# MEASURE RIGIDITY FOR RANDOM DYNAMICS ON SURFACES AND RELATED SKEW PRODUCTS 

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## 1. Introduction

Given an action of a one-parameter group on a manifold with some degree of hyperbolicity, there are typically many ergodic, invariant measures with positive entropy. For instance, given an Anosov or Axiom A diffeomorphism of a compact manifold, the equilibrium states for Hölder-continuous potentials provide measures with the above properties $\mathrm{BR}, \mathrm{Bow}$. When passing to hyperbolic actions of larger groups, the following phenomenon has been demonstrated in many settings: the only invariant ergodic measures with positive entropy are absolutely continuous (with respect to the ambient Riemannian volume). For instance, consider the action of the semi-group $\mathbb{N}^{2}$ on the additive circle generated by

$$
x \mapsto 2 x \bmod 1 \quad x \mapsto 3 x \bmod 1
$$

Rudolph showed for this action that the only invariant, ergodic probability measures are Lebesgue or have zero-entropy for every one-parameter subgroup [Rud. In KS, Katok and Spatzier generalized the above phenomenon to actions of commuting toral automorphisms.

Outside of the setting of affine actions, Kalinin, Katok, and Rodriguez Hertz, have recently demonstrated a version of abelian measure rigidity for nonuniformly hyperbolic, maximal-rank actions. In [KKRH], the authors consider $\mathbb{Z}^{n}$ acting by $C^{1+\alpha}$ diffeomorphisms on a $(n+1)$-dimensional manifold and prove that any $\mathbb{Z}^{n}$ invariant measure $\mu$ is absolutely continuous assuming that at least one element of $\mathbb{Z}^{n}$ has positive entropy with respect to $\mu$ and that the Lyapunov exponent functionals are in general position.

For affine actions of non-abelian groups, a number of results have recently been obtained by Benoist and Quint in a series of papers BQ1, BQ2, BQ3. For instance, consider a finitely supported measure $\nu$ on the group $\operatorname{SL}(n, \mathbb{Z})$. Let $\Gamma_{\nu} \subset \operatorname{SL}(n, \mathbb{Z})$ be the (semi-)group generated by the support of $\nu$. We note that $\Gamma_{\nu}$ acts naturally on the torus $\mathbb{T}^{n}$. In [BQ1], it is proved that if every finite-index subgroup of (the group generated by) $\Gamma_{\nu}$ acts irreducibly on $\mathbb{R}^{n}$, then every $\nu$-stationary probability measure on $\mathbb{T}^{n}$ is either finitely supported or is Haar; in particular every

[^0]$\nu$-stationary probability measure is $\mathrm{SL}(n, \mathbb{Z})$-invariant. Similar results was obtained in BFLM through completely different methods. In BQ1 the authors obtain similar stiffness results for groups of translations on quotients of simple Lie groups. More recently, in a breakthrough paper [EM] Eskin and Mirzakhani consider the natural action of the upper triangular subgroup $P \subset \mathrm{SL}(2, \mathbb{R})$ on a stratum of abelian differentials on a surface. They show that any such measures are in fact $\mathrm{SL}(2, \mathbb{R})$-invariant and affine in the natural coordinates on the stratum. Furthermore, for certain measures $\mu$ on $\operatorname{SL}(2, \mathbb{R})$, it is shown that all ergodic $\mu$-stationary measures are $\operatorname{SL}(2, \mathbb{R})$-invariant and affine.

In this article, we prove a number of measure rigidity results for dynamics on surfaces. We consider stationary measures for groups acting by diffeomorphisms on surfaces as well as skew products (or non-independent identically distributed (i.i.d.) random dynamics) with surface dynamics in the fibers. All measures will be hyperbolic either by assumption or by entropy considerations. In this setup we prove for hyperbolic stationary measures the following trichotomy: either the stable distributions are non-random, the measure is Sinai-Ruelle-Bowen, or the measure is supported on a finite set and is hence almost-surely invariant.

In the case that $\nu$-almost every (a.e.) diffeomorphism preserves a common smooth measure $m$, we show for any non-atomic stationary measure $\mu$ that either there exists a $\nu$-almost-surely invariant $\mu$-measurable line field (corresponding to the stable distributions for a.e. random composition) or the measure $\mu$ is $\nu$ -almost-surely invariant and coincides with an ergodic component of $m$.

In the proof of the above results, we study skew products with surface fibers over a measure-preserving transformation equipped with a decreasing sub- $\sigma$-algebra $\hat{\mathcal{F}}$. Given an invariant measure $\mu$ for the skew product whose fiber-wise conditional measures are non-atomic, we assume the $\hat{\mathcal{F}}$-measurability of the "past dynamics" and the fiber-wise conditional measures and prove the following dichotomy: either the fiber-wise stable distributions are measurable with respect to a related decreasing sub- $\sigma$-algebra, or the measure $\mu$ is fiber-wise SRB.

We focus here only on actions on surfaces and measures with non-zero exponents though we expect the results to hold in more generality. We rely heavily on the tools from the theory of non-uniformly hyperbolic diffeomorphisms used in KKRH and many ideas developed in EM including a modified version (see EM, Section 16]) of the "exponential drift" arguments from BQ1.

## 2. Preliminary definitions and constructions

Let $M$ be a closed (compact, boundaryless) $C^{\infty}$ Riemannian manifold. We write $\operatorname{Diff}^{r}(M)$ for the group of $C^{r}$-diffeomorphisms from $M$ to itself equipped with its natural $C^{r}$-topology. Fix $r=2$ and consider a subgroup $\Gamma \subset \operatorname{Diff}^{2}(M)$. We say a Borel probability measure $\mu$ on $M$ is $\Gamma$-invariant if

$$
\begin{equation*}
\mu\left(f^{-1}(A)\right)=\mu(A) \tag{2.1}
\end{equation*}
$$

for all Borel $A \subset M$ and all $f \in \Gamma$.
We note that for any continuous action by an amenable group on a compact metric space there always exists at least one invariant measure. However, for actions by non-amenable groups invariant measures need not exist. For this reason, we introduce a weaker notion of invariance. Let $\nu$ be a Borel probability measure on
the group $\Gamma$. We say a Borel probability measure $\mu$ on $M$ is $\nu$-stationary if

$$
\int \mu\left(f^{-1}(A)\right) d \nu(f)=\mu(A)
$$

for any Borel $A \subset M$. By the compactness of $M$, it follows that for any probability $\nu$ on $\Gamma$ there exists a $\nu$-stationary probability $\mu$ (e.g. Kif, Lemma I.2.2].)

We note that if $\mu$ is $\Gamma$-invariant, then $\mu$ is trivially $\nu$-stationary for any measure $\nu$ on $\Gamma$. Given a $\nu$-stationary measure $\mu$ such that equality (2.1) holds for $\nu$-a.e. $f \in \Gamma$, we say that $\mu$ is $\nu$-almost surely (a.s.) $\Gamma$-invariant.

Given a probability $\nu$ on $\operatorname{Diff}^{2}(M)$ one defines the random walk on the group of diffeomorphisms. A path in the random walk induces a sequence of diffeomorphisms from $M$ to itself. As in the case of a single transformation, we study the asymptotic ergodic properties of typical sequences of diffeomorphisms of $M$. We write $\Sigma_{+}=$ $\left(\operatorname{Diff}^{2}(M)\right)^{\mathbb{N}}$ for the space of sequences of diffeomorphisms $\omega=\left(f_{0}, f_{1}, f_{2}, \ldots\right) \in$ $\Sigma_{+}$. Given a Borel probability measure $\nu$ on $\operatorname{Diff}^{2}(M)$, we equip $\Sigma_{+}$with the product measure $\nu^{\mathbb{N}}$. We remark that $\operatorname{Diff}^{2}(M)$ is a Polish space, hence $\Sigma_{+}$is Polish, and the probability $\nu^{\mathbb{N}}$ is Radon. Let $\sigma: \Sigma_{+} \rightarrow \Sigma_{+}$be the shift map

$$
\sigma:\left(f_{0}, f_{1}, f_{2}, \ldots\right) \mapsto\left(f_{1}, f_{2}, \ldots\right)
$$

We have that $\nu^{\mathbb{N}}$ is $\sigma$-invariant. Given a sequence $\omega=\left(f_{0}, f_{1}, f_{2}, \ldots\right) \in \Sigma_{+}$and $n \geq 0$ we define a cocycle

$$
f_{\omega}^{0}:=\mathrm{Id}, \quad f_{\omega}=f_{\omega}^{1}:=f_{0}, \quad f_{\omega}^{n}:=f_{n-1} \circ f_{n-2} \circ \cdots \circ f_{1} \circ f_{0}
$$

We interpret $\left(\Sigma_{+}, \nu^{\mathbb{N}}\right)$ as a parametrization of all paths in the random walk defined by $\nu$. Following existing literature ( $[\boxed{\mathrm{LY} 3}, \boxed{\mathrm{LQ}})$, we denote by $\mathcal{X}^{+}(M, \nu)$ the random dynamical system on $M$ defined by the random compositions $\left\{f_{\omega}^{n}\right\}_{\omega \in \Sigma_{+}}$.

Given a measure $\nu$ on $\operatorname{Diff}^{2}(M)$ and a $\nu$-stationary measure $\mu$, we say a subset $A \subset M$ is $\mathcal{X}^{+}(M, \nu)$-invariant if for $\nu$-a.e. $f$ and $\mu$-a.e. $x \in M$,
(1) $x \in A \Longrightarrow f(x) \in A$ and
(2) $x \in M \backslash A \Longrightarrow f(x) \in M \backslash A$.

We say a $\nu$-stationary probability measure $\mu$ is ergodic if, for every $\mathcal{X}^{+}(M, \nu)$ invariant set $A$, we have either $\mu(A)=0$ or $\mu(M \backslash A)=0$. We note that for a fixed $\nu$-stationary measure $\mu$ we have an ergodic decomposition of $\mu$ into ergodic, $\nu$-stationary measures Kif, Proposition I.2.1].

For a fixed $\nu$ and a fixed $\nu$-stationary probability $\mu$, one can define the $\mu$-metric entropy of the random process $\mathcal{X}^{+}(M, \nu)$, written $h_{\mu}\left(\mathcal{X}^{+}(M, \nu)\right)$. We refer to Kif] for a definition.

In the case that the support of $\nu$ is not bounded in $\operatorname{Diff}^{2}(M)$, we assume the integrability condition

$$
\begin{equation*}
\int \log ^{+}\left(|f|_{C^{2}}\right)+\log ^{+}\left(\left|f^{-1}\right|_{C^{2}}\right) d \nu<\infty \tag{*}
\end{equation*}
$$

where $\log ^{+}(a)=\max \{\log (x), 0\}$ and $|\cdot|_{C^{2}}$ denotes the $C^{2}$-norm. The integrability condition (*) implies the weaker condition

$$
\begin{equation*}
\int \log ^{+}\left(|f|_{C^{1}}\right)+\log ^{+}\left(\left|f^{-1}\right|_{C^{1}}\right) d \nu<\infty \tag{2.2}
\end{equation*}
$$

which guarantees Oseledec's multiplicative ergodic theorem holds. The logintegrability of the $C^{2}$-norms is used later to apply tools from Pesin theory.

Proposition 2.1 (Random Oseledec's multiplicative theorem). Let $\nu$ be the measure on $\operatorname{Diff}^{2}(M)$ satisfying (2.2). Let $\mu$ be an ergodic, $\nu$-stationary probability.

Then there are numbers $-\infty<\lambda_{1}<\lambda_{2}<\cdots<\lambda_{\ell}<\infty$, called Lyapunov exponents such that for $\nu^{\mathbb{N}}$-a.e. sequence $\omega \in \Sigma_{+}$and $\mu$-a.e. $x \in M$ there is a filtration

$$
\begin{equation*}
\{0\}=V_{\omega}^{0}(x) \subsetneq V_{\omega}^{1}(x) \subset \cdots \subsetneq V_{\omega}^{\ell}(x)=T M \tag{2.3}
\end{equation*}
$$

such that for $v \in V_{\omega}^{k}(x) \backslash V_{\omega}^{k-1}(x)$

$$
\lim _{n \rightarrow \infty} \frac{1}{n}\left\|D_{x} f_{\omega}^{n}(v)\right\|=\lambda_{k}
$$

Moreover, $m_{i}:=\operatorname{dim} V_{\omega}^{k}(x)-\operatorname{dim} V_{\omega}^{k-1}(x)$ is constant a.s. and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{n} \log \left|\operatorname{det}\left(D_{x} f_{\xi}^{n}\right)\right|=\sum_{i=1}^{\ell} \lambda_{i} m_{i} . \tag{2.4}
\end{equation*}
$$

The subspaces $V_{\omega}^{i}(x)$ are invariant in the sense that

$$
D_{x} f_{\omega} V_{\omega}^{k}(x)=V_{\sigma(\omega)}^{k}\left(f_{\omega}(x)\right) .
$$

For a proof of the above theorem see, for example, $[\mathrm{LQ}$, Proposition I.3.1]. We write

$$
E_{\omega}^{s}(x):=\bigcup_{\lambda_{j}<0} V_{\omega}^{j}(x)
$$

for the stable Lyapunov subspace for the word $\omega$ at the point $x$.
A stationary measure $\mu$ is hyperbolic if no exponent $\lambda_{i}$ is zero.
We note that the random process $\mathcal{X}^{+}(M, \nu)$ is not invertible. Thus, while stable Lyapunov subspaces are defined for $\nu^{\mathbb{N}}$-a.e. $\omega$ and $\mu$-a.e. $x$, there are no well-defined unstable Lyapunov subspaces for $\mathcal{X}^{+}(M, \nu)$. However, to state results we will need a notion of SRB-measures (also called $u$-measures) for random sequences of diffeomorphisms. We will state the precise definition (Definition 6.8) in Section 6.3 after introducing fiber-wise unstable manifolds for a related skew product construction. Roughly speaking, a $\nu$-stationary measure $\mu$ is SRB if it has absolutely continuous conditional measures along unstable manifolds. Since we have not yet defined unstable manifolds (or subspaces), we postpone the formal definition and give here an equivalent property. The following is an adaptation of LY1.

