of a spatially uniform steady state. The models developed in [6] have been studied in depth by Levine and Sleeman [3]. They gave some heuristic understanding of some of these phenomena and investigated the properties of solutions of a system of chemotaxis

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Throughout this paper, we denote positive constants by $C$. The character “$C$” may differ in different places.

In this paper, we will study the global existence of solutions to the initial boundary value problem (1.1)-(1.3) in $L^\infty([0, \infty), H^2([0, 1]))$. To get $L^2$-estimates of solutions, we construct a nonnegative convex entropy of the corresponding hyperbolic system. The higher energy estimates are obtained by the energy method and a priori assumptions.

The corresponding hyperbolic system of (1.1) is

\begin{equation}
\begin{cases}
  u_t - v_x = 0, \\
  v_t - (uv)_x = 0.
\end{cases}
\end{equation}

The eigenvalues of (1.13) are:

\begin{equation}
\lambda_1 = -\frac{1}{2} u - \frac{1}{2} \sqrt{u^2 + 4v}, \quad \lambda_2 = -\frac{1}{2} u + \frac{1}{2} \sqrt{u^2 + 4v}.
\end{equation}

Therefore, the system (1.13) is strictly hyperbolic when $v > 0$.

Remark 1.1. By the boundary conditions (1.2) and (1.1)$_1$, we have

\begin{equation}
u_t(0, t) = u_t(1, t) = v_x(0, t) = v_x(1, t) = 0.
\end{equation}

Notation. Throughout this paper, we denote positive constants by $C$. Moreover, the character “$C$” may differ in different places. $L^p = L^p([0, 1])$ $(1 \leq p \leq \infty)$ denotes the usual Lebesgue space with the norm

$$
||f||_{L^p} = \left( \int_0^1 |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

$$
|f|_\infty = \sup_{[0, 1]} |f(x)|.
$$

$H^l([0, 1])$ $(l \geq 0)$ denotes the usual $l$th-order Sobolev space with the norm

$$
||f||_l = \left( \sum_{j=0}^l ||\partial_x^j f||^2 \right)^{\frac{1}{2}},
$$

where $|| \cdot || = || \cdot ||_0 = || \cdot ||_{L^2}$. For simplicity, $||f(\cdot, t)||_{L^p}$ and $||f(\cdot, t)||_l$ are denoted by $||f(t)||_{L^p}$ and $||f(t)||_l$ respectively.

We will prove the following global existence theorem.

**Theorem 1.2.** Let $u_0, \ v_0 \in H^2([0, 1])$ and let $||u_0||_2^2 + ||v_0 - 1||_2^2$ be sufficiently small. Then there exists a unique global solution $(u(x, t), v(x, t))$ of (1.1)-(1.3) satisfying

\begin{enumerate}
  \item[(i)] $u, \ v \in L^\infty([0, \infty), H^2([0, 1])), \ v_x \in L^2([0, \infty), H^2([0, 1]));$
  \item[(ii)] $||u(t)||_2^2 + ||v(t) - 1||_2^2 + \int_0^t ||v_x(\tau)||_2^2 d\tau \leq C (||u_0||_2^2 + ||v_0 - 1||_2^2).$
\end{enumerate}
2. $L^2$-ENERGY ESTIMATES

In this section, we give $L^2$-energy estimates of the initial boundary value problem (1.1), (1.2) and (1.3) by a nonnegative convex entropy of the system of hyperbolic conservation laws (1.13). To do this, we first give the following relation on the entropy-entropy flux pair $(\eta(u, v), q(u, v))$ of (1.13) (see [8]):

\begin{equation}
\begin{aligned}
&\eta_u = -q_v, \\
&q_v = -\eta_u - u\eta_v.
\end{aligned}
\end{equation}

Eliminating $q$ from (2.1), we have

\begin{equation}
\eta_{uu} + u\eta_{uv} - v\eta_{vv} = 0.
\end{equation}

Next, we seek the entropy of (1.13) with the following form:

\begin{equation}
\eta(u, v) = \frac{1}{2}u^2 + a(v),
\end{equation}

where $a(v)$ is a nonnegative convex function.

Substituting (2.3) into (2.2), we have

\begin{equation}
1 - va''(v) = 0,
\end{equation}

which implies

\begin{equation}
a(v) = v\ln v - v + k_1v + k_2,
\end{equation}

where $k_1, k_2$ are arbitrary constants.

It is easy to get the flux corresponding to the entropy $\eta(u, v)$ defined by (2.3) and (2.4), namely

\begin{equation}
q(u, v) = -uv \ln v - k_1uv + k_3,
\end{equation}

where $k_3$ is an arbitrary constant.

