ON HOMEOMORPHISMS AND QUASI-ISOMETRIES
OF THE REAL LINE

PARAMESWARAN SANKARAN

(Communicated by Alexander N. Dranishnikov)

Abstract. We show that the group of piecewise-linear homeomorphisms of \( \mathbb{R} \) having bounded slopes surjects onto the group \( QI(\mathbb{R}) \) of all quasi-isometries of \( \mathbb{R} \). We prove that the following groups can be imbedded in \( QI(\mathbb{R}) \): the group of compactly supported piecewise-linear homeomorphisms of \( \mathbb{R} \), the Richard Thompson group \( F \), and the free group of continuous rank.

1. Introduction

We begin by recalling the notion of quasi-isometry. Let \( f : X \to X' \) be a map (which is not assumed to be continuous) between metric spaces. We say that \( f \) is a \( C \)-quasi-isometric embedding if there exists a \( C > 1 \) such that

\[
(*) \quad C^{-1}d(x, y) - C \leq d'(f(x), f(y)) \leq Cd(x, y) + C
\]

for all \( x, y \in X \). Here \( d, d' \) denote the metrics on \( X, X' \), respectively. If, further, every \( x' \in X' \) is within distance \( C \) from the image of \( f \), we say that \( f \) is a \( C \)-quasi-isometry. If \( f \) is a quasi-isometry (for some \( C \)), then there exists a quasi-isometry \( f'' : X' \to X \) (for a possibly different constant \( C' \)) such that \( f' \circ f \) (resp. \( f \circ f' \)) is quasi-isometry equivalent to the identity map of \( X \) (resp. \( X' \)). (Two maps \( f, g : X \to X \) are said to be quasi-isometrically equivalent if there exists a constant \( M \) such that \( d(f(x), g(x)) \leq M \) for all \( x \in X \).) Let \([f]\) denote the equivalence class of a quasi-isometry \( f : X \to X \). The set \( QI(X) \) of all equivalence classes of quasi-isometries of \( X \) is a group under composition: \([f] \circ [g] = [f \circ g] \) for \([f], [g] \in QI(X)\). If \( X' \) is quasi-isometry equivalent to \( X \), then \( QI(X') \) is isomorphic to \( QI(X) \). We refer the reader to [1] for basic facts concerning quasi-isometry. For example \( t \mapsto [t] \) is a quasi-isometry from \( \mathbb{R} \) to \( \mathbb{Z} \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be any homeomorphism of \( \mathbb{R} \). Denote by \( B(f) \) the set of break points of \( f \), i.e., points where \( f \) fails to have derivative and by \( \Lambda(f) \) the set of slopes of \( f \), i.e., \( \Lambda(f) = \{ f'(t) \mid t \in \mathbb{R} \setminus B(f) \} \). Note that \( B(f) \subset \mathbb{R} \) is discrete if \( f \) is piecewise differentiable.

Definition 1.1. We say that a subset \( \Lambda \) of \( \mathbb{R}^+ \), the set of non-zero real numbers, is bounded if there exists an \( M > 1 \) such that \( M^{-1} < |\lambda| < M \) for all \( \lambda \in \Lambda \). We say that a homeomorphism \( f \) of \( \mathbb{R} \) which is piecewise differentiable has bounded slopes if \( \Lambda(f) \) is bounded.
We denote by $PL_\delta(\mathbb{R})$ the set of all those piecewise-linear homeomorphisms $f$ of $\mathbb{R}$ such that $\Lambda(f)$ is bounded. It is clear that $PL_\delta(\mathbb{R})$ is a subgroup of the group $PL(\mathbb{R})$ of all piecewise-linear homeomorphisms of $\mathbb{R}$.

It is easy to see that each $f \in PL_\delta(\mathbb{R})$ is a quasi-isometry. (See Lemma 2.3 below.) One has a natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \to QI(\mathbb{R})$, where $\varphi(f) = [f]$ for all $f \in PL_\delta(\mathbb{R})$.

**Theorem 1.2.** The natural homomorphism $\varphi : PL_\delta(\mathbb{R}) \to QI(\mathbb{R})$, defined as $f \mapsto [f]$, is surjective.

