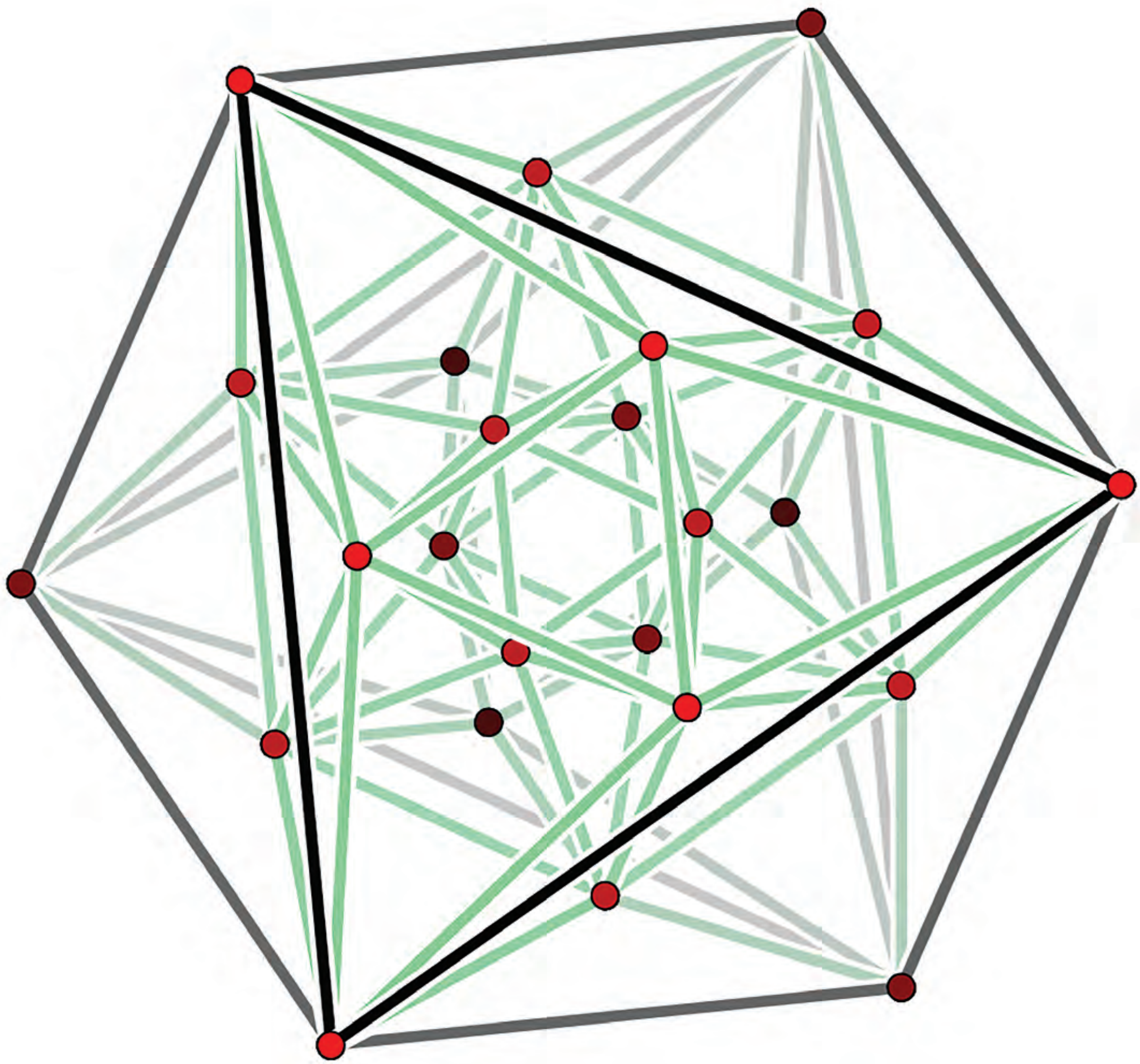
For Example: On Occasion of the Fiftieth Anniversary of Grünbaum's *Convex Polytopes*



Günter M. Ziegler

ABSTRACT. Let's take the fiftieth anniversary of the publication of Grünbaum's *Convex Polytopes* (1967) as an occasion for an excursion into polytope theory, FF looking for examples, images, and problems.

Are examples important? I think so! Just for illustration? Why do we need them if we understand the theory, you might ask. On the other hand, what's the worth of a theory for which there are no examples?

