ON HOMEOMORPHISMS AND QUASI-ISOMETRIES OF THE REAL LINE

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ABSTRACT. We show that the group of piecewise-linear homeomorphisms of \( \mathbb{R} \) having bounded slopes surjects onto the group \( QI(\mathbb{R}) \) of all quasi-isometries of \( \mathbb{R} \). We prove that the following groups can be imbedded in \( QI(\mathbb{R}) \): the group of compactly supported piecewise-linear homeomorphisms of \( \mathbb{R} \), the Richard Thompson group \( F \), and the free group of continuous rank.

1. Introduction

We begin by recalling the notion of quasi-isometry. Let \( f : X \to X' \) be a map (which is not assumed to be continuous) between metric spaces. We say that \( f \) is a \( C \)-quasi-isometric embedding if there exists a \( C > 1 \) such that

\[
C^{-1}d(x,y) - C \leq d'(f(x), f(y)) \leq Cd(x,y) + C
\]

for all \( x, y \in X \). Here \( d, d' \) denote the metrics on \( X, X' \), respectively. If, further, every \( x' \in X' \) is within distance \( C \) from the image of \( f \), we say that \( f \) is a \( C \)-quasi-isometry. If \( f \) is a quasi-isometry (for some \( C \)), then there exists a quasi-isometry \( f' : X' \to X \) (for a possibly different constant \( C' \)) such that \( f' \circ f \) (resp. \( f \circ f' \)) is quasi-isometry equivalent to the identity map of \( X \) (resp. \( X' \)). (Two maps \( f, g : X \to X \) are said to be quasi-isometrically equivalent if there exists a constant \( M \) such that \( d(f(x),g(x)) \leq M \) for all \( x \in X \).) Let \([f]\) denote the equivalence class of a quasi-isometry \( f : X \to X \). The set \( QI(X) \) of all equivalence classes of quasi-isometries of \( X \) is a group under composition: \([f],[g] = [f \circ g] \) for \([f],[g] \in QI(X)\). If \( X' \) is quasi-isometry equivalent to \( X \), then \( QI(X') \) is isomorphic to \( QI(X) \). We refer the reader to [1] for basic facts concerning quasi-isometry. For example \( t \mapsto [t] \) is a quasi-isometry from \( \mathbb{R} \) to \( \mathbb{Z} \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be any homeomorphism of \( \mathbb{R} \). Denote by \( B(f) \) the set of break points of \( f \), i.e., points where \( f \) fails to have derivative and by \( \Lambda(f) \) the set of slopes of \( f \), i.e., \( \Lambda(f) = \{ f(t) \mid t \in \mathbb{R} \setminus B(f) \} \). Note that \( B(f) \subset \mathbb{R} \) is discrete if \( f \) is piecewise differentiable.

Definition 1.1. We say that a subset \( \Lambda \) of \( \mathbb{R}^+ \), the set of non-zero real numbers, is bounded if there exists an \( M > 1 \) such that \( M^{-1} < |\lambda| < M \) for all \( \lambda \in \Lambda \). We say that a homeomorphism \( f \) of \( \mathbb{R} \) which is piecewise differentiable has bounded slopes if \( \Lambda(f) \) is bounded.
We denote by $PL_{\delta}(\mathbb{R})$ the set of all those piecewise-linear homeomorphisms $f$ of $\mathbb{R}$ such that $\Lambda(f)$ is bounded. It is clear that $PL_{\delta}(\mathbb{R})$ is a subgroup of the group $PL(\mathbb{R})$ of all piecewise-linear homeomorphisms of $\mathbb{R}$.

It is easy to see that each $f \in PL_{\delta}(\mathbb{R})$ is a quasi-isometry. (See Lemma 2.3 below.) One has a natural homomorphism $\varphi : PL_{\delta}(\mathbb{R}) \rightarrow QI(\mathbb{R})$, where $\varphi(f) = [f]$ for all $f \in PL_{\delta}(\mathbb{R})$.

**Theorem 1.2.** The natural homomorphism $\varphi : PL_{\delta}(\mathbb{R}) \rightarrow QI(\mathbb{R})$, defined as $f \mapsto [f]$, is surjective.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism, recall that $\text{Supp}(f)$, the support of $f$, is the closure of the set $\{x \in \mathbb{R} \mid f(x) \neq x\}$ of all points moved by $f$. Denote by $PL_{\kappa}(\mathbb{R})$ the group of all piecewise-linear homeomorphisms of $\mathbb{R}$ which have compact support. It is obvious that $PL_{\kappa}(\mathbb{R}) \subset \text{ker}(\varphi)$.