In particular, if we take $k_1 = k_3 = 0, k_2 = 1$, we will get an entropy-entropy flux pair of (1.13):

\begin{equation}
\begin{aligned}
&\eta(u, v) = \frac{1}{2}u^2 + v\ln v - v + 1, \\
&q(u, v) = -uv \ln v.
\end{aligned}
\end{equation}

In the next analysis, we devote ourselves to the estimates of the solution $(u(x, t), v(x, t))$ of (1.1), (1.2) and (1.3) under the a priori assumptions:

\begin{equation}
|u| \leq \varepsilon, \quad |v - 1| \leq \frac{1}{2}, \quad |u_x| \leq \varepsilon, \quad |v_x| \leq \varepsilon,
\end{equation}

where $0 < \varepsilon << 1$.

**Lemma 2.1.** The entropy $\eta(u, v)$ defined by (2.6) satisfies for $|v - 1| \leq \frac{1}{2}$,

\begin{equation}
\frac{1}{2}u^2 + \frac{1}{3}(v - 1)^2 \leq \eta(u, v) \leq \frac{1}{2}u^2 + (v - 1)^2.
\end{equation}

**Proof.** Let

\begin{equation}
a_0(v) = v\ln v - v + 1.
\end{equation}

Then

\begin{equation}
a_0(1) = a'_0(1) = 0, \quad \frac{2}{3} \leq a''_0(v) = \frac{1}{v} \leq 2,
\end{equation}

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which implies
\[
\frac{1}{3}(v-1)^2 \leq a_0(v) = \frac{1}{2}a_0''(\xi)(v-1)^2 \leq (v-1)^2,
\]
where $\xi$ is between 1 and $v$. \qed

From (2.6), (2.9) and (2.10), (2.8) follows. This proves Lemma 2.1.

**Lemma 2.2** ($L^2$-energy estimates). Under the assumptions of Theorem 1.2 and the a priori assumptions (2.7), we have
\[
\int_0^1 \left( \frac{1}{2}u^2 + \frac{1}{3}(v-1)^2 \right) dx + \frac{2}{3} \int_0^t \int_0^1 v_x^2dxdt \leq \int_0^1 \left( \frac{1}{2}u_0^2(x) + (v_0(x) - 1)^2 \right) dx.
\]

**Proof.** Multiplying (1.1) by $\nabla \eta$ and integrating it, we have by the boundary conditions (1.2) and (1.15),
\[
\int_0^1 \eta(x,t)dx + \int_0^t \int_0^1 \frac{v_x^2}{v}dxds = \int_0^1 \eta(x,0)dx,
\]
i.e.,
\[
\int_0^1 \left( \frac{1}{2}u^2 + v \ln v - v + 1 \right) dx + \int_0^t \int_0^1 \frac{v_x^2}{v}dxds = \int_0^1 \left( \frac{1}{2}u_0^2 + v_0 \ln v_0 - v_0 + 1 \right) dx.
\]
This proves Lemma 2.2 by Lemma 2.1 and the a priori assumptions (2.7). \qed

3. **Higher energy estimates**

In this section, we will establish higher energy estimates.

**Lemma 3.1.** Under the assumptions of Theorem 1.2 and the a priori assumptions (2.7), we have
\[
\int_0^1 \left( u_x^2 + \frac{2}{3}v_x^2 \right) dx + \frac{4}{3} \int_0^t \int_0^1 v_{xx}^2dxds \leq C \left( ||u_0||_1^2 + ||v_0 - 1||_1^2 \right),
\]
where $C$ is a positive constant.

**Proof.** Differentiating (1.1) with respect to $x$, we have
\[
\begin{align*}
\left\{ \begin{array}{l}
  u_{xt} - v_{xx} = 0, \\
  v_{xt} - (uv)_{xx} = v_{xxx}.
\end{array} \right.
\end{align*}
\]
For any smooth function $g_1(v)$, take (3.2)$_1 \times u_x + (3.2)$_2 $g_1(v)v_x$, and integrate it to get
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 u_x^2 dx + \frac{1}{2} \frac{d}{dt} \int_0^1 g_1(v)v_x^2 dx \\
= \int_0^1 u_x v_{xx} dx + \frac{1}{2} \int_0^1 g_1'(v)v_x^2v_t dx + \int_0^1 g_1(v)uw_xv_{xx} dx \\
+ 2 \int_0^1 g_1(v)u_xv_x^2 dx + \int_0^1 g_1(v)u_{xx}v_x^2 dx + \int_0^1 g_1(v)v_{xxx}v_x dx.
\]
\[
(3.3)
\]
Next, we estimate the terms in the right side of (3.3) as follows:

\[
\frac{1}{2} \int_0^1 g_1'(v) v_x^2 v_x dx = \frac{1}{2} \int_0^1 g_1'(v) v_x^2 (uv)_x dx + \frac{1}{2} \int_0^1 g_1'(v) v_x^2 v_{xx} dx \\
= \frac{1}{6} \int_0^1 g_1''(v) v_x^4 dx + \frac{1}{6} (g_1'(v) v_x^2)_{0}^1 \\
\]