If $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, recall that $\text{Supp}(f)$, the support of $f$, is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq x\}$ of all points moved by $f$. Denote by $PL_\kappa(\mathbb{R})$ the group of all piecewise-linear homeomorphisms of $\mathbb{R}$ which have compact support. It is obvious that $PL_\kappa(\mathbb{R}) \subset \ker(\varphi)$.

Let $\Gamma$ be a group of homeomorphisms of $S^1$. Any $f \in \Gamma$ can be lifted to obtain a homeomorphism $\tilde{f}$ of $\mathbb{R}$ over the covering projection $p : \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi \sqrt{-1}t)$. The set $\tilde{\Gamma}$ of all homeomorphisms of $\mathbb{R}$ which are lifts of elements of $\Gamma$ is a subgroup of the group $\text{Homeo}(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$. Indeed $\tilde{\Gamma}$ is a central extension of $\Gamma$ by the infinite cyclic group generated by translation by 1: $x \mapsto x + 1$. Denote by $Diff(S^1)$ the group of all $C^\infty$ diffeomorphisms of the circle. When $\Gamma$ is one of the groups $PL(S^1)$, $Diff(S^1)$, any element of $\tilde{f} \in \tilde{\Gamma}$ has bounded slope and is quasi-isometrically equivalent to the identity map of $\mathbb{R}$ (since $\tilde{f}(x + n) = \tilde{f}(x) + n$ for $n \in \mathbb{Z}$).

Recall that Richard Thompson discovered the group

$$F = \langle x_0, x_1, \cdots \mid x_i x_j x_i^{-1} = x_{j+1}, \ i < j \rangle$$

and used it in some constructions in logic related to word problems. The group $F$ is finitely presentable with two generators $x_0, x_1$ and two relations. This group and a closely related larger group $G$ have since then appeared in several contexts including homotopy theory [6], homological group theory [3], Teichmüller theory [9], etc. The group $F$ is isomorphic to the subgroup of piecewise-linear homeomorphisms of $\mathbb{R}$ which are the identity outside the unit interval $I$ such that $B(f)$ is contained in dyadic rationals and $\Lambda(f)$ is contained in the subgroup of $\mathbb{R}^*$ generated by 2. Although $F$ satisfies no (non-trivial) group law, it contains no non-abelian free group. The group $G$ is the group of piecewise-linear homeomorphisms $f$ of the circle $S^1 = I/(0,1)$ with $B(\tilde{f})$ contained in dyadic rationals and $\Lambda(\tilde{f})$ contained in the multiplicative subgroup of $\mathbb{R}^*$ generated by 2 for some lift $\tilde{f}$ of $f$. It is the first known example of a finitely presented infinite simple group. We recommend the beautiful survey article [4] for further information about Richard Thompson’s groups.

We shall prove the following theorem:

**Theorem 1.3.** The following groups can be imbedded in $QI(\mathbb{R})$:

(i) the groups $Diff(S^1)$ and $PL(S^1)$,
(ii) the group $PL_\kappa(\mathbb{R})$,
(iii) the Thompson’s group $F$, and
(iv) the free group of rank $c$, the continuum.

Our proofs are completely elementary. We explain the main idea of the proof of Theorem 1.3. Take for example the group $PL_\kappa(\mathbb{R})$. The first step is to realise
this as a subgroup $\Gamma_1$ of $PL_n(\mathbb{R})$ having support in $(0, 1)$. This is easily achieved by imbedding $\mathbb{R}$ in the interval $(0, 1)$. The group $\Gamma_1$ can be thought of as a group of piecewise-linear homeomorphisms of the circle. Lifting this back to $\mathbb{R}$ via the covering projection, we now obtain a group $\tilde{\Gamma}_1$ which no longer has compact support. However each element of this group is quasi-isometric to $id$. So we conjugate this group by a piecewise-linear homeomorphism whose slope grows exponentially. The result is that the features of each element of $\tilde{\Gamma}_1$ get magnified, resulting in a quasi-isometry not representing 1. The same trick works for $Diff(S^1)$ as well. Parts (iii) and (iv) follow from known embeddings of the relevant groups.