Let $\Gamma$ be a group of homeomorphisms of $S^1$. Any $f \in \Gamma$ can be lifted to obtain a homeomorphism $\tilde{f}$ of $\mathbb{R}$ over the covering projection $p : \mathbb{R} \rightarrow S^1$, $t \mapsto \exp(2\pi \sqrt{-1}t)$. The set $\tilde{\Gamma}$ of all homeomorphisms of $\mathbb{R}$ which are lifts of elements of $\Gamma$ is a subgroup of the group $\text{Homeo}(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$. Indeed $\tilde{\Gamma}$ is a central extension of $\Gamma$ by the infinite cyclic group generated by translation by 1: $x \mapsto x + 1$. Denote by $\text{Diff}(S^1)$ the group of all $C^\infty$ diffeomorphisms of the circle. When $\Gamma$ is one of the groups $PL(S^1)$, $\text{Diff}(S^1)$, any element of $\tilde{f} \in \tilde{\Gamma}$ has bounded slope and is quasi-isometrically equivalent to the identity map of $\mathbb{R}$ (since $\tilde{f}(x + n) = \tilde{f}(x) + n$ for $n \in \mathbb{Z}$).

Recall that Richard Thompson discovered the group

$$F = \langle x_0, x_1, \cdots \mid x_jx_i^{-1} = x_{j+1}, i < j \rangle$$

and used it in some constructions in logic related to word problems. The group $F$ is finitely presentable with two generators $x_0, x_1$ and two relations. This group and a closely related larger group $G$ have since then appeared in several contexts including homotopy theory [6], homological group theory [3], Teichmüller theory [4], etc. The group $F$ is isomorphic to the subgroup of piecewise-linear homeomorphisms of $\mathbb{R}$ which are the identity outside the unit interval $I$ such that $B(f)$ is contained in dyadic rationals and $\Lambda(f)$ is contained in the subgroup of $\mathbb{R}^\times$ generated by 2. Although $F$ satisfies no (non-trivial) group law, it contains no non-abelian free group. The group $G$ is the group of piecewise-linear homeomorphisms $f$ of the circle $S^1 = I/(0, 1)$ with $B(f)$ contained in dyadic rationals and $\Lambda(\tilde{f})$ contained in the multiplicative subgroup of $\mathbb{R}^\times$ generated by 2 for some lift $\tilde{f}$ of $f$. It is the first known example of a finitely presented infinite simple group. We recommend the beautiful survey article [4] for further information about Richard Thompson’s groups.

We shall prove the following theorem:

**Theorem 1.3.** The following groups can be imbedded in $QI(\mathbb{R})$:

(i) the groups $\text{Diff}(S^1)$ and $\bar{PL}(S^1)$,
(ii) the group $PL_{\kappa}(\mathbb{R})$,
(iii) the Thompson’s group $F$, and
(iv) the free group of rank $c$, the continuum.

Our proofs are completely elementary. We explain the main idea of the proof of Theorem 1.3. Take for example the group $PL_{\kappa}(\mathbb{R})$. The first step is to realise...
Proof of Theorem 1.2

This as a subgroup $\Gamma_1$ of $PL_\kappa(\mathbb{R})$ having support in $(0,1)$. This is easily achieved by imbedding $\mathbb{R}$ in the interval $(0,1)$. The group $\Gamma_1$ can be thought of as a group of piecewise-linear homeomorphisms of the circle. Lifting this back to $\mathbb{R}$ via the covering projection, we now obtain a group $\tilde{\Gamma}_1$ which no longer has compact support. However each element of this group is quasi-isometric to $id$. So we conjugate this group by a piecewise-linear homeomorphism whose slope grows exponentially. The result is that the features of each element of $\tilde{\Gamma}_1$ are magnified, resulting in a quasi-isometry not representing $1$. The same trick works for $Diff(S^1)$ as well. Parts (iii) and (iv) follow from known embeddings of the relevant groups.

2. Proof of Theorem 1.2

We first establish the following basic observation.