Proposition 2.2 ( LQ , Theorem VI.1.1]). Let $M$ be a compact manifold, and let $\nu$ be a probability on Diff $^{2}(M)$ satisfying ( $(*)$. Then an ergodic, $\nu$-stationary probability $\mu$ is an SRB-measure if and only if

$$
h_{\mu}\left(\mathcal{X}^{+}(M, \nu)\right)=\sum_{\lambda_{i}>0} m_{i} \lambda_{i} .
$$

We introduce some terminology for invariant measurable subbundles. Given a subgroup $\Gamma \subset \operatorname{Diff}^{2}(M)$, we have induced the action of $\Gamma$ on sub-vector-bundles of the tangent bundle $T M$ via the differential. Consider $\nu$ supported on $\Gamma$ and a $\nu$-stationary Borel probability $\mu$ on $M$.
(1) We say a $\mu$-measurable subbundle $V \subset T M$ is $\nu$-a.s. invariant if $D f(V(x))=$ $V(f(x))$ for $\nu$-a.e. $f \in \Gamma$ and $\mu$-a.e. $x \in M$.
（2）A $\left(\nu^{\mathbb{N}} \times \mu\right)$－measurable family of subbundles $(\omega, x) \mapsto V_{\omega}(x) \subset T_{x} M$ is $\mathcal{X}^{+}(\Gamma, \nu)$－invariant if for $\left(\nu^{\mathbb{N}} \times \mu\right)$－a．e．$(\omega, x)$

$$
D_{x} f_{\omega} V_{\omega}(x)=V_{\sigma(\omega)}\left(f_{\omega}(x)\right)
$$

Note that subbundles in the filtration（2．3）are $\mathcal{X}^{+}(\Gamma, \nu)$－invariant．
（3）We say a $\mathcal{X}^{+}(M, \nu)$－invariant family of subspaces $V_{\omega}(x) \subset T M$ is non－ random if there exists a $\nu$－a．s．invariant $\mu$－measurable subbundle $\hat{V} \subset T M$ with $\hat{V}(x)=V_{\omega}(x)$ for $\left(\nu^{\mathbb{N}} \times \mu\right)$－a．e．$(\omega, x)$ ．

## 3．Statement of results：groups of surface diffeomorphisms

For all results in this paper，we restrict ourselves to the case that $M$ is a closed surface．Equip $M$ with a background Riemannian metric．

Let $\hat{\nu}$ be a Borel probability on the group $\operatorname{Diff}^{2}(M)$ satisfying the integrability hypotheses（柬）．Let $\hat{\mu}$ be an ergodic $\hat{\nu}$－stationary measure on $M$ ．At times，we may assume $h_{\hat{\mu}}\left(\mathcal{X}^{+}(M, \hat{\nu})\right)>0$ ．By the fiber－wise Margulis－Ruelle inequality［BB ap－ plied to the associated skew product（see Section 4．1），positivity of entropy implies that the Oseledec＇s filtration（2．3）is non－trivial and the exponents satisfy

$$
\begin{equation*}
-\infty<\lambda_{1}<0<\lambda_{2}<\infty . \tag{3.1}
\end{equation*}
$$

In particular，the stable Lyapunov subspace $E_{\omega}^{s}(x)$ corresponds to the subspace $V_{\omega}^{1}(x)$ in（2．3）and is one－dimensional．

We state our first main theorem．
Theorem 3．1．Let $M$ be a closed surface，and let $\hat{\nu}$ be a Borel probability measure on $\operatorname{Diff}^{2}(M)$ satisfying（娄）．Let $\hat{\mu}$ be an ergodic，hyperbolic，$\hat{\nu}$－stationary Borel probability measure on $M$ ．Then either
（1）the stable distribution $E_{\omega}^{s}(x)$ is non－random，
（2）$\hat{\mu}$ is finitely supported，and hence $\hat{\nu}$－a．s．is invariant，or
（3）$\hat{\mu}$ is $S R B$ ．
By the above discussion and standard facts about entropy，if $h_{\hat{\mu}}\left(\mathcal{X}^{+}(M, \hat{\nu})\right)>0$ ， then $\hat{\mu}$ is hyperbolic and has no atoms．We thus obtain as a corollary the following dichotomy for positive－entropy stationary measures．

Corollary 3．2．Let $M$ be a closed surface．Let $\hat{\nu}$ be a Borel probability measure on $\operatorname{Diff}^{2}(M)$ satisfying（娄），and let $\hat{\mu}$ be an ergodic，$\hat{\nu}$－stationary Borel probability measure on $M$ with $h_{\hat{\mu}}\left(\mathcal{X}^{+}(M, \hat{\nu})\right)>0$ ．Then either
（1）the stable distribution $E_{\omega}^{s}(x)$ is non－random，or
（2）$\hat{\mu}$ is $S R B$ ．
We also immediately obtain from Theorem 3.1 the following corollary．
Corollary 3．3．Let $\hat{\nu}$ be as in Theorem 3.1 with $\hat{\mu}$ an ergodic，hyperbolic，$\hat{\nu}$－ stationary probability measure．Assume that $\hat{\mu}$ has one exponent of each sign and that there are no $\hat{\nu}$－a．s．invariant，$\hat{\mu}$－measurable line fields on $T M$ ．Then either $\hat{\mu}$ is SRB or $\hat{\mu}$ is finitely supported．

We note that in BQ1，the authors prove an analogous statement．Namely， for homogeneous actions satisfying certain hypotheses，any non－atomic stationary measure $\hat{\mu}$ is shown to be absolutely continuous along some unstable（unipotent） direction．Using the Ratner theory，one concludes that the stationary measure $\hat{\mu}$
is thus the Haar measure and hence invariant for every element of the action. In non-homogeneous settings, such as the one considered here and the one considered in (EM, there is no analogue of the Ratner theory. Thus, in such settings more structure is needed in order to promote the SRB property to absolute continuity or almost-sure invariance of the stationary measure $\hat{\mu}$. The next theorem demonstrates that this promotion is possible assuming the existence of an almost-surely invariant volume.

Theorem 3.4. Let $\Gamma \subset \operatorname{Diff}^{2}(M)$ be a subgroup, and assume $\Gamma$ preserves a probability measure $m$ equivalent to the Riemannian volume on $M$. Let $\hat{\nu}$ be a probability measure on $\operatorname{Diff}^{2}(M)$ with $\hat{\nu}(\Gamma)=1$ and satisfying (㘢). Let $\hat{\mu}$ be an ergodic, hyperbolic, $\hat{\nu}$-stationary Borel probability measure. Then either
(1) $\hat{\mu}$ has finite support,
(2) the stable distribution $E_{\omega}^{s}(x)$ is non-random, or
(3) $\hat{\mu}$ is absolutely continuous and is $\hat{\nu}$-a.s. $\Gamma$-invariant.

Furthermore, in conclusion (3), we will have that $\hat{\mu}$ is-up to normalization-the restriction of $m$ to a positive volume subset.

In particular, in Theorem 3.4 if the stable distribution $E_{\omega}^{s}(x)$ is not non-random, then we have the following stiffness result.
Corollary 3.5. Let $m$ be a probability measure on $M$ equivalent to the Riemannian volume. Let $\hat{\nu}$ be a probability measure on $\operatorname{Diff}^{2}(M)$ satisfying (困) and such that $m$ is $\hat{\nu}$-a.s. invariant. Let $\hat{\mu}$ be an ergodic, hyperbolic, $\hat{\nu}$-stationary Borel probability measure. Assume there are no $\hat{\mu}$-measurable, $\hat{\nu}$-a.s. invariant line fields on TM. Then $\hat{\mu}$ is invariant under $\hat{\nu}$-a.e. $f \in \operatorname{Diff}^{2}(M)$.

## 4. General skew products

In this section, we reformulate the results stated in Section 3 in terms of results about related skew product systems. This allows us to convert the dynamical properties of non-invertible, random dynamics to properties of one-parameter invertible actions and to exploit tools from the theory of non-uniformly hyperbolic diffeomorphisms. A result for a more general skew product system is also introduced.
4.1. Canonical skew product associated to a random dynamical system. Let $M$ and $\hat{\nu}$ be as in Section 3 Consider the product space $\Sigma_{+} \times M$, and define the (non-invertible) skew product $\hat{F}: \Sigma_{+} \times M \rightarrow \Sigma_{+} \times M$ by

$$
\hat{F}:(\omega, x) \mapsto\left(\sigma(\omega), f_{\omega}(x)\right) .
$$

We have the following reinterpretation of $\hat{\nu}$-stationary measures.
Proposition 4.1. Kifl Lemma I.2.3, Theorem I.2.1] For a Borel probability measure $\hat{\mu}$ on $M$ we have that
(1) $\hat{\mu}$ is $\hat{\nu}$-stationary if and only if $\hat{\nu}^{\mathbb{N}} \times \hat{\mu}$ is $\hat{F}$-invariant;
(2) a $\hat{\nu}$-stationary measure $\hat{\mu}$ is ergodic for $\mathcal{X}^{+}(M, \hat{\nu})$ if and only if $\hat{\nu}^{\mathbb{N}} \times \hat{\mu}$ is ergodic for $\hat{F}$.
Let $\Sigma:=\left(\operatorname{Diff}^{r}(M)\right)^{\mathbb{Z}}$ be the space of bi-infinite sequences and equip $\Sigma$ with the product measure $\hat{\nu}^{\mathbb{Z}}$. We again write $\sigma: \Sigma \rightarrow \Sigma$ for the left shift $(\sigma(\xi))_{i}=\xi_{i+1}$. Given

$$
\xi=\left(\ldots, f_{-2}, f_{-1}, f_{0}, f_{1}, f_{2}, \ldots\right) \in \Sigma
$$

define $f_{\xi}:=f_{0}$, and define the (invertible) skew product $F: \Sigma \times M \rightarrow \Sigma \times M$ by

$$
\begin{equation*}
F:(\xi, x) \mapsto\left(\sigma(\xi), f_{\xi}(x)\right) . \tag{4.1}
\end{equation*}
$$

We have the following proposition producing the measure whose properties we will study for the remainder.

Proposition 4.2. Let $\hat{\mu}$ be a $\hat{\nu}$-stationary Borel probability measure. There is a unique $F$-invariant Borel probability measure $\mu$ on $\Sigma \times M$ whose image under the canonical projection $\Sigma \times M \rightarrow \Sigma_{+} \times M$ is $\hat{\nu}^{\mathbb{N}} \times \hat{\mu}$.

Furthermore, $\mu$ projects to $\hat{\nu}^{\mathbb{Z}}$ and $\hat{\mu}$, respectively, under the canonical projections $\Sigma \times M \rightarrow \Sigma$ and $\Sigma \times M \rightarrow M$ and is equal to the weak-* limit

$$
\begin{equation*}
\mu=\lim _{n \rightarrow \infty}\left(F^{n}\right)_{*}\left(\hat{\nu}^{\mathbb{Z}} \times \hat{\mu}\right) . \tag{4.2}
\end{equation*}
$$

See, for example, $[\mathrm{LQ}$, Proposition I.1.2] for a proof of the proposition in this setting. Let $\left\{\mu_{\xi}\right\}_{\xi} \in \Sigma$ be a family of conditional measures of $\mu$ relative to the partition into fibers of $\Sigma \times M \rightarrow \Sigma$. By a slight abuse of notation, consider $\mu_{\xi}$ as a measure on $M$ for each $\xi$. It follows that for $\hat{\nu}^{\mathbb{Z}}$-a.e. $\xi \in \Sigma$ and $\eta \in \Sigma$ with $\eta_{i}=\xi_{i}$ for all $i<0$ that $\mu_{\eta}=\mu_{\xi}$.

Write $\pi: \Sigma \times M \rightarrow \Sigma$ for the canonical projection. We write $h_{\mu}(F \mid \pi)$ for the conditional metric entropy of $(F, \mu)$ conditioned on the sub- $\sigma$-algebra generated by $\pi^{-1}$.

Proposition 4.3 ([Kif, Theorem II.1.4], LQ, Theorem I.2.3]). We have the equality of entropies $h_{\hat{\mu}}\left(\mathcal{X}^{+}(M, \hat{\nu})\right)=h_{\mu}(F \mid \pi)$.
4.2. General skew products. We give a generalization of the setup introduced in Section 4.1. Let $\left(\Omega, \mathcal{B}_{\Omega}, \nu\right)$ be a Polish probability space; that is, $\Omega$ has the topology of a complete separable metric space, $\nu$ is a Borel probability measure, and $\mathcal{B}_{\Omega}$ is the $\nu$-completion of the Borel $\sigma$-algebra. Let $\theta:\left(\Omega, \mathcal{B}_{\Omega}, \nu\right) \rightarrow\left(\Omega, \mathcal{B}_{\Omega}, \nu\right)$ be an invertible, ergodic, measure-preserving transformation. Let $M$ be a closed $C^{\infty}$ manifold. Fix a background $C^{\infty}$ Riemannian metric on $M$, and write $\|\cdot\|$ for the norm on the tangent bundle $T M$ and $d(\cdot, \cdot)$ for the induced distance on $M$. We note that compactness of $M$ guarantees all metrics are equivalent, whence all dynamical object structures defined below are independent of the choice of metric.

We consider a $\nu$-measurable mapping $\Omega \ni \xi \mapsto f_{\xi} \in \operatorname{Diff}^{2}(M)$. Definđ ${ }^{1}$ a cocycle $\mathscr{F}: \Omega \times \mathbb{Z} \rightarrow \operatorname{Diff}^{r}(M)$ over $\theta$, written $\mathscr{F}:(\xi, n) \mapsto f_{\xi}^{n}$, by
(1) $f_{\xi}^{0}:=\mathrm{Id}, f_{\xi}^{1}:=f_{\xi}$,
(2) $f_{\xi}^{n}:=f_{\theta^{n-1}(\xi)} \circ \cdots \circ f_{\theta(\xi)} \circ f_{\xi}$ for $n>0$, and
(3) $f_{\xi}^{n}:=\left(f_{\theta^{n}(\xi)}\right)^{-1} \circ \cdots \circ\left(f_{\theta^{-1}(\xi)}\right)^{-1}=\left(f_{\theta^{n}(\xi)}^{|n|}\right)^{-1}$ for $n<0$.

As above, we will always assume the following integrability condition:

$$
\begin{equation*}
\int \log ^{+}\left(\left|f_{\xi}\right|_{C^{2}}\right)+\log ^{+}\left(\left|f_{\xi}^{-1}\right|_{C^{2}}\right) d \nu(\xi)<\infty \tag{IC}
\end{equation*}
$$

Write $X:=\Omega \times M$ with canonical projection $\pi: X \rightarrow \Omega$. For $\xi \in \Omega$, we write

$$
M_{\xi}:=\{\xi\} \times M=\pi^{-1}(\xi)
$$

[^1]for the fiber of $X$ over $\xi$. On $X$, we define the skew product $F: X \rightarrow X F:(\xi, x) \mapsto$ $\left(\theta(\xi), f_{\xi}(x)\right)$.

Note that $X=\Omega \times M$ has a natural Borel structure. The main object of study for the remainder will be $F$-invariant Borel probability measures on $X$ with marginal $\nu$.