2. Proof of Theorem \[1.2\]

We first establish the following basic observation.

**Lemma 2.1.** Let $f$ be a piecewise differentiable homeomorphism of $\mathbb{R}$ with $\Lambda(f) \subset \mathbb{R}^*$ bounded. Then $f$ is a quasi-isometry.

**Proof.** Replacing $f$ by $-f$ if necessary, one may assume without loss of generality that $f$ is monotone increasing.

Suppose that $\Lambda(f) \subset (1/M, M)$. If $f$ is differentiable everywhere, then it is an $M$-quasi-isometry.

Suppose that $B(f) \neq \emptyset$. Let $a \in \mathbb{R}$. For any $b > a$, let $a_1 < \cdots < a_k$ be the points of $(a, b)$ where $f$ is non-differentiable. Then, applying the mean value theorem,

$$f(b) - f(a) = \sum_{0 \leq i \leq k} f(a_{i+1}) - f(a_i) = \sum_{0 \leq i \leq k} f'(c_i)(a_{i+1} - a_i)$$

for some $c_i \in (a_i, a_{i+1})$. Since $\Lambda(f) \subset (1/M, M)$, it follows that $M^{-1}(b-a) < f(b) - f(a) < M(b-a)$. Since $a, b \in \mathbb{R}$ are arbitrary, we conclude that $f$ is a quasi-isometry.

One has a well-defined map $\varphi : PL_3(\mathbb{R}) \longrightarrow QI(\mathbb{R})$ which is a homomorphism. We now prove that $\varphi$ is surjective.

**Lemma 2.2.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $C$-quasi-isometry that preserves the ends of $\mathbb{R}$. Let $x \in \mathbb{R}$. Then (i) there exists $y$ such that $y - x \leq 4C^2$ is positive integer and $f(y) > f(x)$; (ii) there exists $v$ such that $x - v \leq 4C^2$ is a positive integer and $f(x) > f(v)$.

**Proof.** If $f(x + 1) > f(x)$, then $y = x + 1$ meets our requirements.

Assume that $f(x) > f(y)$ for all $y$ such that $x + 1 \leq y < 4C^2 + x$. Let $z \geq x + 2$ be the smallest real number such that $z - x$ is a positive integer and $f(z) > f(x) \geq f(z - 1)$. Such a $z$ exists since $f(t) \to +\infty$ as $t \to +\infty$. By our assumption $z - x \geq 4C^2 + 1$. Set $u = z - 1$. Then the inequality (*) implies $f(u) < f(x) + C - C^{-1}(u - x) \leq f(x) - 3C$ and $f(z) - f(u) < C(z - u) + C = 2C$. Hence $f(z) < f(u) + 2C < f(x) - C$, i.e., $f(z) - f(x) < -C$. This contradicts our hypothesis that $f(z) > f(x)$, completing the proof of part (i). The proof of part (ii) is similar.

**Proof of Theorem \[1.2\]** Since the subgroup $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ that preserves the ends $\{+\infty, -\infty\}$ of $\mathbb{R}$ is of index 2 and since $PL_3(\mathbb{R})$ contains elements which are orientation reversing, it suffices to show that $QI^+(\mathbb{R})$ is contained in the image of $\varphi$, where $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ is the index 2 subgroup whose elements preserve the ends of $\mathbb{R}$.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $C$-quasi isometry, with $C > 1$, which preserves the ends of $\mathbb{R}$. We assume, as we may, that $C$ is a positive integer.
Set \(x_0 = 0\). We define \(x_k \in \mathbb{Z}\) for any integer \(k\) as follows: Let \(k \geq 1\). Having defined \(x_{k-1}\) inductively, choose \(x_k > x_{k-1}\) to be the smallest integer such that \(f(x_k) > f(x_{k-1})\). For any negative integer \(k\), we define \(x_k\) analogously (by downward induction) as the greatest integer such that \(x_k < x_{k+1}\) and \(f(x_k) < f(x_{k+1})\).