**Lemma 2.1.** Let $f$ be a piecewise differentiable homeomorphism of $\mathbb{R}$ with $\Lambda(f) \subset \mathbb{R}^*$ bounded. Then $f$ is a quasi-isometry.

**Proof.** Replacing $f$ by $-f$ if necessary, one may assume without loss of generality that $f$ is monotone increasing.

Suppose that $\Lambda(f) \subset (1/M, M)$. If $f$ is differentiable everywhere, then it is an $M$-quasi-isometry.

Suppose that $B(f) \neq \emptyset$. Let $a \in \mathbb{R}$. For any $b > a$, let $a_1 < \cdots < a_k$ be the points of $(a,b)$ where $f$ is non-differentiable. Then, applying the mean value theorem, $f(b) - f(a) = \sum_{0 \leq i \leq k} f(a_i) - f(a_i)$ for some $c_i \in (a_i, a_{i+1})$. Since $\Lambda(f) \subset (1/M, M)$, it follows that $M^{-1}(b-a) < f(b) - f(a) < M(b-a)$. Since $a, b \in \mathbb{R}$ are arbitrary, we conclude that $f$ is a quasi-isometry. □

One has a well-defined map $\varphi: PL_0(\mathbb{R}) \rightarrow QI(\mathbb{R})$ which is a homomorphism. We now prove that $\varphi$ is surjective.

**Lemma 2.2.** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a $C$-quasi-isometry that preserves the ends of $\mathbb{R}$. Let $x \in \mathbb{R}$. Then (i) there exists $y$ such that $y-x \leq 4C^2$ is positive integer and $f(y) > f(x)$; (ii) there exists $v$ such that $x-v \leq 4C^2$ is a positive integer and $f(x) > f(v)$.

**Proof.** If $f(x+1) > f(x)$, then $y = x + 1$ meets our requirements.

Assume that $f(x) > f(y)$ for all $y$ such that $x+1 \leq y < 4C^2 + x$. Let $z \geq x + 2$ be the smallest real number such that $z-x$ is a positive integer and $f(z) > f(x) \geq f(z-1)$. Such a $z$ exists since $f(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. By our assumption $z-x \geq 4C^2 + 1$. Set $u = z-1$. Then the inequality (*) implies $f(u) < f(x) + C - C^{-1}(u-x) \leq f(x) - 3C$ and $f(z) - f(u) < C(z-u) + C = 2C$. Hence $f(z) < f(u) + 2C < f(x) - C$, i.e., $f(z) - f(x) < -C$. This contradiction our hypothesis that $f(z) > f(x)$, completing the proof of part (i). The proof of part (ii) is similar. □

**Proof of Theorem 1.2** Since the subgroup $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ that preserves the ends $\{+\infty, -\infty\}$ of $\mathbb{R}$ is of index 2 and since $PL_0(\mathbb{R})$ contains elements which are orientation reversing, it suffices to show that $QI^+(\mathbb{R})$ is contained in the image of $\varphi$, where $QI^+(\mathbb{R}) \subset QI(\mathbb{R})$ is the index 2 subgroup whose elements preserve the ends of $\mathbb{R}$.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a $C$-quasi isometry, with $C > 1$, which preserves the ends of $\mathbb{R}$. We assume, as we may, that $C$ is a positive integer.
Set $x_0 = 0$. We define $x_k \in \mathbb{Z}$ for any integer $k$ as follows: Let $k \geq 1$. Having defined $x_{k-1}$ inductively, choose $x_k > x_{k-1}$ to be the smallest integer such that $f(x_k) > f(x_{k-1})$. For any negative integer $k$, we define $x_k$ analogously (by downward induction) as the greatest integer such that $x_k < x_{k+1}$ and $f(x_k) < f(x_{k+1})$.

Set $y_k := x_{C^k}$, and let $B := \{y_k : k \in \mathbb{Z}\} \subset \mathbb{Z}$. By Lemma 2.2 we see that $B$ is a discrete subset of $\mathbb{R}$ which is $4C^5$-dense in $\mathbb{R}$. Note that for any $k \in \mathbb{Z}$, $y_k - y_{k-1} \geq C^3$.