In his millennium survey on discrete geometry, Gil Kalai from Jerusalem put it this way:

> It is not unusual that a single example or a very few shape an entire mathematical discipline. Examples are the Petersen graph, cyclic polytopes, the Fano plane, the prisoner dilemma, the real n-dimensional projective space and the group of two by two nonsingular matrices. And it seems that overall, we are short of examples. The methods for coming up with useful examples in mathematics (or counterexamples for commonly believed conjectures) are even less clear than the methods for proving mathematical statements. [4, p. 769]

Discrete geometry, and the theory of convex polytopes in particular, thrives on a wealth of examples, which can be constructed, visualized, analyzed, classified, and admired.

The Icosahedron

Euclid's *Elements* ends with the construction and the classification of the five *Platonic solids*: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. What could be more classical than that? The British geometer Peter McMullen said that we should consider them as "wayside shrines at which one should worship on the way to higher things."

The most striking and enigmatic of the five platonic solids is arguably the icosahedron. The Mathematical Association of America (MAA) has it in its logo. However, the version of the MAA logo that was used from the early 1970s until 1984 got the geometry wrong, as Branko Grünbaum noticed in his highly entertaining paper "Geometry strikes again" [2]. Grünbaum also provided instructions for how to draw it correctly, and the MAA was quick to use this. The easiest and most striking way to get coordinates for the vertices of an icosahedron is to refer to the logo of the Berlin research center MATHEON, which recently celebrated its fifteenth anniversary: Write out coordinates $(\pm 1, \pm t, 0)$, $(0, \pm 1, \pm t)$, and $(\pm t, 0, \pm 1)$ for the vertices of the three rectangles, and discover that they form the vertices of a regular icosahedron exactly if t is the golden ratio or its inverse, $\frac{1}{2}(\pm 1 + \sqrt{5})$. Moreover, the boundary curves of the rectangles topologically represent a Borromean link: They cannot be separated, although they are not linked pairwise! See Figure 1.

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Figure 1. A regular icosahedron can be obtained from the corners of the three rectangles of the Matheon logo.

The Miller Solid

The next big step in polytope theory was a second class of examples, still classical: the *Archimedean solids*. They are polyhedra put together from regular polygons, which look the same at all the vertices. By tradition, the Platonic solids and the prisms and antiprisms are excluded. (Note that also the cube is a prism, and the octahedron is an antiprism.) There are only finitely many types. Who defined them first? Pappus of Alexandria credits Archimedes, but his writings are lost. Renaissance artists like Leonardo da Vinci and Albrecht Dürer searched for examples and produced amazing drawings (see Figure 2).

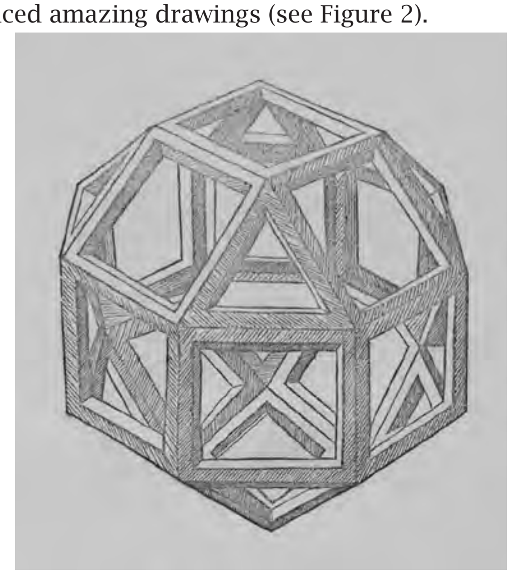

Figure 2. Leonardo da Vinci discovered and drew Archimedean solids, such as the pictured rhombicuboctahedron.

Finally, the astronomer and mathematician Johannes Kepler gave a classification of the thirteen Archimedean solids in his 1619 book *Harmonices Mundi*.

Throughout history, there have been two conflicting definitions of an Archimedean polyhedron, a local one:

> a convex polyhedron whose faces are regular polygons, and which have the same cyclic arrangement at each vertex,

and a global one:

> a convex polyhedron whose faces are regular polygons, and whose symmetry group acts transitively on the vertices.