Set \(y_k := x_{C^2k}\), and let \(B := \{y_k | k \in \mathbb{Z}\} \subset \mathbb{Z}\). By Lemma 2.2 we see that \(B\) is a discrete subset of \(\mathbb{R}\) which is \(4C^3\)-dense in \(\mathbb{R}\). Note that for any \(k \in \mathbb{Z}\), \(y_k - y_{k-1} \geq C^3\).

Since \(f(y_k) > f(y_{k-1})\) for all \(k \in \mathbb{Z}\), there exists a unique piecewise-linear homeomorphism \(g : \mathbb{R} \to \mathbb{R}\) such that \(g(y_k) = f(y_k)\) and is linear on the interval \([y_{k-1}, y_k]\) for every \(k \in \mathbb{Z}\). We claim that \(g\) has bounded slopes. Since \(g\) is linear on each of the intervals \([y_{k-1}, y_k]\), we need only bound \(\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}}\). Indeed,

\[
g(y_k) - g(y_{k-1}) = \frac{f(y_k) - f(y_{k-1})}{y_k - y_{k-1}} < C + \frac{C}{y_k - y_{k-1}} \leq C + C^{-2}
\]

as \(y_k - y_{k-1} \geq C^3\). Similarly,

\[
\frac{g(y_k) - g(y_{k-1})}{y_k - y_{k-1}} > C^{-1} - C^{-2}.
\]

It follows that \(\Lambda(g) \subset [C^{-1} - C^{-2}, C + C^{-2}]\) and \(g \in PL_3(\mathbb{R})\).

Since \(f\) and \(g\) agree on the quasi-dense set \(B\), we see that \([f] = [g]\). This completes the proof. \(\square\)

**Remark 2.3.**  (i) By setting \(g(y_k)\) equal to a rational number sufficiently close to \(f(y_k)\) in the above proof, we see that since \(y_k \in \mathbb{Z}\), the element \(g \in PL_3(\mathbb{R})\) has rational slopes. Consequently it follows that \(\varphi\) restricted to the subgroup \(PL_\mathbb{Q}^+(\mathbb{R})\) of \(PL_3(\mathbb{R})\) consisting of those \(g \in PL_3(\mathbb{R})\) having slopes in \(\mathbb{Q}\) and \(B(g)\) contained in \(\mathbb{Q}\) is surjective.

(ii) The kernel of \(\varphi\) contains the group of all piecewise-linear homeomorphisms which have slope 1 outside a compact interval. This latter group equals the derived group \(PL'(\mathbb{R})\), where \(PL(\mathbb{R})\) denotes the subgroup of \(PL_3(\mathbb{R})\) consisting of homeomorphisms \(f\) for which \(B(f)\) is finite. Also \(PL_\mathbb{Q}(\mathbb{R}) = PL(\mathbb{Q})\). See [2].

3. **Proof of Theorem 1.3**

Let \(h_1 : \mathbb{R} \to (0, 1)\) be the homeomorphism defined by \(h_1(-x) = 1 - h_1(x)\) for every \(x \in \mathbb{R}\), \(h_1(n) = 1 - 1/(n + 2)\) for each integer \(n \geq 0\), and is linear on each interval \([n, n + 1]\) for \(n \in \mathbb{Z}\). If \(f\) is any compactly supported (piecewise-linear) homeomorphism of \(\mathbb{R}\), then \(h_1 \circ f \circ h_1^{-1}\) is a compactly supported (piecewise-linear) homeomorphism of \((0, 1)\). Since \(S^1 = I/(0, 1)\), we also get an embedding \(\bar{\eta} : PL_\mathbb{Q}(\mathbb{R}) \to PL(S^1)\), where \(\bar{\eta}(f)\) is defined to be the extension of \(h_1 \circ f \circ h_1^{-1}\) to \(S^1\).

We define \(\eta : PL_\mathbb{R}(\mathbb{R}) \to PL_\mathbb{R}(\mathbb{R})\) as the imbedding \(f \mapsto \eta(f)\), where \(\eta(f)(n) = n\) for \(n \in \mathbb{Z}\) and \(\eta(f)(x) = n + h_1 f h_1^{-1}(x - n)\) for \(n < x < n + 1\).