Since $f(y_k) > f(y_{k-1})$ for all $k \in \mathbb{Z}$, there exists a unique piecewise-linear homeomorphism $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(y_k) = f(y_k)$ and is linear on the interval $[y_{k-1}, y_k]$ for every $k \in \mathbb{Z}$. We claim that $g$ has bounded slopes. Since $g$ is linear on each of the intervals $[y_{k-1}, y_k]$, we need only bound $\frac{g(y_k)-g(y_{k-1})}{y_k-y_{k-1}}$. Indeed,

$$\frac{g(y_k)-g(y_{k-1})}{y_k-y_{k-1}} = \frac{f(y_k)-f(y_{k-1})}{y_k-y_{k-1}} < C + \frac{C}{y_k-y_{k-1}} \leq C + C^{-2}$$

as $y_k - y_{k-1} \geq C^3$. Similarly,

$$\frac{g(y_k)-g(y_{k-1})}{y_k-y_{k-1}} = \frac{g(y_{k-1})-g(y_k)}{y_{k-1}-y_k} \geq C^{-1} - C^{-2}.$$

It follows that $\Lambda(g) \subset [C^{-1} - C^{-2}, C + C^{-2}]$ and $g \in PL_3(\mathbb{R})$.

Since $f$ and $g$ agree on the quasi-dense set $B$, we see that $[f] = [g]$. This completes the proof. \qed

Remark 2.3. (i) By setting $g(y_k)$ equal to a rational number sufficiently close to $f(y_k)$ in the above proof, we see that since $y_k \in \mathbb{Z}$, the element $g \in PL_3(\mathbb{R})$ has rational slopes. Consequently it follows that $\varphi$ restricted to the subgroup $PL_3^\ast(\mathbb{R})$ of $PL_3(\mathbb{R})$ consisting of those $g \in PL_3(\mathbb{R})$ having slopes in $\mathbb{Q}^\ast$ and $B(g)$ contained in $\mathbb{Q}$ is surjective.

(ii) The kernel of $\varphi$ contains the group of all piecewise-linear homeomorphisms with slope 1 outside a compact interval. This latter group equals the derived group $PL''(\mathbb{R})$, where $PL(F) \mathbb{R}$ denotes the subgroup of $PL_3(\mathbb{R})$ consisting of homeomorphisms $f$ for which $B(f)$ is finite. Also $PL_\omega(\mathbb{R}) = PL''(\mathbb{R})$. See \[2\].

3. Proof of Theorem \[1,3\]

Let $h_1 : \mathbb{R} \rightarrow (0, 1)$ be the homeomorphism defined by $h_1(x) = 1 - h_1(x)$ for every $x \in \mathbb{R}$, $h_1(n) = 1 - 1/(n + 2)$ for each integer $n \geq 0$, and is linear on each interval $[n, n + 1]$ for $n \in \mathbb{Z}$. If $f$ is any compactly supported (piecewise-linear) homeomorphism of $\mathbb{R}$, then $h_1 \circ f \circ h_1^{-1}$ is a compactly supported (piecewise-linear) homeomorphism of $(0, 1)$. Since $S^1 = I/\{0, 1\}$, we also get an embedding $\tilde{\eta} : PL_\kappa(\mathbb{R}) \rightarrow PL(S^1)$, where $\tilde{\eta}(f)$ is defined to be the extension of $h_1 \circ f \circ h_1^{-1}$ to $S^1$.

We define $\eta : PL_\kappa(\mathbb{R}) \rightarrow PL(\mathbb{R})$ as the imbedding $f \mapsto \eta(f)$, where $\eta(f)(n) = n$ for $n \in \mathbb{Z}$ and $\eta(f)(x) = n + h_1 h_1^{-1}(x - n)$ for $n < x < n + 1$.

Let $h_0 : \mathbb{R} \rightarrow \mathbb{R}$ be the piecewise-linear homeomorphism defined as follows: $h_0(x) = -h_0(x)$ if $x \in \mathbb{R}$, $h_0(x) = x$ for $0 \leq x \leq 1$ and maps the interval $[n, n + 1]$ onto $[2^{n-1}, 2^n]$ linearly for each positive integer $n$.

Suppose $f : S^1 \rightarrow S^1$ is an orientation-preserving piecewise-linear homeomorphism or a diffeomorphism. Let $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ be any lift of $f$ so that $p \circ \tilde{f} = f \circ p$, where $p : \mathbb{R} \rightarrow S^1$ is the covering projection $t \mapsto \exp(2\pi \sqrt{-1}t)$. Then $[f] = 1$ in
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Proof of Theorem 1.3. We use the above notations throughout the proof.