Grünbaum pointed out that the definitions are not equivalent: there is one single example that satisfies the first one but not the second. In a Kepler-style pseudo-Greek naming scheme this would be the "pseudorhombicuboctahedron." Coxeter and Grünbaum call this 14th polyhedron "Miller's Solid" after J. C. P. Miller (1906–1980), who worked with Coxeter, but it turns out that a planar diagram appears already in a paper from 1906 by Duncan M'Laren Young Sommerville. Of course this is my favorite Archimedean solid, even if it is only pseudo-Archimedean.

The 24-Cell

Our next example (and, in the author's opinion, the most beautiful one of all of them) is a 4-dimensional polytope. One shouldn't be scared of such objects, as we can describe them in coordinates, analyze them using linear algebra, and even study them in pictures known as *Schlegel diagrams*, (see Figure 3), after Victor Schlegel (1843–1905).

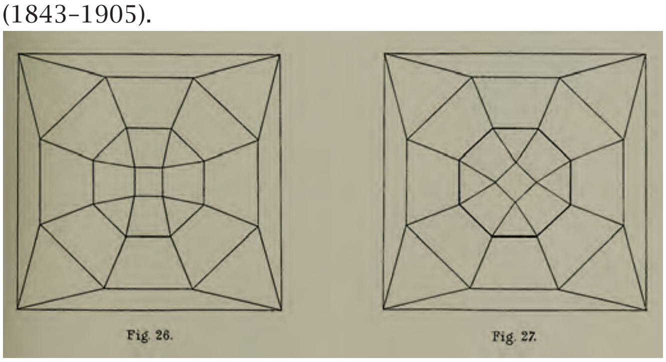

Figure 3. The Schlegel diagram of the rhombicuboctahedron and the pseudorhombicuboctahedron, which satisfies one definition of Archimedean solid but not another.

The Swiss geometer Ludwig Schläfli (1814–1895) classified, around 1850, the d-dimensional regular polytopes in all dimensions $d \geq 3$. It turns out that the most interesting case is in dimension $d = 4$, where there are six different types—even one more than in dimension 3! And the most interesting and exciting one of these six is the "24-cell": its vertices may be given by the centers of the 2-dimensional faces of a 4-dimensional cube. Thus, for example, one can take the 24 vectors in $\mathbb{R}^4$ that have two 0s and two ± 1s as coordinates; its faces then turn out to be 24 perfect regular octahedra. See Figure 4 for representations of a 24-cell.

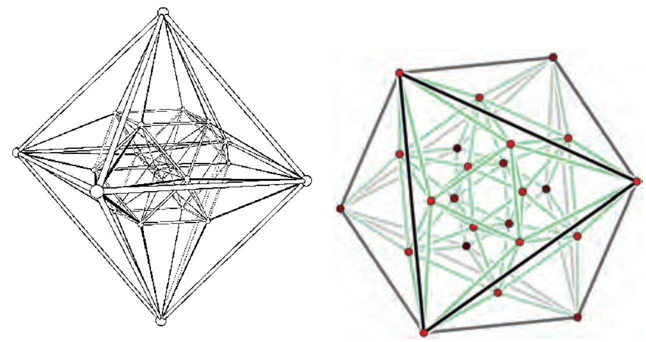

Figure 4. The regular 24**-cell in 4D.**

This turns out to be a polytope with remarkable properties. For example, it is the only centrally symmetric ($P = -P$) regular polytope that is isomorphic to its own dual! (The d-dimensional simplex is self-dual as well, but it is not centrally symmetric.)

The 24-cell has many more remarkable properties: for example, it is 2-*simplicial*, as all of its 2-faces are triangles, and it also has the dual property, being 2-*simple* (all edges lie in exactly three of the octahedron facets), as it is self-dual. Such 2-simplicial 2-simple 4-polytopes are rare. A few years ago, only eight examples of such *2s2s polytopes* were known: The simplex, the hypersimplex and its dual, a glued hypersimplex and its dual (constructed by Tom Braden), the 24-cell, and a 720-cell (derived from the 120-cell by Gábor Gévay) and its dual. Eppstein, Kuperberg, and Ziegler first managed to construct infinitely many examples. Now we know that a 2s2s 4-polytope with n vertices exists if and only if $n = 5$ or $n \geq 9$.

But still, the construction methods we have are not very flexible. We still cannot prove a conjecture by David Walkup that any 4-dimensional convex body can be approximated by 2s2s 4-polytopes.