Let \(h_0 : \mathbb{R} \to \mathbb{R}\) be the piecewise-linear homeomorphism defined as follows: \(h_0(-x) = -h_0(x)\) \(\forall x \in \mathbb{R}\), \(h_0(x) = x\) for \(0 \leq x \leq 1\) and maps the interval \([n, n + 1]\) onto \([2n-1, 2n]\) linearly for each positive integer \(n\).

Suppose \(f : S^1 \to S^1\) is an orientation-preserving piecewise-linear homeomorphism or a diffeomorphism. Let \(\bar{f} : \mathbb{R} \to \mathbb{R}\) be any lift of \(f\) so that \(\bar{p} \circ \bar{f} = f \circ p\), where \(p : \mathbb{R} \to S^1\) is the covering projection \(t \mapsto \exp(2\pi \sqrt{-1}t)\). Then \([\bar{f}] = 1\) in
HOMEOMORPHISMS AND QUASI-ISOMETRIES OF THE REAL LINE 1879

Let \( f \) be one of the groups \( PL(S^1) \) or \( Diff(S^1) \) and let \( \tilde{f} \) be the group of homeomorphisms of \( p \) which are lifts of elements of \( \Gamma \) with respect to the covering projection \( p \). For \( \tilde{f} \in \tilde{\Gamma} \) set \( f_0 := h_0 f h_0^{-1} \). Clearly, \( f \mapsto f_0 \) is a monomorphism of groups \( \tilde{\Gamma} \rightarrow Homeo(\mathbb{R}) \). We claim that for any \( f \in \tilde{\Gamma} \), \( f_0 \) is a quasi-isometry.

To see this, we assume without loss of generality that \( f \) is orientation preserving. It is clear that \( f_0 \) is differentiable outside a discrete subset of \( \mathbb{R} \). We claim that \( f_0 \) has bounded slopes. Since \( f_0 \) has continuous derivatives on each interval on which \( f_0 \) has derivatives, it suffices to show that the set \( \{ f_0'(t) \} \) as \( t \) varies in \( \mathbb{R} \setminus B \) is bounded, where \( B \) is any discrete set which contains \( B(f_0) \). We set\( B := B(h_0) \cup h_0 B(\tilde{f}) \cup h_0 \tilde{f}^{-1} B(h_0) \).

Let \( 0 < m < M \) be such that \( m < \tilde{f}(x) < M \) for \( x \in \mathbb{R} \). Let \( t \in \mathbb{R} \setminus B \) and set \( s = h_0^{-1}(t), u = \tilde{f}(s) \) so that \( h_0^{-1}, \tilde{f}, h_0 \) are differentiable at \( t, s, u \), respectively. Consequently \( f_0 \) is differentiable at \( t \).

Since \( |u - s| = |\tilde{f}(s) - s| < q \), where \( q := ||\tilde{f}(0)|| + 2 \), we see that \( 2^{-q} < h_0'(u)/h_0'(s) < 2^q \). Using the chain rule, it follows that \( f_0'(t) = h_0'(u) \tilde{f}'(s)(h_0^{-1})'(t) = \tilde{f}'(s) h_0'(u)/h_0'(s) \) lies in the interval \( (2^{-q}m, 2^q M) \). It follows from Lemma 2.3 that \( f_0 \) is a quasi-isometry.

It is clear that the map \( \psi : \tilde{\Gamma} \rightarrow QI(\mathbb{R}) \) defined as \( \tilde{f} \mapsto [f_0] \) is a homomorphism.

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** We use the above notations throughout the proof.