Definition 4.4. A probability measure $\mu$ on $X$ is called $\mathscr{F}$-invariant if it is $F$ invariant and satisfies

$$
\pi_{*} \mu=\nu
$$

Such a measure $\mu$ is said to be ergodic if it is $F$-ergodic.
Let $\left\{\mu_{\xi}\right\}_{\xi \in \Omega}$ denote the family of conditional probability measures with respect to the partition induced by the projection $\pi: X \rightarrow \Omega$. Using the canonical identification of fibers $M_{\xi}=\{\xi\} \times M$ in $X$ with $M$, by an abuse of notation we consider the map $\xi \mapsto \mu_{\xi}$ as a measurable map from $\Omega$ to the space of Borel probabilities on $M$.
4.2.1. Fiber-wise Lyapunov exponents. We define $T X$ to be the fiber-wise tangent bundle

$$
T X:=\Omega \times T M
$$

and $D F: T X \rightarrow T X$ to be the fiber-wise differential

$$
D F:(\xi,(x, v)) \mapsto\left(\theta(\xi),\left(f_{\xi}(x), D_{x} f_{\xi} v\right)\right) .
$$

Let $\mu$ be an ergodic, $\mathscr{F}$-invariant probability. We have that $D F$ defines a linear cocycle over the (invertible) measure-preserving system $F:(X, \mu) \rightarrow(X, \mu)$. By the integrability condition (IC), we can apply Oseledec's theorem to $D F$ to obtain a $\mu$-measurable splitting

$$
\begin{equation*}
T_{(\xi, x)} X:=\{\xi\} \times T_{x} M=\bigoplus_{j} E^{j}(\xi, x) \tag{4.3}
\end{equation*}
$$

and numbers $\lambda_{\mu}^{j}$ so that for $\mu$-a.e. $(\xi, x)$, and every $v \in E^{j}(\xi, x) \backslash\{0\}$,

$$
\lim _{n \rightarrow \pm \infty} \frac{1}{n} \log \left\|D F^{n}(v)\right\|=\lim _{n \rightarrow \pm \infty} \frac{1}{n} \log \left\|D_{x} f_{\xi}^{n} v\right\|=\lambda_{\mu}^{j} .
$$

It follows from standard arguments that if the fiber-wise exponents of $D F$ are all positive (or negative), then the fiber-wise conditional measure $\mu_{\xi}$ are purely atomic.
4.3. Reformulation of Theorem [3.1, Let $M$ and $\hat{\nu}$ be as in Section 3] and let $\hat{\mu}$ be an ergodic, hyperbolic, $\hat{\nu}$-stationary measure. Let $F: \Sigma \times M \rightarrow \Sigma \times M$ denote the canonical skew product, and let $\mu$ be the measure given by Proposition 4.2. We have a $\mu$-measurable splitting of $\Sigma \times T M$ into measurable bundles

$$
\{\xi\} \times T_{x} M=E^{s}(\xi, x) \oplus E^{u}(\xi, x) .
$$

Note that, a priori, one of the bundles $E^{s}(\xi, x)$ or $E^{u}(\xi, x)$ might be trivial; however, by Remark 4.9 below, Theorem 3.1 follows trivially in these cases.

For $\sigma \in\{s, u\}$ and $(\xi, x) \in \Sigma \times M$ we write $E_{\xi}^{\sigma}(x) \subset T M$ for the subspace with $E^{\sigma}(\xi, x)=\{\xi\} \times E_{\xi}^{\sigma}(x)$. Projectivizing the tangent bundle $T M$, we obtain a measurable function

$$
(\xi, x) \mapsto E_{\xi}^{\sigma}(x) .
$$

For $\xi=\left(\ldots, \xi_{-2}, \xi_{-1}, \xi_{0}, \xi_{1}, \xi_{2}, \ldots\right) \in \Sigma$ write $\Sigma_{\text {loc }}^{-}(\xi)$ and $\Sigma_{\text {loc }}^{+}(\xi)$ for the local stable and unstable sets

$$
\begin{aligned}
& \Sigma_{\text {loc }}^{-}(\xi):=\left\{\eta \in \Sigma \mid \eta_{i}=\xi_{i} \text { for all } i \geq 0\right\} \\
& \Sigma_{\text {loc }}^{+}(\xi):=\left\{\eta \in \Sigma \mid \eta_{i}=\xi_{i} \text { for all } i<0\right\}
\end{aligned}
$$

Write $\hat{\mathcal{F}}$ for (the completion of) the Borel sub- $\sigma$-algebra of $\Sigma$ containing sets that are a.s. saturated by local unstable sets: $C \in \hat{\mathcal{F}}$ if and only if $C=\hat{C} \bmod \hat{\nu}^{\mathbb{Z}}$ where $\hat{C}$ is Borel in $\Sigma$ with

$$
\hat{C}=\bigcup_{\xi \in \hat{C}} \Sigma_{\mathrm{loc}}^{+}(\xi) .
$$

Similarly, we define $\hat{\mathcal{G}}$ to be the sub- $\sigma$-algebra of $\Sigma$ whose atoms are local stable sets. Writing $\mathcal{B}_{M}$ for the Borel $\sigma$-algebra on $M$ we define sub- $\sigma$-algebras on $\mathcal{F}$ and $\mathcal{G}$ on $X$ to be, respectively, the $\mu$-completions of the $\sigma$-algebras $\hat{\mathcal{F}} \otimes \mathcal{B}_{M}$ and $\hat{\mathcal{G}} \otimes \mathcal{B}_{M}$.

We note that, by construction, the assignments $\Omega \rightarrow \operatorname{Diff}^{2}(M)$ given by $\xi \mapsto f_{\xi}$ and $\xi \mapsto f_{\xi}^{-1}$ are, respectively, $\hat{\mathcal{G}}$ - and $\hat{\mathcal{F}}$-measurable. Furthermore, observing that the stable line fields $E_{\xi}^{s}(x)$ depend only on the value of $f_{\xi}^{n}$ for $n \geq 0$, we have the following straightforward but crucial observation.

Proposition 4.5. The map $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $\mathcal{G}$-measurable, and the map $(\xi, x) \mapsto$ $E_{\xi}^{u}(x)$ is $\mathcal{F}$-measurable.

We have the following claim, which follows from the explicit construction of $\mu$ in (4.2).

Proposition 4.6. The intersection $\mathcal{F} \cap \mathcal{G}$ is equivalent modulo $\mu$ to the $\sigma$-algebra $\{\varnothing, \Sigma\} \otimes \mathcal{B}_{M}$.
Proof. Let $A \in \mathcal{F} \cap \mathcal{G}$. Since $A \in \mathcal{G}$, we have that $A \doteq \hat{A}$ where $\hat{A}$ is a Borel subset of $\Sigma \times M$ such that for any $(\xi, y) \in \hat{A}$ and $\eta \in \Sigma_{\text {loc }}^{-}(\xi)$,

$$
(\eta, y) \in \hat{A}
$$

We write $\left\{\mu_{(\xi, x)}^{\mathcal{F}}\right\}$ and $\left\{\mu_{(\xi, x)}^{\Sigma}\right\}$, respectively, for families of conditional probabilities given by the partition of $\Sigma \times M$ into atoms of $\mathcal{F}$ and the partition $\{\Sigma \times\{x\} \mid$ $x \in M\}$ of $\Sigma \times M$. It follows from the construction of $\mu$ given by (4.2) that $\mu_{(\xi, x)}^{\mathcal{F}}$ may be taken to be the form

$$
\begin{equation*}
d \mu_{(\xi, x)}^{\mathcal{F}}(\eta, y)=d \hat{\nu}^{\mathbb{N}}\left(\eta_{0}, \eta_{1}, \ldots\right) \delta_{x}(y) \delta_{\left(\xi_{-1}\right)}\left(\eta_{-1}\right) \delta_{\left(\xi_{-2}\right)}\left(\eta_{-2}\right) \cdots \tag{4.4}
\end{equation*}
$$

for every $(\xi, x) \in X$.
Since $A \in \mathcal{F}$, we have $\hat{A} \in \mathcal{F}$. Thus, for $\mu$-a.e. $(\xi, x) \in \hat{A}$,

$$
\mu_{(\xi, x)}^{\mathcal{F}}(\hat{A})=1
$$

Furthermore, it follows from (4.4) and the form of $\hat{A}$ that if

$$
\mu_{(\xi, x)}^{\mathcal{F}}(\hat{A})=1
$$

then

$$
\mu_{\left(\xi^{\prime}, x\right)}^{\mathcal{J}}(\hat{A})=1
$$

for any $\xi^{\prime} \in \Sigma$. It follows that

$$
\mu_{(\xi, x)}^{\Sigma} \hat{A}=1
$$

for a.e. $(\xi, x) \in \hat{A}$. In particular, $\hat{A} \stackrel{\circ}{\doteq} \Sigma \tilde{A}$ for some set $\tilde{A} \in \mathcal{B}_{M}$.

We remark that if $\xi$ projects to $\omega$ under the natural projection $\Sigma \rightarrow \Sigma_{+}$, then the subspace $E_{\xi}^{s}(x)$ and the subspace $E_{\omega}^{s}(x)$ given by Proposition 2.1 coincide almost surely. It then follows from Proposition 4.6 that the bundle $E_{\omega}^{s}(x)$ in Theorem 3.1 is non-random if and only if the bundle $E_{\xi}^{s}(x)$ is $\mathcal{F}$-measurable. Thus, Theorem 3.1 follows from the following 2 results.

Theorem 4.7. Let $\hat{\nu}$ and $\hat{\mu}$ be as in Theorem 3.1. Let $F: \Sigma \times M \rightarrow \Sigma \times M$ be the canonical skew product, and let $\mu$ be as in Proposition 4.2. Assume the fiber-wise conditional measures $\mu_{\xi}$ are non-atomic. Then either $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $\mathcal{F}$-measurable or $\mu$ is fiber-wise SRB.

Recall that a measure is non-atomic if there is no point with positive mass. By the ergodicity of $\mu$ under the dynamics of $F$ it follows that either $\mu_{\xi}$ is non-atomic a.s. or there is a $N \in \mathbb{N}$ such that $\mu_{\xi}$ is supported on exactly $N$ points a.s. We consider the case that $\mu_{\xi}$ is finitely supported separately.

Theorem 4.8. Let $\hat{\nu}$ and $\hat{\mu}$ be as in Theorem 3.1. Assume the fiber-wise conditional measures $\mu_{\xi}$ are finitely supported $\hat{\nu}^{\mathbb{Z}}$-a.s. Then either $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $\mathcal{F}$-measurable or the measure $\hat{\mu}$ is finitely supported and $\hat{\nu}$-a.s. invariant.

Remark 4.9. In the proof of Theorem 3.1 below, we may assume that $\hat{\mu}$ has one exponent of each sign. Indeed if $\hat{\mu}$ has only negative exponents, then the measurability of $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is trivial. Furthermore, if $\hat{\mu}$ has only positive exponents, then a standard argument shows that $\hat{\mu}$ is finitely supported and $\hat{\nu}$-a.s. invariant. Indeed, if all exponents are positive, then the measures $\mu_{\xi}$ are finitely supported for a.e. $\xi$. That $\hat{\mu}$ is $\hat{\nu}$-a.s. invariant follows, for instance, from the invariance principle in $\overline{\mathrm{AV}}$, the $\hat{\mathcal{F}}$-measurability of the measure $\mu_{\xi}$, and an argument similar to Proposition 4.6 above.
4.4. Statement of results: general skew products. We introduce a generalization of Theorem4.7, the proof of which consumes Sections 710, Let $\theta:\left(\Omega, \mathcal{B}_{\Omega}, \nu\right) \rightarrow$ $\left(\Omega, \mathcal{B}_{\Omega}, \nu\right)$ be as in Section 4.2] Let $M$ be a closed $C^{\infty}$ surface, and let $\mathscr{F}$ be a cocycle generated by a $\nu$-measurable map $\xi \mapsto f_{\xi}$ satisfying the integrability hypothesis (ICI). Fix $\mu$ an ergodic, $\mathscr{F}$-invariant, hyperbolic, Borel probability measure on $X=\Omega \times M$. For the general setting we will further assume the measures $\mu_{\xi}$ are non-atomic $\nu$-a.s. It follows that from the hyperbolicity and non-atomicity of the fiber-wise measures $\mu_{\xi}$ that the fiber-wise derivative $D F$ has two exponents $\lambda^{s}$ and $\lambda^{u}$, one of each sign.

We say a sub- $\sigma$-algebra $\hat{\mathcal{F}} \subset \mathcal{B}_{\Omega}$ is decreasing (for $\theta$ ) if

$$
\theta(\hat{\mathcal{F}})=\{\theta(A) \mid A \in \hat{\mathcal{F}}\} \subset \hat{\mathcal{F}} .
$$

(Note that $\hat{\mathcal{F}}$ is decreasing under the forwards dynamics if the partition into atoms is an increasing partition in the sense of [Y1]. Alternatively, $\hat{\mathcal{F}}$ is decreasing if the $\operatorname{map} \theta^{-1}: \Omega \rightarrow \Omega$ is $\hat{\mathcal{F}}$-measurable.) As a primary example, the sub- $\sigma$-algebra of $\Sigma$ generated by local unstable sets is decreasing (for $\sigma: \Sigma \rightarrow \Sigma$ ).

Let $\hat{\mathcal{F}}$ be a decreasing sub- $\sigma$-algebra, and write $\mathcal{F}$ for the $\mu$-completion of $\hat{\mathcal{F}} \otimes \mathcal{B}_{M}$ where $\mathcal{B}_{M}$ is the Borel algebra on $M$. As in the previous section, to compare stable distributions in different fibers over $\Omega$ write $E_{\xi}^{s}(x) \subset T_{x} M$ for the subspace with $E^{s}(\xi, x)=\{\xi\} \times E_{\xi}^{s}(x)$. We then consider $(\xi, x) \mapsto E_{\xi}^{s}(x)$ as a measurable map from $X$ to the projectivization of $T M$.

With the above setup, we now state the following result which generalizes Theorem 4.7
Theorem 4.10. Assume $\mu$ is hyperbolic and the conditional measures $\left\{\mu_{\xi}\right\}$ are non-atomic a.s. Further assume
(1) $\xi \mapsto f_{\xi}^{-1}$ is $\hat{\mathcal{F}}$-measurable, and
(2) $\xi \mapsto \mu_{\xi}$ is $\hat{\mathcal{F}}$-measurable.

Then either $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $\mathcal{F}$-measurable or $\mu$ is fiber-wise SRB. (See Definition 6.7.)

Note that the hypothesis that $\xi \mapsto f_{\xi}^{-1}$ is $\hat{\mathcal{F}}$-measurable combined with the fact that $\hat{\mathcal{F}}$ is decreasing implies $\xi \mapsto f_{\theta^{-j}(\xi)}^{-1}$ is $\hat{\mathcal{F}}$-measurable for all $j \geq 0$. It follows that $\xi \mapsto f_{\xi}^{n}$ is $\hat{\mathcal{F}}$-measurable for all $n \leq 0$. It then follows that $\mathcal{F}$ is a decreasing sub- $\sigma$-algebra of $\mathcal{B}_{X}$.