Steinitz's Theorem and Grünbaum's Book

General polyhedra, in contrast to the regular (Platonic) or semiregular (Archimedean) ones, were studied only much later. What we now know as Euler's equation, $f_0 - f_1 + f_2 = 2$, was developed by Descartes and Euler, though Euler had to admit that he couldn't prove it. The combinatorial characterization of general polyhedra by 3-connected planar graphs was developed by Ernst Steinitz (1871–1928). However, the original version of Steinitz's theorem looked cumbersome and complicated. Only Grünbaum in his 1967 book *Convex Polytopes* presented Steinitz's theorem as we know it today:

Theorem 1 (Steinitz's theorem, Grünbaum's version [3, Sec. 13.1]). *A graph is realizable as a* 3*-dimensional polytope if and only if it is planar and* 3*-connected.*

Indeed, polytope theory as we know it today was largely shaped by Grünbaum's monumental monograph, which appeared fifty years ago: It is a treasure trove of results, methods, examples, and problems! In this paper we present examples of all four: results, methods, examples, and problems, all of them coming from Grünbaum's book.

Branko Grünbaum was born in 1929 in what was then Yugoslavia. In 1949 he emigrated to Jerusalem, where in 1957 he got his PhD with Aryeh Dvoretzky. After two years at the Institute in Princeton, Grünbaum returned to Jerusalem.

Figure 5. Polytope theory was largely shaped by Grünbaum's *Convex Polytopes*. Top: Grünbaum with his wife, Zdenka, and their son, Rami, about 1960. Bottom: Grünbaum in 1988.

In 1966, after a year at Michigan State, he finally settled at the University of Washington.

How did the book come about? Grünbaum writes:

> In Summer of 1963 I spent three months in Seattle, as guest of Vic Klee, ...This was a serendipitous time for what became "Convex Polytopes". Vic Klee was finishing his path-breaking papers on the Dehn–Sommerville equations and the Upper Bound Conjecture. I was working through the Steinitz-Rademacher book, translating Steinitz's complicated process of establishing the "Fundamentalsatz der konvexen Typen" into the easy-to-follow proof of the graph-theoretical formulation. On returning to Jerusalem I organized a seminar to study convex polyhedra and polytopes. The first topic I assigned dealt with Klee's recent results on f-vectors, based on preprints of his papers. The students came back to say they are not able to understand Klee's papers. Since I had to concede that the papers are hard to follow, I reformulated them in a much more understandable form—that worked for the students. My proof of Steinitz's theorem was easy to follow, and was presented to the seminar. During the following academic year (1964/65) I gave a course on combinatorics of convex polytopes, for which I prepared lecture notes.

These lecture notes, in particular, contained Grünbaum's version of Steinitz's theorem. This theorem is deep: By now we have three different types of proofs: the Steinitz-type proofs by local modifications [3, Sect. 13.1], the Tutte rubber band proofs, and the Koebe–Andreev–Thurston circle packing proofs. Each of these is different, each of them provides extra information and consequences the others don't, all of them are nontrivial. But mystery remains: For example, can any type of polyhedron be realized with small integer coordinates, say with vertices in $\{0, 1, \ldots, n^2\}^3$, if it has n vertices? We don't know!

Grünbaum's lecture notes with Steinitz's theorem in them quickly grew into a full-fledged monograph.

The Cyclic Polytopes

The cyclic polytopes appeared already in the opening Kalai quote. Their construction is deceptively simple: You take the convex hull of n points on the curve $\gamma(t) := (t, t^2, \ldots, t^d)$ in d-dimensional space and call the result a *cyclic polytope* $C_d(n)$.

For $d = 2$ these are n points on the standard parabola, so their convex hull is a convex n-gon. For $d = 3$ we get the *twisted cubic*. But for $d \geq 4$ things become interesting, as the resulting polytope turns out to be *neighborly*; that is, any two vertices are connected by an edge (and indeed more: any $\lfloor d/2 \rfloor$ of the points form the vertices of a face). Moreover, it turns out that the combinatorics of the polytope does not depend at all on which points you take on the moment curve: this is known as "Gale's evenness criterion" [3, p. 62], named after David Gale.

Apparently it was Constantin Carathéodory who discovered the cyclic polytopes in 1907 and found them to be neighborly. Theodore Motzkin studied them and computed their number of facets. However, in his 1956 abstract Motzkin also claimed that the cyclic polytopes are the only neighborly ones. We now know that this is dramatically wrong: there are *huge* numbers of neighborly d-dimensional polytopes on n vertices if d and n are large.