(3.4)

\[
\int_0^1 g_1(v) u v_x v_{xx} dx = -\frac{1}{2} \int_0^1 (g_1(v) u v_x^2 v_x + \frac{1}{2} (g_1(v) u v_x^2)_{0}^1 \\
= \frac{1}{2} \int_0^1 g_1(v) u v_x^2 v_x dx - \frac{1}{2} \int_0^1 g_1(v) u v_x^2 v_{xx} dx \\
\]

(3.5)

\[
\int_0^1 g_1(v) v v_x u_x dx = \int_0^1 (g_1(v) v v_x)_x u_x dx + (g_1(v) v v_x u_x)_{0}^1 \\
= \int_0^1 g_1(v) v v_x v_x dx - \int_0^1 g_1(v) v v_x u_x dx \\
\]

(3.6)

and

\[
\int_0^1 g_1(v) v_x v_{xx} dx = \int_0^1 (g_1(v) v_x v_{xx} dx + (g_1(v) v_x v_{xx})_{0}^1 \\
= \frac{1}{3} \int_0^1 g_1''(v) v_x^4 dx - \int_0^1 g_1(v) v_x^2 v_{xx} dx \\
\]

(3.7)

Substituting (3.4)-(3.7) into (3.3) and using the boundary conditions (1.2) and (1.15), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u_x^2 + g_1(v) v_x^2 \right) dx + \int_0^1 g_1'(v) v_x^2 v_{xx} dx \\
= \frac{1}{6} \int_0^1 g_1''(v) v_x^4 dx - \int_0^1 (g_1(v) v - 1) u_x v_{xx} dx - \frac{1}{2} \int_0^1 (g_1'(v) v - g_1(v)) u_x v_x^2 dx. \\
\]

Choosing

\[
g_1(v) = \frac{1}{v} > 0, \]

we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u_x^2 + \frac{v_x^2}{v} \right) dx + \int_0^1 \frac{v_x^2}{v} dx = \frac{1}{3} \int_0^1 \frac{v_x^4}{v^3} dx + \int_0^1 \frac{u_x v_x^2}{v} dx. \\
\]
From the \textit{a priori} assumptions (2.7), we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u_x^2 + \frac{v_x^2}{v} \right) dx + \int_0^1 \frac{v_x^2}{v} \, dx \leq \frac{4}{3} \varepsilon^2 \int_0^1 \frac{v_x^2}{v} \, dx + \varepsilon \int_0^1 \frac{v_x^2}{v} \, dx
\]
(3.11)
\[
= \left( \frac{4}{3} \varepsilon^2 + \varepsilon \right) \int_0^1 \frac{v_x^2}{v} \, dx.
\]
Integrating (3.11) in $t$ over $[0, t]$, we can obtain
\[
\int_0^1 \left( u_x^2 + \frac{v_x^2}{v} \right) dx + 2 \int_0^t \int_0^1 \frac{v_x^2}{v} \, dx \, ds
\]
(3.12)
\[
\leq \int_0^1 \left( u_{0x}^2 + \frac{v_{0x}^2}{v_0} \right) dx + 2 \left( \frac{4}{3} \varepsilon^2 + \varepsilon \right) \int_0^t \int_0^1 \frac{v_x^2}{v} \, dx \, ds,
\]
which implies (3.1) by (2.11) and the \textit{a priori} assumptions (2.7).
\square

The proof of Lemma 3.1 is completed.

**Lemma 3.2.** Under the assumptions of Theorem 1.2 and the \textit{a priori} assumptions (2.7), we have
\[
\int_0^1 \left( u_{xx}^2 + \frac{2}{3} v_{xx}^2 \right) dx + \frac{1}{3} \int_0^t \int_0^1 u_{xxx}^2 \, dx \, ds \leq C \left( \|u_0\|_2^2 + \|v_0 - 1\|_2^2 \right),
\]
(3.13)
where $C$ is a positive constant.

Before proving Lemma 3.2, we give the following result.