(i) We prove that \( \psi : \tilde{\Gamma} \rightarrow QI(\mathbb{R}) \) is a monomorphism where \( \Gamma = PL(S^1) \) or \( Diff(S^1) \). Suppose that \( \tilde{f} \in \tilde{\Gamma}, \tilde{f} \neq id \). We shall show that that \( |f_0 - id| \) is unbounded. Choose \( x \) in the interval \([0, 1)\) such that \( \tilde{f}(x) \neq x \). Set \( k = [\tilde{f}(x)] \) so that \( \tilde{f}(x) = k + y, 0 \leq y < 1 \). Replacing \( \tilde{f} \) by its inverse if necessary, we assume without loss of generality that \( x < \tilde{f}(x) \). This implies that \( k \geq 0 \) with equality only if \( y > x \). For any positive integer \( n \), we have

\[
 f_0(2^n + 2^n x) = h_0 f_0^{-1}(2^n + 2^n x) = \tilde{h}_0(n + 1 + x) = \tilde{h}_0(n + 1 + \tilde{f}(x)) = \tilde{h}_0(n + 1 + k + y) = 2^n + 2^n k + 2^n y.
\]

If \( k = 0 \), then \( y > x \) and so \( f_0(2^n + 2^n x) = 2^n(y - x) \). Thus \( |f_0 - id| \) is unbounded.

If \( k > 0 \), then \( f_0(2^n + 2^n x) - (2^n + 2^n x) = 2^n + 2^n k y - 2^n - 2^n x \geq 2^n + 1 - 2^n = 2^n(1 - x) \). As \( 0 \leq x < 1 \), again it follows that \( |f_0 - id| \) is unbounded.

(ii) As observed earlier, \( \eta : PL_\delta(\mathbb{R}) \rightarrow PL(\mathbb{R}) \) is a monomorphism. It is evident that the image of \( \eta \) is contained in \( PL(S^1) \). Since \( \psi : PL(S^1) \rightarrow QI(\mathbb{R}) \) is a monomorphism by (i), assertion (ii) follows.

(iii) Now statement (iii) follows from (ii) above and the fact that Thompson’s group \( F \) is isomorphic to the subgroup of \( PL_\delta(\mathbb{R}) \) of all piecewise-linear homeomorphisms which have support in \([0, 1]\) having break points contained in the set of dyadic rationals in \([0, 1]\) and slopes contained in the multiplicative subgroup of \( \mathbb{R}^* \) generated by 2.

(iv) To prove (iv), recall that Grabowski [8] has shown that the free group of rank \( c \) the continuum embeds in the group of compactly supported \( C_1 \) diffeomorphisms \((1 \leq k \leq \infty)\) of any positive-dimensional manifold. In particular, this is true of
$Diff(S^1)$. It follows easily that $\overline{Diff(S^1)}$ also contains a free group of rank the continuum. By part (i), this completes the proof. □

**Lemma 3.1.** The group $QI^+(\mathbb{R})$ is torsion-free.

**Proof.** Let $f \in PL_\delta(\mathbb{R})$ be such that $[f] \neq 1 \in QI^+(\mathbb{R})$. Thus $f - id$ is unbounded. Choose a sequence $(a_n)$ of real numbers such that $a_n \to +\infty$ as $n \to +\infty$ and $|f(a_n) - a_n| \to +\infty$. Let $k > 1$ be any integer. Suppose that $f(a_n) > a_n$. Since $f$ is order preserving, for each $n$ we have $a_n < f(a_n) < \cdots < f^k(a_n)$. In particular $f^k(a_n) - a_n > f(a_n) - a_n$. Similarly, $a_n - f^k(a_n) > a_n - f(a_n)$ in case $a_n > f(a_n)$. Therefore $|f^k(a_n) - a_n| > |f(a_n) - a_n|$ for all $n$ and hence $f^k - id$ is unbounded. Hence $[f^k] \neq 1$ in $QI^+(\mathbb{R})$ for $k > 1$. □

**Remark 3.2.** Thompson’s group $G$ does not imbed in $QI(\mathbb{R})$ since it has an element of order 3, whereas it follows from Lemma 3.1 that all torsion elements in $QI(\mathbb{R})$ are of order 2.

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