We recall that in the case that $F$ is the canonical skew product for a random dynamical system and $\hat{\mathcal{F}}$ is the sub- $\sigma$-algebra generated by local unstable sets, writing $\xi=\left(\ldots, f_{-1}, f_{0}, f_{1}, \ldots\right)$ the $\hat{\mathcal{F}}$-measurability of $\xi \mapsto f_{\xi}^{-1}=\left(f_{-1}\right)^{-1}$ follows from construction. The $\hat{\mathcal{F}}$-measurability of $\xi \mapsto \mu_{\xi}$ follows from the construction of the measure $\mu$ given by (4.2) in Proposition 4.2. Theorem 4.7 then follows immediately from Theorem 4.10

## 5. Some applications

We present a number of applications of our main theorems.
5.1. Groups of measure-preserving diffeomorphisms. Fix $M$ a closed surface. Let $\mu$ be a Borel probability measure on $M$. Let $\operatorname{Diff}_{\mu}^{2}(M)$ denote the group of $C^{2}, \mu$-preserving diffeomorphisms of $M$. Given $f \in \operatorname{Diff}_{\mu}^{2}(M)$ write $\lambda^{i}(f, \mu, x)$ for the $i$ th Lyapunov exponents of $f$ with respect to the measure $\mu$ at the point $x$. If $f$ is ergodic (for $\mu$ ) we write $\lambda^{i}(f, \mu)$ for the $\mu$-almost surely constant value of $\lambda^{i}(f, \mu, x)$. A diffeomorphism $f \in \operatorname{Diff}_{\mu}^{2}(M)$ is hyperbolic (relative to $\mu$ ) if $\lambda^{i}(f, \mu, x) \neq 0$ for almost every $x$ and every $i$.

Note that if $f \in \operatorname{Diff}_{\mu}^{2}(M)$ is hyperbolic and $\mu$ contains no atoms, then $(f, \mu)$ has one exponent of each sign $\lambda^{s}(f, \mu, x)<0<\lambda^{u}(f, \mu, x)$. For such $f$, we write $T_{x} M=E_{f}^{s}(x) \oplus E_{f}^{u}(x)$ for the $\mu$-measurable Oseledec's splitting induced by $(f, \mu)$.
Theorem 5.1. Let $\mu$ be a Borel probability measure on $M$ with no atoms. Suppose $\operatorname{Diff}_{\mu}^{2}(M)$ contains an ergodic, hyperbolic element $f$. Write $\Gamma=\operatorname{Diff}_{\mu}^{2}(M)$.
(a) If the union $E_{f}^{u} \cup E_{f}^{s}$ is not $\Gamma$-invariant and neither $E_{f}^{s}$ nor $E_{f}^{u}$ is $\Gamma$ invariant, then $\mu$ is absolutely continuous.
(b) If the union $E_{f}^{u} \cup E_{f}^{s}$ is not $\Gamma$-invariant and $E_{f}^{u}$ is $\Gamma$-invariant, then $\mu$ is an SRB measure for $f$.
(c) If the union $E_{f}^{u} \cup E_{f}^{s}$ is not $\Gamma$-invariant and $E_{f}^{s}$ is $\Gamma$-invariant, then $\mu$ is an SRB measure for $f^{-1}$.
In the case that $\lambda^{s}(f, \mu) \neq-\lambda^{u}(f, \mu)$ we can give more precise results using the following lemma.
Lemma 5.2. Let $\mu$ be non-atomic, and let $f \in \operatorname{Diff}_{\mu}^{2}(M)$ be ergodic and hyperbolic. Suppose $\lambda^{s}(f, \mu) \neq-\lambda^{u}(f, \mu)$. Then any $g \in \operatorname{Diff}_{\mu}^{2}(M)$ that preserves the union $E_{f}^{u} \cup E_{f}^{s}$ preserves the individual distributions $E_{f}^{s}$ and $E_{f}^{u}$.

Proof. Suppose $g \in \operatorname{Diff}_{\mu}^{2}(M)$ preserves the union $E_{f}^{u} \cup E_{f}^{s}$ almost surely but

$$
\begin{equation*}
D_{x} g\left(E_{f}^{s}(x)\right)=E_{f}^{u}(g(x)) \tag{5.1}
\end{equation*}
$$

for a positive measure set of $x$. Let $\mathbb{P} T M$ denote the projectivized tangent bundle. Let $\nu=t \delta_{g}+(1-t) \delta_{f}$ and $\Sigma=\{f, g\}^{\mathbb{Z}}$. On $\Sigma \times \mathbb{P} T M$ consider the measure

$$
d \eta(\xi, x, E)=d \nu^{\mathbb{Z}}(\xi) d \mu(x) d\left(.5 \delta_{E_{f}^{u}(x)}+.5 \delta_{E_{f}^{s}(x)}\right)(E)
$$

Write $D F$ for the derivative skew product $D F: \Sigma \times T M \rightarrow \Sigma \times T M$ and $\mathbb{P} D F$ for the projectivized derivative skew product $\mathbb{P} D F: \Sigma \times \mathbb{P} T M \rightarrow \Sigma \times \mathbb{P} T M$. Then $\eta$ is $\mathbb{P} D F$-invariant and is ergodic by (5.1). Let $\Phi: \Sigma \times \mathbb{P} T M \rightarrow \mathbb{R}$ be

$$
\Phi(\xi, x, E)=\log \left\|D_{x} f_{\xi} \upharpoonright_{E}\right\| .
$$

Then for $\mu$-a.e. $x$ and $v \in E_{f}^{s}(x) \cup E_{f}^{u}(x)$ with $v \neq 0$ we have

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{n} \log \left\|D_{x} f_{\xi}^{n}(v)\right\|=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{n-1} \Phi\left(\mathbb{P} D F^{j}(\xi, x,[v])\right)=\int \Phi d \eta \\
& \quad=(1-t)\left(.5 \lambda^{u}(f, \mu)+.5 \lambda^{s}(f, \mu)\right)+.5 t \int_{E \in\left\{E_{f}^{u}(x), E_{f}^{s}(x)\right\}} \log \left\|D_{x} g \upharpoonright_{E}\right\| d \mu(x) .
\end{aligned}
$$

As the $C^{1}$ norm of $g$ is bounded, for $t>0$ sufficiently small, either all fiber-wise exponents of $D F$ are negative or all fiber-wise exponents of $D F$ are positive. This contradicts that $\mu$ is non-atomic.

From the above lemma we have the following theorem.
Theorem 5.3. Let $\mu$ be the Borel probability measure on $M$ with no atoms. Suppose $\operatorname{Diff}_{\mu}^{2}(M)$ contains an ergodic, hyperbolic element $f$ with $\lambda^{s}(f, \mu) \neq-\lambda^{u}(f, \mu)$. Then with $\Gamma=\operatorname{Diff}_{\mu}^{2}(M)$ either
(a) both $E_{f}^{u}$ and $E_{f}^{s}$ are $\Gamma$-invariant or
(b) exactly one $E_{f}^{u}$ and $E_{f}^{s}$ is $\Gamma$-invariant in which case $\mu$ is $S R B$ for $f$ or $f^{-1}$.

Note that the hypothesis that $\lambda^{s}(f, \mu) \neq-\lambda^{u}(f, \mu)$ implies that $\mu$ is not absolutely continuous.
5.2. Smooth stabilizers of measures invariant by Anosov maps. As a consequence of the results in the previous section, we obtain a strengthening of the result from Bro .

Let $f: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ be Anosov. Then there is a hyperbolic $A \in \mathrm{GL}(2, \mathbb{Z})$ such that any lift $\tilde{f}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ of $f$ is of the form $f(x)=A x+\eta(x)$ where $\eta: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ is $\mathbb{Z}^{2}$ is periodic. Given $B \in \mathrm{GL}(2, \mathbb{Z})$ let $L_{B}: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ be the induced map. Then there is a (non-unique) homeomorphism $h: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ with $h \circ f=L_{A} \circ h$.

Let $\mu$ be a fully supported, ergodic, $f$-invariant measure. Let $K \subset \mathbb{R}^{2}$ be the set $K=\left\{v: h_{*} \mu\right.$ is $T_{v}$-invariant $\} \subset \mathbb{R}^{2}$ where $T_{v}: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ is the translation by $v$. Then $K$ descends to a closed $L_{A}$-invariant subgroup of $\mathbb{T}^{2}$ and so either is discrete or is all of $\mathbb{R}^{2}$. The latter case can happen only if the measure $\mu$ is the measure of maximal entropy for $f$. It follows that the group $(A-I)^{-1} K$ either is discrete or is all of $\mathbb{R}^{2}$. Let $\hat{K}$ be the smallest subgroup of $\mathbb{R}^{2}$ that is invariant under the
centralizer $C_{\mathrm{GL}(2, \mathbb{Z})}(A)$ of $A$ in $\mathrm{GL}(2, \mathbb{Z})$ and contains $(A-I)^{-1} K$. Note that $\hat{K}$ descends to a subgroup of $\mathbb{T}^{2}$. Then $\hat{K}$ either is $\mathbb{R}^{2}$ or is discrete. Let $T_{\hat{K}}$ denote the corresponding group of translations on $\mathbb{T}^{2}$. Then $T_{\hat{K}}$ is finite if $\mu$ is not the measure of maximal entropy.

Recall that the centralizer of $A$ is of the form $C_{\mathrm{GL}(2, \mathbb{Z})}(A)=\langle \pm M\rangle$ for some $M \in \mathrm{GL}(2, \mathbb{Z})$.

Theorem 5.4. Let $f: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ be a $C^{2}$ Anosov diffeomorphism, and let $\mu$ be a fully supported, ergodic, $f$-invariant measure. If $\mu$ is not absolutely continuous, then for every $g \in \operatorname{Diff}_{\mu}^{2}(M)$ there is a $B \in C_{\mathrm{GL}(2, \mathbb{Z})}(A)$ and $v \in(A-I)^{-1}(K)$ with

$$
h \circ g \circ h^{-1}(x)=L_{B}(x)+v ;
$$

in particular, $\operatorname{Diff}_{\mu}^{2}\left(\mathbb{T}^{2}\right)$ is isomorphic to a subgroup of

$$
C_{\mathrm{GL}(2, \mathbb{Z})}(A) \ltimes T_{\hat{K}} .
$$

Moreover, if $\mu$ is not the measure of maximal entropy (for $f$ ), then $T_{\hat{K}}$ is finite, whence $\operatorname{Diff}_{\mu}^{2}\left(\mathbb{T}^{2}\right)$ is virtually $\mathbb{Z}$.

Recall that a group is virtually $\mathbb{Z}$ if it contains a finite-index subgroup isomorphic to $\mathbb{Z}$. Theorem 5.4 follows exactly from the argument in Bro with only minor modifications coming from Theorem 5.1.

Proof. Recall that if $f$ is Anosov, then the measurable distributions $E_{f}^{s}$ and $E_{f}^{u}$ appearing in Oseledec's splitting coincide with continuous transverse distributions.

Consider first $g \in \operatorname{Diff}_{\mu}^{2}(M)$ such that $D g$ does not interchange $E_{f}^{s}$ and $E_{g}^{u}$ on a set of full measure (and hence at every point). Then, if $\mu$ is not absolutely continuous, by Theorem 5.1 at least one of the (continuous) distributions $E_{f}^{s}$ or $E_{f}^{u}$ is preserved (on a set of full measure and hence everywhere) by $g$. Then, as the integral foliations to $E_{f}^{s}$ and $E_{f}^{u}$ are unique, it follows that either the stable or the unstable foliation of $f$ is preserved by every such $g$.

It is then shown in Bro that $g$ necessarily preserves both the stable and the unstable foliations for $f$ and hence preserves the corresponding tangent line fields $E_{f}^{s}$ and $E_{f}^{u}$. If there exists $g \in \operatorname{Diff}_{\mu}^{2}(M)$ such that $g$ interchanges $E_{f}^{s}$ and $E_{f}^{u}$, then we may restrict to an index-2 subgroup preserving $E_{f}^{s}$ and $E_{f}^{u}$ and the corresponding foliations.

The remainder of the proof of Theorem 5.4 and a more detailed description of the structure of $\operatorname{Diff}_{\mu}^{2}\left(\mathbb{T}^{2}\right)$ proceeds exactly as in [Bro and will not be repeated here.
5.3. Perturbations of algebraic systems. Let $A, B \in \mathrm{GL}(2, \mathbb{Z})$ be hyperbolic matrices. Write $E_{A}^{s}$ and $E_{A}^{u}$, respectively, for the stable and unstable eigenspaces of $A$. We say that $\{A, B\}$ satisfy a joint cone condition if there are disjoint open cones $C^{s}$ and $C^{u}$, containing $\left\{E_{A}^{s}, E_{B}^{s}\right\}$ and $\left\{E_{A}^{u}, E_{B}^{u}\right\}$, respectively, with $A^{-1} C^{s} \subset C^{s}$, $B^{-1} C^{s} \subset C^{s}, A C^{u} \subset C^{u}$, and $B C^{u} \subset C^{u}$ and a number $\kappa>1$ such that if $v \in C^{u}$, then $\|B v\|>\kappa\|v\|$ and $\|A v\|>\kappa\|v\|$, and if $w \in C^{s}$, then $\left\|B^{-1} w\right\|>\kappa\|w\|$ and $\left\|A^{-1} w\right\|>\kappa\|w\|$.

Given $A \in \mathrm{GL}(2, \mathbb{Z})$ let $L_{A}: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}$ be the induced diffeomorphism.

Proposition 5.5. Suppose that $A$ and $B$ do not commute and satisfy a joint cone condition. Then for sufficiently small $C^{2}$ perturbations $f$ of $L_{A}$ and $g$ of $L_{B}$, for $\nu=p \delta_{f}+(1-p) \delta_{g}$ with $p \in(0,1)$ the only ergodic, $\nu$-stationary measures are $\operatorname{SRB}$ or finitely supported.

Moreover, for every such $f$ and a generic $g$, the only $\nu$-stationary measure is SRB.

Note that in the setting of the above proposition, stationary measures with the SRB property are unique. The proof of the proposition will be given in Section 13.3 ,

Theorem 5.6. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ be an infinite subgroup that is not virtually $\mathbb{Z}$. Let $S=\left\{A_{1}, \ldots, A_{n}\right\}$ be a finite set generating $\Gamma$. Consider $0<p_{k}<1$ with $\sum_{k=1}^{n} p_{k}=1$, and let $\nu_{0}=\sum p_{k} \delta_{L_{A_{k}}}$. Then there is an open set $U \subset \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right)$ with $\nu_{0}(U)=1$ such that for every probability $\nu$ on $U$ sufficiently close to $\nu_{0}$, any ergodic, $\nu$-stationary measure either is atomic or is hyperbolic with one exponent of each sign and is SRB.

The proof of the theorem will be given in Section 13.3
Let $m$ denote the Lebesgue area on $\mathbb{T}^{2}$. If we restrict the above to the setting of area-preserving perturbations, we obtain the following nonlinear counterpart to [BQ1. Note in particular that we obtain stiffness of all stationary measures.

Theorem 5.7. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ be an infinite subgroup that is not virtually $\mathbb{Z}$. Let $S=\left\{A_{1}, \ldots, A_{n}\right\}$ be a finite set generating $\Gamma$. Consider $0<p_{k}<1$ with $\sum_{k=1}^{n} p_{k}=1$, and let $\nu_{0}=\sum p_{k} \delta_{L_{A_{k}}}$. Then there is an open set $U \subset \operatorname{Diff}_{m}^{2}\left(\mathbb{T}^{2}\right)$ with $\nu_{0}(U)=1$ such that for every probability $\nu$ on $U$ sufficiently close to $\nu_{0}$, any ergodic, $\nu$-stationary measure is hyperbolic with one exponent of each sign and either coincides with $m$ or is atomic.