**Proposition 3.3.** The smooth function $v(x, t)$ obtained by Theorem 1.2 satisfies the following properties:
\[
\left| \int_0^1 \frac{1}{v^2} v_x^3 \, dx \right| \leq \int_0^1 \frac{1}{v_x} v_x^2 \, dx + C \int_0^1 \frac{1}{v_x} v_x^2 \, dx,
\]
(3.14)
where $C$ is a positive constant.

**Proof.** From the Gagliardo-Nirenberg-Moser inequality, we have
\[
\|v_{xx}(t)\|_{L^3}^3 \leq C \|v_x(t)\|_{L^6}^2 \|v_{xxx}(t)\|_{L^2}^2,
\]
(3.15)
where $C$ is a positive constant.

(3.15) and Young’s inequality show that
\[
\|v_{xx}(t)\|_{L^3}^3 \leq C \|v_x(t)\|_{L^6}^6 + \frac{2}{3} \|v_{xxx}(t)\|_{L^2}^2.
\]
(3.16)
Therefore, we have from the \textit{a priori} assumptions (2.7),
\[
\left| \int_0^1 \frac{1}{v^2} v_x^3 \, dx \right| \leq C \|v_x(t)\|_{L^6}^6 + \frac{2}{3} \|v_{xxx}(t)\|_{L^2}^2
\]
\[
\leq C \|v_x\|_{L^\infty}^4 \int_0^1 \frac{v_x^2}{v} \, dx + \int_0^1 \frac{1}{v_x} v_x^2 \, dx
\]
(3.17)
\[
\leq C \int_0^1 \frac{v_x^2}{v} \, dx + \int_0^1 \frac{1}{v_x} v_x^2 \, dx.
\]
This proves Proposition 3.3. \square
Proof of Lemma 3.2. Differentiating (3.2) with respect to $x$, we have

\begin{equation}
\left\{ \begin{array}{l}
u_{xxt} - v_{xxx} = 0, \\
v_{xxt} - (uv)_{xxx} = v_{xxxx}.
\end{array} \right.
\end{equation}

(3.18)

For any smooth function $g_2(v)$, taking (3.18)$_1 \times u_{xx} + (3.18)_2 \times g_2(v)v_{xx}$, and integrating it, we have

\begin{equation}
\frac{1}{2} \frac{d}{dt} \int_0^1 (u_{xx}^2 + g_2(v)v_{xx}^2) \, dx = \int_0^1 u_{xx}v_{xxx} \, dx + \frac{1}{2} \int_0^1 g_2'(v)v_{x}^2 \, dx \\
+ \int_0^1 g_2(v)v_{xx}(uv)_{xxx} \, dx + \int_0^1 g_2(v)v_{xxx}v_{xxxx} \, dx.
\end{equation}

(3.19)

Next, we estimate the terms in the right side of (3.19) as follows:

\begin{equation}
\frac{1}{2} \int_0^1 g_2'(v)v_{x}^2 \, dx = \frac{1}{2} \int_0^1 g_2'(v)[(uv)_x + v_{xx}]v_{xx}^2 \, dx
\end{equation}

(3.20)

\begin{equation}
\int_0^1 g_2(v)v_{xx}(uv)_{xxx} \, dx = - \int_0^1 (uv)_{xx}(g_2(v)v_{xx}) \, dx + (uv)_{xxx}g_2(v)v_{xx} \bigg|_0^1
\end{equation}

(3.21)

\begin{equation}
\int_0^1 g_2(v)v_{xxx}v_{xxxx} \, dx = - \int_0^1 (g_2(v)v_{xxx})_xv_{xxx} \, dx + (g_2(v)v_{xxx}v_{xxxx}) \bigg|_0^1
\end{equation}

(3.22)
Substituting (3.20)-(3.22) into (3.19), we obtain by the boundary conditions (1.2) and (1.15),

(3.23) $$\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u^2_{xx} + v^2_{xx} \right) dx + \int_0^1 g_2(v) v^2_{xx} dx$$

$$= \int_0^1 u_{xx} v_{xxx} dx - \int_0^1 v g_2(v) u_{xx} v_{xxx} dx + \frac{1}{2} \int_0^1 g'_2(v) u v_x v^2_{xx} dx$$

$$+ \frac{1}{2} \int_0^1 g'_2(v) u v_x v^2_{xx} dx + \frac{1}{2} \int_0^1 g'_2(v) v^3_{xx} dx - \int_0^1 u g_2(v) v_{xxx} v_{xx} dx$$

$$- 2 \int_0^1 g_2(v) u_x v_x v_{xxx} dx - \int_0^1 u g'_2(v) v_x v^2_{xx} dx - 2 \int_0^1 g'_2(v) u v^2_x v_x dx$$