On a different aspect, Motzkin was right, however: He claimed that the cyclic polytope $C_d(n)$ has the maximal number of k-dimensional faces among all convex d-dimensional polytopes with n vertices. This became known as the "Upper Bound Conjecture," until Peter McMullen finally proved it in 1970.

Perles's Irrational Polytope

Grünbaum's book, on the way to the publisher, was delayed once more:

> When it seemed that the book is finally ready, I started getting long letters from Micha A. Perles.…Written in Hebrew, Perles was systematically developing the material on Gale transforms and diagrams, as well as improvements on many other results.

What Perles designed (and called "Gale diagrams," though they really should be called "Perles diagrams") was a magnificent linear algebra technique by which low-dimensional point configurations (say configurations of points in the plane) can be used to represent polytopes of low codimension (in the sense that their number of vertices is only little more than the dimension). This allowed him to classify and count d-dimensional polytopes with at most $d+3$ vertices. The most striking instance was Perles's irrational polytope: A configuration of twelve points in the plane that occupy nine different locations determined by a regular pentagon by Perles's technique represented an 8-dimensional polytope with twelve vertices, and the fact that the planar configuration cannot be drawn without using the golden ratio $\frac{1}{2}(-1+\sqrt{5})$ as a cross-ratio translated into the fact that the 8-dimensional polytope cannot be realized with rational vertex coordinates; see [5]. What a discovery!

With this final gem added, Grünbaum's book was eventually published by Wiley-Interscience in 1967, labeled "with the cooperation of Victor Klee, M. A. Perles, and G. C. Shephard." It was very well received, but it also went out of print quickly, and Wiley didn't want to reprint it, and later Grünbaum didn't want it reprinted without updates. After one or two failed attempts, a second edition with short notes on each chapter was eventually published in 2003. It received the AMS Steele Prize for Mathematical Exposition in 2005; the citation said that the book "has served both as a standard reference and as an inspiration for three and a half decades of research in the theory of polytopes" and its second edition "will extend the book's influence to future generations of mathematicians."

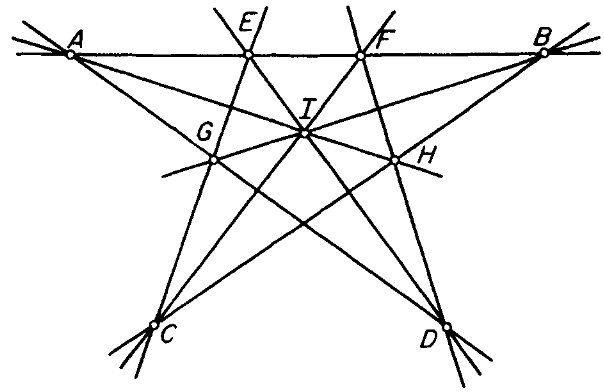

Figure 5.5.1

Figure 6. Gale represented interesting examples of polytopes by planar configurations. The Gale diagram for Perles's polytope without a rational realization [3, p. 93].

Are Examples Important?

Fields Medalist Gerd Faltings once said in a TV interview, "Whenever in my career I tried to work out an example, it has led me astray." Indeed, there are the theory guys in mathematics, who may do amazing things without looking at a single example—these are the "birds" in Freeman Dyson's categories of "birds and frogs" [1]—and there are the mathematicians whose work starts with interesting examples. I guess I am a frog.

ACKNOWLEDGMENTS. Thanks to Branko Grünbaum for decades of inspiration by his books and for his emails about the genesis of *Convex Polytopes*, to Isabella Novik for support, and to MSRI Berkeley and Lufthansa Business Class for wonderful writing environments.

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Figure 2 Da Vinci's regular rhombicubotahedron, with twenty-six open faces (*Vigintisex basium planum vacuum*) courtesy of the S.C. Williams Library, Archives and Special Collections; S.C. Williams Library, Stevens Institute of Technology, Hoboken, NJ.

Figure 3 Figures from Sommerville, D. (1906). XXIX. Semi-regular Networks of the Plane in Absolute Geometry. *Transactions of the Royal Society of Edinburgh*, **41**(3), 725–747 © Royal Society of Edinburgh 1906, published by Cambridge

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ABOUT THE AUTHOR

Günter M. Ziegler's work on convex polytopes has produced various intriguing classes of examples, among them "neighborly cubical polytopes," "projected deformed products of polygons," and "simple 4-polytopes without small separators." His *Lectures on Polytopes* appeared in 1994. In 2003 he co-edited the second edition of Grünbaum's *Convex Polytopes*.