In particular, every $\nu$-stationary measure is preserved by every $g \in \operatorname{Diff}_{m}^{2}\left(\mathbb{T}^{2}\right)$ in the support of $\nu$.

The theorem follows from Theorem 5.6 and (the proof of) Theorem 3.4. In the proof of Theorem 5.6 it is shown that for all $\nu$ sufficiently close to $\nu_{0}$, every ergodic $\nu$-stationary measure $\mu$ has a positive exponent. That $\mu$ also has a negative exponent follows from (2.4). Moreover, for such $\nu$, a positive $\nu$-measure set of $f \in \operatorname{Diff}_{m}^{2}\left(\mathbb{T}^{2}\right)$ is Anosov, whence $m$ is ergodic for such $f$ and hence ergodic for $\nu$.

Finally, we consider stationary measures for perturbations of rotations. Let $R_{1}, \ldots, R_{\ell}$ be $\ell$ rotations in $\mathbb{R}^{3}$ generating a dense subgroup of $\mathrm{SO}(3, \mathbb{R})$. We identify each $R_{i}$ with a diffeomorphism of $S^{2} \subset \mathbb{R}^{3}$. Let $m$ denote the unique $\mathrm{SO}(3, \mathbb{R})$ invariant measure on $S^{2}$.
Theorem 5.8. For $k \in \mathbb{N}$ sufficiently large, for each $1 \leq i \leq \ell$ there is a neighborhood $R_{i} \in U_{i} \subset \operatorname{Diff}_{m}^{k}\left(S^{2}\right)$ such that given any $g_{i} \in U_{i}$ and $\nu=\frac{1}{\ell} \sum_{i=1}^{\ell} \delta_{g_{i}}$, any ergodic $\nu$-stationary measure on $S^{2}$ either is finitely supported or coincides with $m$.

Proof. In DK it is shown that either the diffeomorphisms $g_{i}$ are simultaneously smoothly conjugated to $R_{i}$ or every $\nu$-stationary measure is hyperbolic. In the first case, the only stationary measures for the corresponding $R_{i}$ is $m$ and thus using the conjugacy and that each $g_{i}$ preserves $m$, the only $\nu$-stationary measure is $m$.

In the latter case, it is also shown in DK, Corollary 4] that the stable line field is not non-random. The result in this case follows from Theorem 3.4 and, as is also shown in [DK], shows that $m$ is ergodic for the perturbed system.
5.4. Other applications. From Theorem 3.4 we immediately obtain the main results of BQ1 BFLM for measures $\nu$ on $\operatorname{SL}(2, \mathbb{Z})$ acting on $\mathbb{T}^{2}$ that satisfy a logintegrability condition $\int \log \|A\| d \nu(A)<\infty$. In BQ1 the measure $\nu$ is assumed finitely supported. In BFLM a stronger integrability hypothesis is needed. Using the methods of this paper, the results of $\mathrm{BQ1}$ are expected to hold under a logintegrability hypothesis.

Consider a flat surface $S$ with Veech group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$. As was pointed out to the authors by J. Athreya, Theorem 3.4 implies that if the Veech group is infinite and non-elementary, then for any finitely supported measure $\nu$ generating $\Gamma$, all ergodic $\nu$-stationary measures $\mu$ on $S$ either are finitely supported or are the invariant area. There are technicalities in applying Theorem 3.4 directly as the action is non-differentiable at the cone points. This mild difficulty will not be addressed here.

## 6. Background and notation

In this section, we continue to work in the setting introduced in Sections 4.2 and 4.4. We outline extensions of a number of standard facts from the theory of non-uniformly hyperbolic diffeomorphisms to the setting of the fiber-wise dynamics for skew products. As previously remarked, Theorem 3.1holds trivially if the fiberwise exponents are all of the same sign. Moreover, the hypotheses of Theorem 4.10 rule out that all exponents are of the same sign. We thus assume for the remainder that we have one Lyapunov exponent of each sign $\lambda^{s}<0<\lambda^{u}$. For the remainder, fix $0<\epsilon_{0}<\min \left\{1, \lambda^{u} / 200,-\lambda^{s} / 200\right\}$.
6.1. Fiber-wise non-uniformly hyperbolic dynamics. We present a number of extensions of the theory of non-uniformly hyperbolic diffeomorphisms to the fiber-wise dynamics of skew products.
6.1.1. Subexponential estimates. We have the following standard results that follow from the integrability hypothesis (IC) and tempering kernel arguments (cf. $\overline{B P}$, Lemma 3.5.7]).

Lemma 6.1. There is a subset $\Omega_{0} \subset \Omega$ with $\nu\left(\Omega_{0}\right)=1$ and a measurable function $D: \Omega_{0} \rightarrow[1, \infty)$ such that for $\nu$-a.e. $\xi \in \Omega_{0}$ and $n \in \mathbb{Z}$.
(1) $\left|f_{\xi}\right|_{C^{1}} \leq D(\xi)$,
(2) $\left|f_{\xi}^{-1}\right|_{C^{1}} \leq D(\xi)$,
(3) $\operatorname{Lip}\left(D f_{\xi}\right) \leq D(\xi)$ and $\operatorname{Lip}\left(D f_{\xi}^{-1}\right) \leq D(\xi)$,
(4) $D\left(\theta^{n}(\xi)\right) \leq e^{|n| \epsilon_{0}} D(\xi)$ for all $n \in \mathbb{Z}$.

Here $\operatorname{Lip}\left(D f_{\xi}\right)$ denotes the Lipschitz constant of the map $x \mapsto D_{x} f_{\xi}$ for fixed $\xi$.
Lemma 6.2. There is a measurable function $L: X \rightarrow[1, \infty)$ such that for $\mu$-a.e. $(\xi, x) \in X$ and $n \in \mathbb{Z}$ :
(1) For $v \in E_{\xi}^{s}(x)$,

$$
L(\xi, x)^{-1} \exp \left(n \lambda^{s}-|n| \frac{1}{2} \epsilon_{0}\right)\|v\| \leq\left\|D f_{\xi}^{n} v\right\| \leq L(\xi, x) \exp \left(n \lambda^{s}+|n| \frac{1}{2} \epsilon_{0}\right)\|v\|
$$

(2) For $v \in E_{\xi}^{u}(x)$,

$$
L(\xi, x)^{-1} \exp \left(n \lambda^{u}-|n| \frac{1}{2} \epsilon_{0}\right)\|v\| \leq\left\|D f_{\xi}^{n} v\right\| \leq L(\xi, x) \exp \left(n \lambda^{u}+|n| \frac{1}{2} \epsilon_{0}\right)\|v\|
$$

(3) $\angle\left(E_{\theta^{n}(\xi)}^{s}\left(D f_{\xi}^{n}(x)\right), E_{\theta^{n}(\xi)}^{u}\left(D f_{\xi}^{n}(x)\right)\right)>\frac{1}{L(\xi, x)} \exp \left(-|n| \epsilon_{0}\right)$.

Furthermore for $n \in \mathbb{Z}$

$$
L\left(F^{n}(\xi, x)\right) \leq L(\xi, x) e^{\epsilon_{0}|n|} .
$$

Here $\angle$ denotes the Riemannian angle between two subspaces.
6.1.2. Lyapunov charts. We introduce families of two-sided Lyapunov charts. The construction depends on the construction of a Lyapunov norm which we present in Section 9.2. We note that in Section 11.2.1, in the case that $\Omega=\left(\operatorname{Diff}^{2}(M)\right)^{\mathbb{Z}}$ we will need one-sided charts that depend only on the future itinerary of $\xi \in\left(\operatorname{Diff}^{2}(M)\right)^{\mathbb{Z}}$. Given $v \in \mathbb{R}^{2}$ decompose $v=v_{1}+v_{2}$ according to the standard basis and write $|v|_{i}=\left|v_{i}\right|$ and $|v|=\max \left\{|v|_{i}\right\}$. Write $\mathbb{R}^{2}(r)$ for the ball of radius $r$ centered at 0 .

From standard constructions (see [LY1, Appendix], LQ, VI.3]) for every $0<$ $\epsilon_{1}<\epsilon_{0}$, there is a measurable function $\ell: \Omega \times M \rightarrow[1, \infty)$ and a full measure set $\Lambda \subset \Omega \times M$ such that
(1) for $(\xi, x) \in \Lambda$ there is a neighborhood $U_{(\xi, x)} \subset M$ of $x$ and a $C^{\infty}$ diffeomorphism $\phi(\xi, x): U_{(\xi, x)} \rightarrow \mathbb{R}^{2}\left(\ell(\xi, x)^{-1}\right)$ with
(a) $\phi(\xi, x)(x)=0$;
(b) $D \phi(\xi, x) E_{\xi}^{s}(x)=\mathbb{R} \times\{0\}$;
(c) $D \phi(\xi, x) E_{\xi}^{u}(x)=\{0\} \times \mathbb{R}$;
(2) writing

$$
\tilde{f}(\xi, x)=\phi(F(\xi, x)) \circ f_{\xi} \circ \phi(\xi, x)^{-1}, \quad \tilde{f}^{-1}(\xi, x)=\phi\left(F^{-1}(\xi, x)\right) \circ f_{\xi}^{-1} \circ \phi(\xi, x)^{-1}
$$

where we have defined
(a) $\tilde{f}(\xi, x)(0)=0$;
(b) $D_{0} \tilde{f}(\xi, x)=\left(\begin{array}{cc}\alpha & 0 \\ 0 & \beta\end{array}\right)$ where $e^{\lambda^{s}-\epsilon_{1}} \leq \alpha \leq e^{\lambda^{s}+\epsilon_{1}}$ and $e^{\lambda^{u}-\epsilon_{1}} \leq \beta \leq$ $e^{\lambda^{u}+\epsilon_{1}} ;$
writing $\operatorname{Lip}(\cdot)$ for the Lipschitz constant of a map on its domain
(c) $\operatorname{Lip}\left(\tilde{f}(\xi, x)-D_{0} \tilde{f}(\xi, x)\right)<\epsilon_{1}$;
(d) $\operatorname{Lip}(D \tilde{f}(\xi, x))<\ell(\xi, x)$;
(3) properties similar to (2a)-(2d) hold for $\tilde{f}^{-1}(\xi, x)$;
(4) there is a uniform $k_{0}$ with $k_{0}^{-1} \leq \operatorname{Lip}(\phi(\xi, x)) \leq \ell(\xi, x)$;
(5) $\ell\left(F^{n}(\xi, x)\right) \leq \ell(\xi, x) e^{|n| \epsilon_{1}}$ for all $n \in \mathbb{Z}$.

Let

$$
\lambda_{0}=\max \left\{\lambda^{u},-\lambda^{s}\right\}+2 \epsilon_{1}
$$

Then, the domains of $\tilde{f}(\omega, x)$ and $\tilde{f}^{-1}(\omega, x)$ contain the ball in $\mathbb{R}^{2}$ of norm $\ell(\xi, x)^{-1}$ $e^{-\lambda_{0}-\epsilon_{1}}$. Note also that the domain of $\phi(\xi, x)$ contains a ball of radius $\ell(\xi, x)^{-2}$ centered at $x$.

Write $\mathbb{R}^{s}=\mathbb{R} \times\{0\}$ and $\mathbb{R}^{u}=\{0\} \times \mathbb{R}$. Recall that $g: D \subset \mathbb{R} \rightarrow \mathbb{R}$ is $k$-Lipschitz if $|g(x)-g(y)| \leq k|x-y|$ for $x, y \in D$. We have the following observation.

Lemma 6.3. Let $D \subset \mathbb{R}^{s}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell(\xi, x)^{-1}\right)$. Let $g: D \rightarrow \mathbb{R}^{u}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell(\xi, x)^{-1}\right)$ be a 1-Lipschitz function. Then

$$
\tilde{f}^{-1}(\xi, x)(\operatorname{graph}(g))
$$

is the graph of a 1-Lipschitz function

$$
\hat{g}: \hat{D} \rightarrow \mathbb{R}^{u}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-1}(\xi, x)\right)^{-1}\right)
$$

for some $\hat{D} \subset \mathbb{R}^{s}\left(\ell\left(F^{-1}(\xi, x)\right)^{-1}\right)$.
6.1.3. Stable manifold theorem. Relative to the charts $\phi(\xi, x)$ above, one may apply either the Perron-Irwin method or the Hadamard graph transform method to construct stable manifolds. The existence of stable manifolds for diffeomorphisms of manifolds with non-zero exponents is due to Pesin Pes. In the case of random dynamical systems, given the family of charts above, the statements and proofs hold with minor modifications (see, for example, LQ ). See Section 11.2 .1 for some details in the construction of stable manifolds relative to one-sided charts.

Theorem 6.4 (Local stable manifold theorem).
(1) For $(\xi, x) \in \Lambda$ there is a $C^{1,1}$ function

$$
\left.h^{s}(\xi, x): \mathbb{R}^{s}\left(\ell(\xi, x)^{-1}\right) \rightarrow \mathbb{R}^{u}\left(\ell(\xi, x)^{-1}\right)\right)
$$

with
(1) $h^{s}(\xi, x)(0)=0$;
(2) $D_{0} h^{s}(\xi, x)=0$;
(3) $\left\|D h^{s}(\xi, x)\right\| \leq 1 / 3$;
(4) $\tilde{f}(\xi, x)\left(\operatorname{graph}\left(h^{s}(\xi, x)\right)\right) \subset \operatorname{graph}\left(h^{s}(F(\xi, x))\right) \subset \mathbb{R}^{2}\left(\ell(F(\xi, x))^{-1}\right)$; in particular, $\operatorname{graph}\left(h^{s}(\xi, x)\right)$ is in the domain of $\tilde{f}(\xi, x)$.
Setting

$$
V^{s}(\xi, x):=\phi(\xi, x)^{-1}\left(\operatorname{graph}\left(h^{s}(\xi, x)\right)\right)
$$

we have
(5) $f_{\xi}\left(V^{s}(\xi, x)\right) \subset V^{s}(F(\xi, x))$
(6) for $z, y \in V^{s}(\xi, x)$ and $n \geq 0$

$$
d\left(f_{\xi}^{n}(z), f_{\xi}^{n}(y)\right) \leq \ell(\xi, x) k_{0} \exp \left(\left(\lambda^{s}+2 \epsilon_{1}\right) n\right) d(y, z)
$$

We define $V^{s}(\xi, x) \subset M$ to be the local stable manifold at $x$ for $\xi$ relative to the above charts. We similarly construct local unstable manifolds $V^{u}(\xi, x)$. Similar to 6.4 above, for $z, y \in V^{u}(\xi, x)$ and $n \geq 0$

$$
d\left(f_{\xi}^{-n}(z), f_{\xi}^{-n}(y)\right) \leq \ell(\xi, x) k_{0} \exp \left(\left(-\lambda^{u}+2 \epsilon_{1}\right) n\right) d(y, z)
$$

We remark that the family of local stable manifolds $\left\{V^{s}(\xi, x)\right\}$ forms a measurable family of embedded submanifolds.