(i) We prove that \( \psi : \tilde{\Gamma} \rightarrow QI(\mathbb{R}) \) is a monomorphism where \( \Gamma = PL(S^1) \) or \( Diff(S^1) \). Suppose that \( \tilde{f} \in \tilde{\Gamma} \), \( \tilde{f} \neq id \). We shall show that that \( |f_0 - id| \) is unbounded. Choose \( x \) in the interval \( [0, 1] \) such that \( \tilde{f}(x) \neq x \). Set \( k = \lceil \tilde{f}(x) \rceil \) so that \( \tilde{f}(x) = k + y, 0 \leq y < 1 \). Replacing \( \tilde{f} \) by its inverse if necessary, we assume without loss of generality that \( x < \tilde{f}(x) \). This implies that \( k \geq 0 \) with equality only if \( y > x \). For any positive integer \( n \), we have \( f_0(2^n + 2^nx) = h_0 f h_0^{-1}(2^n + 2^n x) = h_0 f(n + 1 + x) = h_0(n + 1 + k + y) = 2^{n+k} + 2^{n+y} \).

If \( k = 0 \), then \( y > x \) and so \( f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^n(y - x) \). Thus \( |f_0 - id| \) is unbounded.

If \( k > 0 \), then \( f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^{n+k} + 2^{n+k} y - 2^n - 2^n x \geq 2^{n+1} - 2^n - 2^n x = 2^n(1 - x) \). As \( 0 \leq x < 1 \), again it follows that \( |f_0 - id| \) is unbounded.

(ii) As observed earlier, \( \eta : PL_k(\mathbb{R}) \rightarrow PL_\eta(\mathbb{R}) \) is a monomorphism. It is evident that the image of \( \eta \) is contained in \( PL(S^1) \). Since \( \psi : PL(S^1) \rightarrow QI(\mathbb{R}) \) is a monomorphism by (i), assertion (ii) follows.

(iii) Now statement (iii) follows from (ii) above and the fact that Thompson’s group \( F \) is isomorphic to the subgroup of \( PL_k(\mathbb{R}) \) of all piecewise-linear homeomorphisms which have support in \([0, 1]\) having break points contained in the set of dyadic rationals in \([0, 1]\) and slopes contained in the multiplicative subgroup of \( \mathbb{R} \) generated by \( 2 \).

(iv) To prove (iv), recall that Grabowski [8] has shown that the free group of rank \( c \) the continuum embeds in the group of compactly supported \( C^k \) diffeomorphisms \((1 \leq k \leq \infty)\) of any positive-dimensional manifold. In particular, this is true of
$Diff(S^1)$. It follows easily that $\overline{Diff}(S^1)$ also contains a free group of rank the continuum. By part (i), this completes the proof. □

**Lemma 3.1.** The group $QI^+(\mathbb{R})$ is torsion-free.

**Proof.** Let $f \in PL_\delta(\mathbb{R})$ be such that $[f] \neq 1 \in QI^+(\mathbb{R})$. Thus $f-\text{id}$ is unbounded. Choose a sequence $(a_n)$ of real numbers such that $a_n \to +\infty$ as $n \to +\infty$ and $|f(a_n) - a_n| \to +\infty$. Let $k > 1$ be any integer. Suppose that $f(a_n) > a_n$. Since $f$ is order preserving, for each $n$ we have $a_n < f(a_n) < \cdots < f^k(a_n)$. In particular $f^k(a_n) - a_n > f(a_n) - a_n$. Similarly, $a_n - f^k(a_n) > a_n - f(a_n)$ in case $a_n > f(a_n)$. Therefore $[f^k(a_n) - a_n] > [f(a_n) - a_n] \forall n$ and hence $f^k - \text{id}$ is unbounded. Hence $[f^k] \neq 1$ in $QI^+(\mathbb{R})$ for $k > 1$. □

**Remark 3.2.** Thompson’s group $G$ does not imbed in $QI(\mathbb{R})$ since it has an element of order 3, whereas it follows from Lemma 3.1 that all torsion elements in $QI(\mathbb{R})$ are of order 2.

**Acknowledgements**

Part of this work was done while the author was visiting the University of Calgary, Alberta, Canada, during the spring and summer of 2003. It is a pleasure to thank Professors K. Varadarajan and P. Zvengrowski for their invitation and hospitality as well as financial support through their NSERC grants, making this visit possible.

**References**


INSTITUTE OF MATHEMATICAL SCIENCES, CITT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA

E-mail address: sankaran@imsc.res.in