$$+ \int_0^1 g'_2(v) v^2_x u_x dx + \int_0^1 v g''_2(v) u^2_x u_x dx + \int_0^1 v g'_2(v) v^2_x u_x dx$$

$$+ \int_0^1 v g'_2(v) u v_x v_{xxx} u_x dx - \int_0^1 g'_2(v) v_x v_{xxx} u_x dx$$

$$+(g_2(v) v_x (v u_{xx} + v_{xxx}))|_0^1.$$  

If choosing $g_2(v) = \frac{1}{v}$, then we get

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u^2_{xx} + \frac{1}{v^2} v^2_{xx} \right) dx + \int_0^1 \frac{1}{v} v^2_{xx} dx$$

$$= \frac{1}{2} \int_0^1 \frac{u}{v^2} v_x v^2_{xx} dx - \frac{1}{2} \int_0^1 \frac{1}{v^2} u_x v^2_{xx} dx - \frac{1}{2} \int_0^1 \frac{1}{v^2} v^3_{xx} dx$$

$$- \int_0^1 \frac{u}{v} v_{xx} v_{xxx} dx - 3 \int_0^1 \frac{1}{v} u_x v_x v_{xxx} dx + 3 \int_0^1 \frac{1}{v} v^2_x u_x v_x dx$$

$$- \int_0^1 \frac{u}{v} u_x v^2_{xx} dx + \int_0^1 \frac{1}{v^2} u v_x v_{xxx} dx + \left( \frac{v_{xx}}{v} (v u_{xx} + v_{xxx}) \right)|_0^1.$$  

From (3.2) and the boundary conditions (1.2) and (1.15), we have

$$v_{xxx}|_0^1 = -((uv)_{xx})|_0^1 = -(uv_{xx})|_0^1 - 2(u_x v_x)|_0^1 - (u_{xx} v)|_0^1 = -(u_{xx} v)|_0^1.$$  

Thus

(3.25) $$\left( \frac{v_{xx}}{v} (v u_{xx} + v_{xxx}) \right)|_0^1 = 0.$$  

By (3.24), (3.25), (3.14) and using the Cauchy-Schwarz inequality, we have

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \left( u^2_{xx} + \frac{1}{v^2} v^2_{xx} \right) dx + \int_0^1 \frac{1}{v} v^2_{xx} dx$$

$$\leq \frac{3}{4} \int_0^1 \frac{1}{v} v^2_{xx} dx + C \int_0^1 \frac{1}{v} (v^2_x + v^2_{xx}) dx.$$  

Integrating (3.26) in $t$ over $[0, t]$ and using Lemmas 2.2 and 3.1, and the $a$ priori assumptions (2.7), we get (3.13).

This proves Lemma 3.2. \qed
4. The proof of Theorem 1.2

The global existence in Theorem 1.2 follows from a local existence theorem (see \cite{2, 5}) and the a priori estimate (1.16) obtained by (2.11), (3.1) and (3.13).

Now, we have to show that the a priori assumptions (2.7) can be closed since, under the a priori assumptions (2.7), we have proved that (1.16) holds.

In fact, by Sobolev’s embedding theorem $W^{1,1}(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ and Hölder’s inequality, we have

\[
|v(x, t) - 1| \leq C \int_0^1 |v(x, t) - 1| \, dx + C \int_0^1 \left| \frac{\partial}{\partial x} (v(x, t) - 1) \right| \, dx \\
\leq C \left( \int_0^1 (v(x, t) - 1)^2 \, dx \right)^{\frac{1}{2}} + C \left( \int_0^1 u_x^2(x, t) \, dx \right)^{\frac{1}{2}} \\
\leq C \left( \|u_0\|_1^2 + \|v_0 - 1\|_1^2 \right)^{\frac{1}{2}},
\]

which implies

\[
|v(x, t) - 1| \leq C \left( \|u_0\|_2^2 + \|v_0 - 1\|_2^2 \right)^{\frac{1}{2}}.
\]

Similarly, we have

\[
|u(x, t)|, \ |u_x(x, t)|, \ |v_x(x, t)| \leq C \left( \|u_0\|_2^2 + \|v_0 - 1\|_2^2 \right)^{\frac{1}{2}}.
\]

By (4.1) and (4.2), it is easy to see that the a priori assumptions (2.7) hold provided $\|u_0\|_2^2 + \|v_0 - 1\|_2^2$ is sufficiently small. Therefore, the a priori assumptions (2.7) are always true provided $\|u_0\|_2^2 + \|v_0 - 1\|_2^2$ is sufficiently small.

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References


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