We define the global stable and unstable manifolds at $x$ for $\xi$ by

$$
\begin{align*}
& W_{\xi}^{s}(x):=\left\{y \in M \left\lvert\, \limsup _{n \rightarrow \infty} \frac{1}{n} \log d\left(f_{\xi}^{n}(x), f_{\xi}^{n}(y)\right)<0\right.\right\},  \tag{6.1}\\
& W_{\xi}^{u}(x):=\left\{y \in M \left\lvert\, \limsup _{n \rightarrow-\infty} \frac{1}{n} \log d\left(f_{\xi}^{n}(x), f_{\xi}^{n}(y)\right)<0\right.\right\} . \tag{6.2}
\end{align*}
$$

For $\mu$-a.e. $(\xi, x)$ we have the nested union $W_{\xi}^{s}(x)=\bigcup_{n \geq 0}\left(f_{\xi}^{n}\right)^{-1}\left(V^{s}\left(F^{n}(\xi, x)\right)\right)$. It follows for such $(\xi, x)$ that $W_{\xi}^{s}(x)$ is a $C^{1,1}$-injectively immersed curve tangent to $E_{\xi}^{s}(x)$. We write

$$
W^{s}(\xi, x):=\{\xi\} \times W_{\xi}^{s}(x), \quad W^{u}(\xi, x):=\{\xi\} \times W_{\xi}^{u}(x)
$$

for the associated fiber-wise stable and unstable manifolds in $X=\Omega \times M$.

The above family of charts and construction of local stable and unstable manifold depends on $f_{\xi}^{n}$ for all $n \in \mathbb{Z}$. However, from (6.2) it is clear that $W_{\xi}^{u}(x)$ depends only on $f_{\xi}^{n}$ for all $n \leq 0$. This fact will be used heavily in the sequel. In Section 11.2.1 we will use one-side charts to construct local stable manifolds that depend only on $f_{\xi}^{n}$ for all $n \geq 0$.
6.2. Affine parameters. Since each stable manifold $W_{\xi}^{s}(x)$ is a curve, it has a natural parametrization via the Riemannian arc length. We define an alternative parametrization, defined on almost every stable manifold, that conjugates the non-linear dynamics $f_{\xi}^{n} \upharpoonright_{W_{\xi}^{s}(x)}$ and the linear dynamics $D f_{\xi}^{n} \upharpoonright_{E_{\xi}^{s}(x)}$. We sketch the construction and refer the reader to [KK, Section 3.1] for additional details.

Proposition 6.5. For almost every $(\xi, x)$ and any $y \in W_{\xi}^{s}(x)$, there is a $C^{1,1}$ diffeomorphism

$$
H_{(\xi, y)}^{s}: W_{\xi}^{s}(x) \rightarrow T_{y} W_{\xi}^{s}(x)
$$

such that
(1) restricted to $W_{\xi}^{s}(x)$ the parametrization intertwines the nonlinear dynamics $f_{\xi}$ with the differential $D_{y} f_{\xi}$,

$$
D_{y} f_{\xi} \circ H_{(\xi, y)}^{s}=H_{F(\xi, y)}^{s} \circ f_{\xi} \upharpoonright_{W_{\xi, r}^{s}(x)}
$$

(2) $H_{(\xi, y)}^{s}(y)=0$ and $D_{y} H_{(\xi, y)}^{s}=\mathrm{Id}$;
(3) if $z \in W_{\xi}^{s}(x)$, then the change of coordinates

$$
H_{(\xi, y)}^{s} \circ\left(H_{(\xi, z)}^{s}\right)^{-1}: T_{z} W_{\xi}^{s}(x) \rightarrow T_{y} W_{\xi}^{s}(x)
$$

is an affine map with derivative

$$
D_{v}\left(H_{(\xi, y)}^{s} \circ\left(H_{(\xi, z)}^{s}\right)^{-1}\right)=\rho_{(\xi, y)}(z)
$$

for any $v \in T_{z} W_{\xi}^{s}(x)$ where $\rho_{(\xi, y)}(z)$ is defined below.
We take $(\xi, x)$ to be in the full measure $F$-invariant set such that for any $y, z \in$ $W_{\xi}^{s}(x)$ there is some $k \geq 0$ with $f_{\xi}^{k}(z)$ and $f_{\xi}^{k}(y)$ contained in $V^{s}\left(F^{k}(\xi, x)\right)$, and we sketch the construction of $H_{(\xi, y)}^{s}$. First consider any $y, z \in V^{s}(\xi, x)$ and define

$$
J(\xi, z):=\left\|D_{z} f_{\xi} v\right\| \cdot\|v\|^{-1}
$$

for any non-zero $v \in T_{z} W_{\xi}^{s}(x)$ where $\|\cdot\|$ denotes the Riemannian norm on $M$. We define

$$
\begin{equation*}
\rho_{(\xi, y)}(z):=\prod_{k=0}^{\infty} \frac{J\left(F^{k}(\xi, z)\right)}{J\left(F^{k}(\xi, y)\right)} . \tag{6.3}
\end{equation*}
$$

Following [KK, Section 3.1], the right-hand side of (6.3) converges uniformly in $z$ to a Lipschitz function. The only modifications needed in our setting are the subexponential growth of $\left\|D f_{\xi}\right\|$ and the Lipschitz constant of $D f_{\xi}$ along orbits given by Lemma 6.1, as well the sub-exponential growth in $n$ of the Lipschitz variation of the tangent spaces $T_{z} V^{s}\left(F^{n}(\xi, x)\right)$ in $z$. The growth of the Lipschitz constant of $T_{z} V^{s}\left(F^{n}(\xi, x)\right)$ follows from the proof of the stable manifold theorem (for example in $(\overline{L Q})$ or by an argument similar to [LY1, Lemma 4.2.2]. We may extend the definition of $\rho_{(\xi, y)}(z)$ to any $z, y \in W_{\xi}^{s}(x)$ using that $f_{\xi}^{k}(z)$ and $f_{\xi}^{k}(y)$ are contained in $V^{s}\left(F^{k}(\xi, x)\right)$ for some $k \geq 0$.

We now define the affine parameter $H_{(\xi, y)}^{s}: W_{\xi}^{s}(x) \rightarrow T_{y} W_{\xi}^{s}(x)$ as follows. We define $H_{(\xi, y)}^{s}$ to be orientation-preserving and

$$
\left\|H_{(\xi, y)}^{s}(z)\right\|:=\int_{y}^{z} \rho_{(\xi, y)}(t) d t
$$

where $\int_{y}^{z} \psi(t) d t$ is the integral of the function $\psi$, along the curve from $y$ to $z$ in $W_{\xi}^{s}(x)$, with respect to the Riemannian arc-length on $W_{\xi}^{s}(x)$.

It follows from computations in [KK] Lemma 3.2, Lemma 3.3] that the map $H_{(\xi, y)}^{s}$ constructed above satisfies the properties above. We similarly construct unstable affine parameters $H_{(\xi, x)}^{u}$ with analogous properties.
Remark 6.6. The unstable line fields $E_{\xi}^{u}(x)$, unstable manifolds, and corresponding affine parameters are constructed using only the dynamics of $f_{\xi}^{n}$ for $n \leq 0$. Recall that we assume $\xi \mapsto f_{\xi}^{-1}$ is $\hat{\mathcal{F}}$-measurable and that $\theta(\hat{\mathcal{F}}) \subset \hat{\mathcal{F}}$. It follows that $\xi \mapsto f_{\xi}^{n}$ is $\hat{\mathcal{F}}$-measurable for all $n \leq 0$. Thus, the line fields $(\xi, x) \mapsto E_{\xi}^{u}(x)$, the unstable manifolds $(\xi, x) \mapsto W_{\xi}^{u}(x)$, and the corresponding affine parameters $H_{(\xi, x)}^{u}$ are $\mathcal{F}$-measurable.
6.2.1. Parametrization of stable and unstable manifolds. We use the affine parameters $H^{s}$ and the background Riemannian norm on $M$ to parametrize local stable manifolds. For $(\xi, x) \in X$ such that affine parameters are defined, write

$$
W_{\xi, r}^{s}(x):=\left(H_{x}^{s}\right)^{-1}\left(\left\{v \in E_{\xi}^{s}(x) \mid\|v\|<r\right\}\right)
$$

for the local stable manifold in $M$ and

$$
W_{r}^{s}(\xi, x):=\{\xi\} \times W_{\xi, r}^{s}(x)
$$

for the corresponding fiber-wise local stable manifold. We use similar notation for local unstable manifolds.

We fix, once and for all, a family $v_{(\xi, x)}^{\sigma} \in E_{\xi}^{\sigma}(x) \subset T_{x} M$ such that
(1) $(\xi, x) \mapsto v_{(\xi, x)}^{s}$ is $\mu$-measurable;
(2) $(\xi, x) \mapsto v_{(\xi, x)}^{u}$ is $\mathcal{F}$-measurable;
(3) $\left\|v_{(\xi, x)}^{s}\right\|=\left\|v_{(\xi, x)}^{u}\right\|=1$.

The family $\left\{v_{(\xi, x)}^{s}\right\}$ and $\left\{v_{(\xi, x)}^{u}\right\}$ induce, respectively, $\mu$ - and $\mathcal{F}$-measurable trivializations of the stable and unstable bundles. Recall that the affine parameters on unstable manifolds are constant along atoms of $\mathcal{F}$. We then obtain, respectively, $\mu$ - and $\mathcal{F}$-measurable maps $(\xi, x) \mapsto \mathcal{I}_{(\xi, x)}^{s}$ and $(\xi, x) \mapsto \mathcal{I}_{(\xi, x)}^{u}$ from $X$ to the space of $C^{1}$-embeddings of $\mathbb{R}$ into $M$ given by

$$
\begin{equation*}
\mathcal{I}_{(\xi, x)}^{s}: t \mapsto\left(H_{(\xi, x)}^{s}\right)^{-1}\left(t v_{(\xi, x)}^{s}\right), \quad \mathcal{I}_{(\xi, x)}^{u}: t \mapsto\left(H_{(\xi, x)}^{u}\right)^{-1}\left(t v_{(\xi, x)}^{u}\right) . \tag{6.4}
\end{equation*}
$$

6.3. Families of conditional measures. The family of fiber-wise unstable manifolds $\left\{W^{u}(\xi, x)\right\}_{(\xi, x) \in X}$ forms a partition of a full measure subset of $X$. However, such a partition is generally non-measurable. To define conditional measures we consider a measurable partition $\mathcal{P}$ of $X$ such that for $\mu$-a.e. $(\xi, x) \in X$ there is an $r>0$ such that $W_{r}^{u}(\xi, x) \subset \mathcal{P}(\xi, x) \subset W^{u}(\xi, x)$. Such a partition is said to be $u$ subordinate. Let $\left\{\tilde{\mu}_{(\xi, x)}^{\mathcal{P}}\right\}_{(\xi, x) \in X}$ denote a family of conditional probability measures with respect to such a partition $\mathcal{P}$.

Definition 6.7. An $\mathscr{F}$-invariant measure $\mu$ is fiber-wise $S R B$ if for any $u$ subordinate measurable partition $\mathcal{P}$ with corresponding family of conditional measures $\left\{\tilde{\mu}_{(\xi, x)}^{\mathcal{P}}\right\}_{(\xi, x) \in X}$, the measure $\tilde{\mu}_{(\xi, x)}^{\mathcal{P}}$ is absolutely continuous with respect to a Riemannian volume on $W^{u}(\xi, x)$ for a.e. ( $\left.\xi, x\right)$.

In the setting introduced in Section 3 we have the following.
Definition 6.8. Let $M$ be a closed manifold, let $\hat{\nu}$ be a Borel measure on $\operatorname{Diff}^{2}(M)$, and let $\hat{\mu}$ be a $\hat{\nu}$-stationary probability measure. We say $\hat{\mu}$ is $S R B$ if the measure $\mu$ given by Proposition 4.2 is fiber-wise SRB for the associated canonical skew product (4.1).

Remark 6.9. In fact, it follows from the proof of Proposition 2.2 (see also BL for a related statement for general skew products) that $\mu$ is fiber-wise SRB if and only if the conditional measures $\left\{\tilde{\mu}_{(\xi, x)}^{\mathcal{P}}\right\}_{(\xi, x) \in X}$ are equivalent to Riemannian volume on $W^{u}(x, \xi)$ restricted to $\mathcal{P}(\xi, x)$. Furthermore, with respect to the affine parameters introduced in Section 6.2 the conditional measures coincide up to normalization with the Haar measure. See [YY1, Corollary 6.1.4].

Following a standard procedure, by fixing a normalization, for a.e. $(\xi, x)$ we define a locally finite, infinite measure $\mu_{(\xi, x)}^{u}$ on the curve $W^{u}(\xi, x)$ that restricts to $\left\{\tilde{\mu}_{(\xi, x)}^{\mathcal{P}}\right\}_{(\xi, x) \in X}$, up to normalization, for any $u$-subordinate partition $\mathcal{P}$ of $X$. We choose the normalization $\mu_{(\xi, x)}^{u}\left(W_{1}^{u}(\xi, x)\right)=1$. Such a measure will be locally finite in the internal topology of $W^{u}(p)$ induced, for instance, by the affine parameters. We remark that the fiber entropy vanishes if and only if the measures $\mu_{(\xi, x)}^{u}$ and $\mu_{(\xi, x)}^{s}$ have support $\{(\xi, x)\}$ for almost every $(\xi, x) \in X$.
6.4. Relationships between entropy, exponents, and dimension. Given $(\xi, x)$ $\in X$ we define the following pointwise dimensions:
(1) $\operatorname{dim}^{u}(\mu,(\xi, x)):=\lim _{r \rightarrow 0} \frac{\log \left(\mu_{(\xi, x)}^{u}\left(W_{r}^{u}(\xi, x)\right)\right)}{\log r}$,
(2) $\operatorname{dim}^{s}(\mu,(\xi, x)):=\lim _{r \rightarrow 0} \frac{\log \left(\mu_{(\xi, x)}^{s}\left(W_{r}^{s}(x) \xi\right)\right)}{\log r}$,
(3) $\operatorname{dim}(\mu,(\xi, x)):=\lim _{r \rightarrow 0} \frac{\log \left(\mu_{\xi}\{y \in M: d(x, y)<r\}\right)}{\log r}$.

We note that $\operatorname{dim}(\mu,(\xi, x))$ is the pointwise dimension of conditional measure $\mu_{\xi}$ at the point $x$. In the case that $\mu$ is obtained from a stationary measure $\hat{\mu}$ from Proposition 4.2, this need not coincide with the pointwise dimension of $\hat{\mu}$ at $x$. We have that $\operatorname{dim}^{u}(\mu,(\xi, x))$ and $\operatorname{dim}^{s}(\mu,(\xi, x))$ are well defined and are furthermore constant a.e. by the ergodicity of $\mu$. Write $\operatorname{dim}^{s / u}(\mu)$ for these constants. Note that $\operatorname{dim}(\mu,(\xi, x))$ may not be defined if there are zero exponents.

We have the following proposition. Recall we write $\pi: X \rightarrow \Omega$ for the natural projection and $h_{\mu}(F \mid \pi)$ for the conditional metric entropy of $(F, \mu)$ conditioned on the sub- $\sigma$-algebra generated by $\pi^{-1}$.

Proposition 6.10. In our setting,
(1) $h_{\mu}(F \mid \pi)=\lambda^{u} \operatorname{dim}^{u}(\mu)=-\lambda^{s} \operatorname{dim}^{s}(\mu)$;
(2) $\operatorname{dim}(\mu,(\xi, x))=\operatorname{dim}^{u}(\mu)+\operatorname{dim}^{s}(\mu)$ for $\mu$-a.e. $(\xi, x)$.

Conclusion (1) follows from a generalization to the case of skew products of the Ledrappier-Young entropy formula [Y2. This generalization appears in [X] in the case of i.i.d. random dynamics; modifications for the case of general skew products are outlined in QQX. Conclusion (2) follows from the results of QQX generalizing to the random setting the dimension formula for hyperbolic measures proven in [BPS.

In our setting, we then have the following equivalent characterizations of the fiber-wise Sinai-Ruelle-Bowen (SRB) property.

Lemma 6.11. The following are equivalent:
(1) $\mu$ is fiber-wise SRB;
(2) $h_{\mu}(F \mid \pi)=\lambda^{u}$;
(3) the measures $\mu_{(\xi, x)}^{u}$ are equivalent to Riemannian arc-length on $W_{\xi}^{u}(x)$ almost everywhere;
(4) $\operatorname{dim}^{u}(\mu)=1$.
6.5. The family $\bar{\mu}_{(\xi, x)}$. Using the affine parameters $H_{(\xi, x)}^{u}: W_{\xi}^{u}(x) \rightarrow E_{\xi}^{u}(x)$ and the trivialization (6.4), we define a family of locally finite Borel measures on $\mathbb{R}$ by

$$
\begin{equation*}
\bar{\mu}_{(\xi, x)}:=\left(\mathcal{I}_{(\xi, x)}^{u}\right)_{*}^{-1} \mu_{\xi, x)}^{u} . \tag{6.5}
\end{equation*}
$$

We equip the space of locally finite Borel measures on $\mathbb{R}$ with its standard Borel structure (dual to compactly supported continuous functions). We thus obtain a measurable function from $X$ to the locally finite Borel measures on $\mathbb{R}$. Since the family of measures $(\xi, x) \mapsto \mu_{(\xi, x)}^{u}$ and parametrizations $\mathcal{I}^{u}$ are $\mathcal{F}$-measurable, it follows that

$$
(\xi, x) \mapsto \bar{\mu}_{(\xi, x)}
$$

is $\mathcal{F}$-measurable.
The family $\left\{\bar{\mu}_{(\xi, x)}\right\}_{(\xi, x) \in X}$ will be our primary focus in the sequel. In particular, the SRB property of $\mu$ will follow by showing that for $\mu$-a.e. $p$, the measure $\bar{\mu}_{(\xi, x)}$ is the Lebesgue (Haar) measure on $\mathbb{R}$ (normalized on $[-1,1]$ ).

## 7. Main proposition and proof of Theorem 4.10

7.1. Main proposition. The major technical result in the proof of Theorem 4.10 is the following key proposition, whose proof occupies Sections 9 and 10 Given two locally finite measures $\eta_{1}$ and $\eta_{2}$ on $\mathbb{R}$ we write $\eta_{1} \simeq \eta_{2}$ if there is some $c>0$ with $\eta_{1}=c \eta_{2}$.

Proposition 7.1. Assume in Theorem 4.10 that $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable. Then there exists $M>0$ such that for every sufficiently small $\varepsilon>0$ there exists a measurable set $G_{\varepsilon} \subset X$ with

$$
\mu\left(G_{\varepsilon}\right)>0
$$

such that for any $(\xi, x) \in G_{\varepsilon}$ there is an affine map

$$
\psi: \mathbb{R} \rightarrow \mathbb{R}
$$

with
(1) $\frac{1}{M} \leq|D \psi| \leq M$;
(2) $\frac{\varepsilon}{M} \leq|\psi(0)| \leq M \varepsilon$;
(3) $\psi_{*} \bar{\mu}_{(\xi, x)} \simeq \bar{\mu}_{(\xi, x)}$.

Furthermore, writing

$$
G:=\left\{(\xi, x) \in X \mid(\xi, x) \in G_{1 / N} \text { for infinitely many } N\right\}
$$

we have $\mu(G)>0$.
Remark 7.2. Given the space of locally finite Borel measures on $\mathbb{R}$, the set of measures satisfying (1)-(3) of Proposition 7.1 for fixed $\varepsilon$ and $M$ is closed. By restricting to measurable sets on which $(\xi, x) \mapsto \bar{\mu}_{(\xi, x)}$ is continuous, for any fixed $M$ defining $G_{\varepsilon}$ to be the set of $(\xi, x)$ such that $\bar{\mu}_{(\xi, x)}$ satisfies (1)-(3) above it follows that $G_{\varepsilon}$ is measurable. Thus, the proof of Proposition 7.1 reduces to showing that $G_{\varepsilon}$ and $G$ have a positive measure for some $M$.
7.2. Proof of Theorem 4.10. Theorem 4.10 follows from Proposition 7.1 by standard arguments. We sketch these below and refer to [KK] for more details.

Lemma 7.3. Under the hypotheses of Proposition 7.1, for a.e. $(\xi, x) \in X, \bar{\mu}_{(\xi, x)}$ is invariant under the group of translations. In particular, for a.e. $(\xi, x) \in X, \bar{\mu}_{(\xi, x)}$ is the Lebesgue measure on $\mathbb{R}$ normalized on $[-1,1]$.
Proof. Let $\operatorname{Aff}(\mathbb{R})$ denote the group of invertible affine transformations of $\mathbb{R}$. For $(\xi, x) \in X$, let $\mathcal{A}(\xi, x) \subset \operatorname{Aff}(\mathbb{R})$ be the group of affine transformations $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with

$$
\psi_{*} \bar{\mu}_{(\xi, x)} \simeq \bar{\mu}_{(\xi, x)}
$$

We have that $\mathcal{A}(\xi, x)$ is a closed subgroup of the Lie group $\operatorname{Aff}(\mathbb{R})$. (See the proof of [KK, Lemma 3.10].) By Proposition [7.1, for $(\xi, x) \in G$, the group $\mathcal{A}(\xi, x)$ contains elements of the form $t \mapsto \lambda_{j} t+v_{j}$ with $v_{j} \neq 0,\left|v_{j}\right| \rightarrow 0$ as $j \rightarrow \infty$, and $\lambda_{j} \in \mathbb{R}$ such that $\left|\lambda_{j}\right|$ is uniformly bounded away from 0 and $\infty$. Then, for $(\xi, x) \in G, \mathcal{A}(\xi, x)$ contains at least one map of the form

$$
t \mapsto \lambda t
$$

for some accumulation point $\lambda$ of $\left\{\lambda_{j}\right\} \subset \mathbb{R}$. We may thus find a subsequence of

$$
\left\{t \mapsto \lambda^{-1} \lambda_{j} t+v_{j}\right\}
$$

converging to the identity in $\mathcal{A}(\xi, x)$. It follows that $\mathcal{A}(\xi, x)$ is not discrete. In particular, for every $(\xi, x) \in G$ the group $\mathcal{A}(\xi, x)$ contains a one-parameter subgroup of $\operatorname{Aff}(\mathbb{R})$.

For $(\xi, x) \in X$ denote by $\mathcal{C}_{(\xi, x)}: \mathbb{R} \rightarrow \mathbb{R}$ the linear map

$$
\mathcal{C}_{(\xi, x)}=\left(\mathcal{I}_{F(\xi, x)}^{u}\right)^{-1} \circ F \circ \mathcal{I}_{(\xi, x)}^{u},
$$

where $\mathcal{I}_{(\xi, x)}^{u}$ denotes the parametrization (6.4). As $\left(\mathcal{C}_{(\xi, x)}\right)_{*} \bar{\mu}_{(\xi, x)} \simeq \bar{\mu}_{F(\xi, x)}$ we have that

$$
\mathcal{A}(F(\xi, x))=\mathcal{C}_{(\xi, x)} \mathcal{A}(\xi, x) \mathcal{C}_{(\xi, x)}^{-1}
$$

Let $\mathcal{A}_{0}(\xi, x) \subset \mathcal{A}(\xi, x)$ denote the identity component of $\mathcal{A}(\xi, x)$. Then $\mathcal{A}_{0}(F(\xi, x))$ is isomorphic to $\mathcal{A}_{0}(\xi, x)$ for a.e. $(\xi, x) \in X$. Since $\mu(G)>0$, it follows by ergodicity that $\mathcal{A}_{0}(\xi, x)$ contains a one-parameter subgroup for a.e. $(\xi, x) \in X$.

The one-parameter subgroups of $\mathrm{Aff}(\mathbb{R})$ either are pure translations or are conjugate to scaling. We show that $\mathcal{A}(\xi, x)$ contains the group of translations for a.e. $(\xi, x) \in X$. Suppose for purposes of contradiction that $\mathcal{A}_{0}(\xi, x)$ were conjugate to scaling for a positive measure set of $(\xi, x) \in X$. By ergodicity, it follows that $\mathcal{A}_{0}(\xi, x)$ is conjugate to scaling for a.e. $(\xi, x) \in X$. For such $(\xi, x)$, there are $t_{0} \in \mathbb{R}, \gamma \in \mathbb{R}_{+}$with

$$
\mathcal{A}_{0}(\xi, x)=\left\{t \mapsto t_{0}+\gamma^{s}\left(t-t_{0}\right) \mid s \in \mathbb{R}\right\} .
$$

In particular, for such $(\xi, x)$ the action of $\mathcal{A}_{0}(\xi, x)$ on $\mathbb{R}$ contains a unique fixed point $t_{0}(\xi, x)$.

For $(\xi, x) \in G$ the fixed point $t_{0}(\xi, x)$ is non-zero since, as observed above, there are $\psi \in \mathcal{A}(\xi, x)$ arbitrarily close to the identity with $\psi(0) \neq 0$. Furthermore, writing $\psi: t \mapsto t_{0}(\xi, x)+\gamma^{s}\left(t-t_{0}(\xi, x)\right)$ we have

$$
\mathcal{C}_{(\xi, x)} \circ \psi \circ \mathcal{C}_{(\xi, x)}^{-1}: t \mapsto \pm\left\|D F \upharpoonright_{E^{u}(\xi, x)}\right\| t_{0}(\xi, x)+\gamma^{s}\left(t- \pm\left\|D F \upharpoonright_{E^{u}(\xi, x)}\right\| t_{0}(\xi, x)\right)
$$

where the sign depends on whether $\mathcal{C}_{(\xi, x)}: \mathbb{R} \rightarrow \mathbb{R}$ preserves orientation. It follows for $(\xi, x) \in G$ that $\left|t_{0}\left(F^{n}(\xi, x)\right)\right|=\left\|D F^{n}{ }_{E^{u}(\xi, x)}\right\|\left|t_{0}(\xi, x)\right|$ becomes arbitrarily large, contradicting Poincaré recurrence.

Therefore, for almost every $(\xi, x) \in X$, the group $\mathcal{A}(\xi, x)$ contains the group of translations. We finish the proof by showing that for such $(\xi, x)$, the measure $\bar{\mu}_{(\xi, x)}$ is invariant under the group of translations. For $s \in \mathbb{R}$ define $T_{s}: \mathbb{R} \rightarrow \mathbb{R}$ by $T_{s}: t \mapsto t+s$ and define $c_{(\xi, x)}: \mathbb{R} \rightarrow \mathbb{R}$ by $c_{(\xi, x)}(s)=\bar{\mu}_{(\xi, x)}([-s-1,-s+1])$. Then

$$
\frac{d\left(T_{s}\right)_{*} \bar{\mu}_{(\xi, x)}}{d \bar{\mu}_{(\xi, x)}}=c_{(\xi, x)}(s)
$$

As the group $\mathcal{A}(\xi, x)$ contains all translations, the measure $\bar{\mu}_{(\xi, x)}$ has no atoms and we have that $c_{(\xi, x)}: \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Note that

$$
\mathcal{C}_{(\xi, x)} \circ T_{s} \circ \mathcal{C}_{(\xi, x)}^{-1}=T_{ \pm\left\|D F \upharpoonright_{E^{u}(\xi, x)}\right\| s}
$$

and for $n \in \mathbb{Z}$

$$
\begin{equation*}
c_{(\xi, x)}(s)=c_{F^{n}(\xi, x)}\left( \pm\left\|D F^{n} \upharpoonright_{E^{u}(\xi, x)}\right\| s\right) \tag{7.1}
\end{equation*}
$$

where the signs depend on whether $D F$ or $D F^{n}$ preserves the orientation on unstable subspaces. Define the set

$$
B_{r, \varepsilon}:=\left\{(\xi, x) \in X:\left|c_{(\xi, x)}(s)-1\right|<\varepsilon \text { for all }|s|<r\right\}
$$

For each $\varepsilon>0$ pick $r$ so that $\mu\left(B_{r, \varepsilon}\right)>0$. Applying (7.1), by ergodicity almost every point visits $B_{r, \varepsilon}$ infinitely often as $n \rightarrow-\infty$ contradicting (7.1) unless $\mid c_{(\xi, x)}(s)-$ $1 \mid<\varepsilon$ for all $s$ and a.e. $(\xi, x) \in X$. Taking $\varepsilon \rightarrow 0$ shows that $c_{(\xi, x)}(s)=1$ for all $s \in \mathbb{R}$ and a.e. $(\xi, x) \in X$ completing the proof of the lemma.

Theorem 4.10 now follows as an immediate corollary of Proposition 7.1 and Lemma 7.3

## 8. SUSPENSION FLOW

As in the proof of BQ1 and EM, Sections 15 and 16], to prove Proposition [7.1]we introduce a suspension flow. On the product space $\mathbb{R} \times X$ consider the identification

$$
(t, \xi, x) \sim(t-1, F(\xi, x))=\left(t-1, \theta(\xi), f_{\xi}(x)\right)
$$

and define the quotient space $Y=(\mathbb{R} \times X) / \sim$. We denote by $[t, \xi, x]$ the element of the quotient space $Y$. The space $Y$ is equipped with a natural flow

$$
\Phi^{t}: Y \rightarrow Y, \quad \Phi^{t}([s, \xi, x])=[s+t, \xi, x] .
$$

We have that $\Phi^{t}$ preserves a Borel probability measure

$$
d \omega([t, \xi, x])=d \mu(\xi, x) d t
$$

It is convenient to consider the measurable parametrization $[0,1) \times X \rightarrow Y$ given by $(\varsigma, \xi, x) \mapsto[\varsigma, \xi, x]$. In these coordinates the flow $\Phi^{t}: Y \rightarrow Y$ is given by

$$
\Phi^{t}(\varsigma, \xi, x)=\left(\{\varsigma+t\}, F^{\lfloor\varsigma+t\rfloor}(\xi, x)\right),
$$

where $\lfloor\ell\rfloor$ denotes the integer part of $\ell$ and $\{\ell\}=\ell-\lfloor\ell\rfloor$. When we write $(\varsigma, \xi, x) \in$ $Y$, it is implied that $0 \leq \varsigma<1$ and that $(\varsigma, \xi, x)$ is identified with $[\varsigma, \xi, x]$. Given $(\varsigma, \xi) \in[0,1) \times \Omega$ we write $M_{(\varsigma, \xi)}=\{\varsigma\} \times\{\xi\} \times M$. We will also write $\Theta^{t}:[0,1) \times \Omega \rightarrow$ $[0,1) \times \Omega$ for the induced suspension flow.

Note that the parametrization $Y=[0,1) \times X$ makes $Y$ into a Polish space with respect to which the measure $\omega$ is Radon. To discuss convergence and continuity, we equip $[0,1)$ and $\Omega$ with complete, separable metrics and endow $Y=[0,1) \times \Omega \times M$ with the product metric.

We use the parametrization $[0,1) \times X \rightarrow Y$ to extend the definition of local and global unstable manifolds. Given $p=(\varsigma, \xi, x) \in Y$ write

- $W_{(\varsigma, \xi), r}^{u}(x)=W_{\xi, r}^{u}(x) \subset M ; W_{(\varsigma, \xi)}^{u}(x)=W_{\xi}^{u}(x) \subset M$;
- $W_{r}^{u}(p)=\{\varsigma\} \times W_{r}^{u}(\xi, x)=\{\varsigma\} \times \xi \times W_{\xi, r}^{u}(x) \subset Y$;
- $W^{u}(p)=\{\varsigma\} \times W^{u}(\xi, x)=\{\varsigma\} \times \xi \times W_{\xi}^{u}(x) \subset Y$.

We similarly extend the definition of local and global stable manifolds, affine parameters, frames for the stable and unstable spaces introduced in Section 6.2.1, and the induced parametrizations $\mathcal{I}^{u}$ and $\mathcal{I}^{s}$. Given $p=(\varsigma, \xi, x) \in Y$, we write $\omega_{p}^{u}=\delta_{\varsigma} \times \mu_{(\xi, x)}^{u}$ for the locally finite measures on $W^{u}(p)$ normalized on $W_{1}^{u}(p)$ and $\bar{\omega}_{p}:=\left(\mathcal{I}_{p}^{u}\right)_{*}^{-1}\left(\omega_{p}^{u}\right)=\bar{\mu}_{(\xi, x)}$ for the corresponding measure on $\mathbb{R}$.

Although the flow $\Phi^{t}: Y \rightarrow Y$ is, at best, measurable, the restriction

$$
\Phi^{t}: M_{(\varsigma, \xi)} \rightarrow M_{\Theta^{t}(\varsigma, \xi)}
$$

is a $C^{2}$-diffeomorphism. Define a fiber-wise tangent bundle

$$
T Y:=[0,1) \times T X=[0,1) \times \Omega \times T M
$$

and the fiber-wise differential $D \Phi^{t}: T Y \rightarrow T Y$

$$
D \Phi^{t}:(\varsigma, \xi,(x, v)) \mapsto\left(\{\varsigma+t\}, \theta^{\lfloor\varsigma+t\rfloor}(\xi),\left(f_{\xi}^{\lfloor\varsigma+t\rfloor}(x), D_{x} f_{\xi}^{\lfloor\varsigma+t\rfloor}(v)\right)\right)
$$

We trivially extend norms on $T M$ to $T Y$ by identifying $\{(\varsigma, \xi)\} \times T_{x} M$ with $T_{x} M$.

## 9. Preparations for the proof of Proposition 7.1

We begin with a number of constructions and technical lemmas that will be used in the proof of Proposition 7.1 .
9.1. Modification of $\hat{\mathcal{F}}$. Recall we assume the function $\xi \mapsto f_{\xi}^{-1}$ is $\hat{\mathcal{F}}$-measurable which implies the entire past dynamics $\xi \mapsto f_{\xi}^{n}$ is $\hat{\mathcal{F}}$-measurable for all $n \leq 0$. It is convenient for technical reasons below to allow the first future iterate $f_{\xi}$ to be measurable on $\hat{\mathcal{F}}$ as well. As $f_{\xi}=\left(f_{\theta(\xi)}^{-1}\right)^{-1}$, this can be accomplished by replacing $\hat{\mathcal{F}}$ with $\theta^{-1}(\hat{\mathcal{F}}) \subset \hat{\mathcal{F}}$. Then $\theta^{-1}(\hat{\mathcal{F}})$ is a decreasing sub- $\sigma$-algebra for which $\xi \mapsto f_{\xi}^{n}$ is measurable for all $n \leq 1$. Moreover, as $f_{\xi}$ is constant on atoms of $\theta^{-1}(\hat{\mathcal{F}})$, we have that

$$
(\xi, x) \mapsto E_{\xi}^{s}(x)
$$

is $\mathcal{F}$-measurable if and only if $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $F^{-1}(\mathcal{F})$-measurable.
Thus for the remainder, we replace $\hat{\mathcal{F}}$ and $\mathcal{F}$ with $\theta^{-1}(\hat{\mathcal{F}})$ and $F^{-1}(\mathcal{F})$, respectively. With this new notation, we then have that $\xi \mapsto f_{\xi}^{n}$ is $\hat{\mathcal{F}}$-measurable for all $n \leq 1$.
9.2. Lyapunov norms. From Lemma 6.2, for each $p \in Y$ we observe the hyperbolicity of the cocycle $D \Phi^{t}$ after a finite amount of time. We define here two norms, called Lyapunov norms, with respect to which the hyperbolicity of $D \Phi^{t}$ is seen immediately. We remark that while the induced Riemannian norm $\|\cdot\|$ on $T Y$ is constant in the first parameter of the parametrization $[0,1) \times X \rightarrow Y$, the Lyapunov norms defined below will vary in the parameter $\varsigma$.

We first define the Lyapunov norms for the skew product $F$ on $X$. For $(\xi, x) \in X$, $\sigma \in\{s, u\}$, and $v \in E_{\xi}^{\sigma}(x)$ define the two-sided Lyapunov norm

$$
\begin{equation*}
\|v\|_{\epsilon_{0}, \pm,(\xi, x)}^{\sigma}:=\left(\sum_{n \in \mathbb{Z}}\left\|D f_{\xi}^{n} v\right\|^{2} e^{-2 \lambda^{\sigma} n-2 \epsilon_{0}|n|}\right)^{1 / 2} \tag{9.1}
\end{equation*}
$$

and the past one-sided Lyapunov norm

$$
\begin{equation*}
\|v\|_{\epsilon_{0},-,(\xi, x)}^{\sigma}:=\left(\sum_{n \leq 0}\left\|D f_{\xi}^{n} v\right\|^{2} e^{-2 \lambda^{\sigma} n-2 \epsilon_{0}|n|}\right)^{1 / 2} \tag{9.2}
\end{equation*}
$$

It follows from Lemma 6.2 that the sums above converge for almost every $(\xi, x) \in X$. Observe that for $v \in E_{\xi}^{\sigma}(x)$,

$$
\|v\| \leq\|v\|_{\epsilon_{0},-,(\xi, x)}^{\sigma}, \quad\|v\| \leq\|v\|_{\epsilon_{0}, \pm,(\xi, x)}^{\sigma} .
$$

Remark 9.1. Recall that we have $\xi \rightarrow f_{\xi}^{-n}$ is $\hat{\mathcal{F}}$-measurable for all $n \geq 0$ whence the assignment $(\xi, x) \mapsto E_{\xi}^{u}(x)$ is $\mathcal{F}$-measurable. Recall the $\mathcal{F}$-measurable family of vectors $v_{(\xi, x)}^{u} \in E_{\xi}^{u}(x)$ built in Section6.2.1. It follows from construction that $(\xi, x) \mapsto$ $\left\|v_{(\xi, x)}^{u}\right\|_{\epsilon_{0},-,(\xi, x)}^{u}$ is $\mathcal{F}$-measurable. Moreover, as discussed in Section 9.1] since we assume $\xi \mapsto f_{\xi}$ is $\hat{\mathcal{F}}$-measurable, we have that $(\xi, x) \mapsto\left\|\left\|D_{x} f_{\xi} v_{(\xi, x)}^{u}\right\|_{\epsilon_{0},-, F(\xi, x)}^{u}\right.$ is $\mathcal{F}$-measurable. This will be the primary reason for using the one-sided Lyapunov norms rather than two-sided Lyapunov norms below.

We have the following bounds on hyperbolicity. For the one-sided norm $\|\cdot\| \|_{\epsilon_{0},-,(\xi, x)}^{\sigma}$, the bounds are of most use when $\sigma=u$. (One can similarly define the future one-sided norm $\left\|\|\cdot\|_{\epsilon_{0},+,(\xi, x)}^{s}\right.$ which is more natural for the stable bundle.)

Lemma 9.2. For $\mu$-a.e. $(\xi, x) \in X, v \in E_{\xi}^{\sigma}(x), n \in \mathbb{Z}$, and $k \geq 0$ we have

$$
\begin{aligned}
& \text { (1) } e^{n \lambda^{\sigma}-|n| \epsilon_{0}}\|v v\|_{\epsilon_{0}, \pm,(\xi, x)}^{\sigma} \leq\left\|D f_{\xi}^{n} v\right\|_{\epsilon_{0}, \pm, F^{n}(\xi, x)}^{\sigma} \leq e^{n \lambda^{\sigma}+|n| \epsilon_{0}}\|v\|_{\epsilon_{0}, \pm,(\xi, x)}^{\sigma}, \\
& \text { (2) } e^{k \lambda^{\sigma}-k \epsilon_{0}}\|v\|_{\epsilon_{0},-,(\xi, x)}^{\sigma} \leq\left\|D f_{\xi}^{k} v\right\| \|_{\epsilon_{0},-, F^{k}(\xi, x)}^{\sigma}
\end{aligned}
$$

9.2.1. Extensions to $Y$. We extend the Lyapunov norms to $T Y$ as follows: For $p=(\varsigma, \xi, x) \in Y$ and $w \in E_{\xi}^{\sigma}(x)=E_{(\varsigma, \xi)}^{\sigma}(x) \subset T_{x} M$, define

$$
\begin{equation*}
\|w\|_{\epsilon_{0},-, p}^{\sigma}=\left(\|w\|_{\epsilon_{0},-,(\xi, x)}^{\sigma}\right)^{1-\varsigma}\left(\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-, F(\xi, x)}^{\sigma}\right)^{\varsigma} . \tag{9.3}
\end{equation*}
$$

Identifying $E^{\sigma}(p)=\{(\varsigma, \xi)\} \times E_{\xi}^{\sigma}(x) \subset T Y$ with $E_{(\varsigma, \xi)}^{\sigma}(x) \subset T M$, we extend the definition of $\|\cdot\| \|_{\epsilon_{0},-, p}^{\sigma}$ to $E^{\sigma}(p)$. We similarly extend the two-sided Lyapunov norms to $T Y$.

Given $t \in \mathbb{R}$ we write

$$
\left\|\mid D \Phi^{t} \upharpoonright_{E^{\sigma}(p)}\right\|\left\|_{\epsilon_{0},-}, \quad\right\| D \Phi^{t} \Gamma_{E^{\sigma}(p)}\| \|_{\epsilon_{0}, \pm}
$$

to indicate the operator norm of $D \Phi^{t}: E^{\sigma}(p) \rightarrow E^{\sigma}\left(\Phi^{t}(p)\right)$ with respect to the corresponding norms.

We have the following extension of Lemma 9.2
Lemma 9.3. For $\omega$-a.e. $p=(\varsigma, \xi, x) \in Y, v \in E^{\sigma}(p), t \in \mathbb{R}$, and $s \geq 0$ we have
(1) $e^{t \lambda^{\sigma}-|t| \epsilon_{0}}\|v\|_{\epsilon_{0}, \pm, p}^{\sigma} \leq\left\|D \Phi^{t} v\right\|_{\epsilon_{0}, \pm, \Phi^{t}(p)}^{\sigma} \leq e^{t \lambda^{\sigma}+|t| \epsilon_{0}}\|v v\|_{\epsilon_{0}, \pm, p}^{\sigma}$,
(2) $e^{s \lambda^{\sigma}-s \epsilon_{0}}\|v\|_{\epsilon_{0},-, p}^{\sigma} \leq\left\|D \Phi^{s} v\right\|_{\epsilon_{0},-, \Phi^{s}(p)}^{\sigma}$.

We have the following estimate which allows us to compare the Lyapunov norm with the induced Riemannian norm. Recall the functions $D: \Omega \rightarrow \mathbb{R}$ and $L: X \rightarrow \mathbb{R}$ in Lemmas 6.1 and 6.2. Let $c_{1}=e^{\epsilon_{0}}\left(1-e^{-\epsilon_{0}}\right)^{1 / 2}$.

Lemma 9.4. For any $w \in E^{u}(\varsigma, \xi, x)$,

$$
\|w\| \leq\|w\|_{\epsilon_{0},-, p}^{u} \leq L(\xi, x) D(\xi) c_{1}\|w\| .
$$

In particular, defining $\hat{L}: Y \rightarrow[1, \infty)$ by

$$
\hat{L}(\varsigma, \xi, x)=L(\xi, x) D(\xi) c_{1}
$$

we have

$$
\hat{L}\left(\Phi^{t}(p)\right) \leq e^{2 \epsilon_{0}(|t|+1)} \hat{L}(p)
$$

and

$$
\begin{equation*}
\hat{L}(p)^{-1}\left\|D \Phi^{t} \upharpoonright_{E^{u}(p)}\right\| \leq\left\|D \Phi^{t} \Gamma_{E^{u}(p)}\right\|\left\|_{\epsilon_{0},-}^{u} \leq e^{2 \epsilon_{0}(|t|+1)} \hat{L}(p)\right\| D \Phi^{t} \Gamma_{E^{u}(p)} \| . \tag{9.4}
\end{equation*}
$$

(A similar estimate holds for the two-sided norms.)
Proof. Recall that for $w \in E_{\xi}^{u}(x)$, we have $\|w\| \leq\|w\|_{\epsilon_{0},-,(\xi, x)}^{u}$ and

$$
\|w\|_{\epsilon_{0},-,(\xi, x)}^{u}<\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-, F(\xi, x)}^{u} .
$$

The lower bound then follows.

For the upper bound we have for $(\xi, x) \in X$ and $w \in E_{\xi}^{u}(x)$ that

$$
\begin{aligned}
\|w\|_{\epsilon_{0},-,(\xi, x)}^{u} & \leq\left(\sum_{n \leq 0}(L(\xi, x))^{2}\|w\|^{2} e^{2 n \lambda^{u}+|n| \epsilon_{0}} e^{-2 n \lambda^{u}-2 \epsilon_{0}|n|}\right)^{1 / 2} \\
& =L(\xi, x)\left(1-e^{-\epsilon_{0}}\right)^{-1 / 2}\|w\|
\end{aligned}
$$

Similarly, from Lemma 6.2 we have

$$
\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-,(\xi, x)}^{u} \leq L(\xi, x) e^{\epsilon_{0}}\left(1-e^{-\epsilon_{0}}\right)^{-1 / 2}\left\|D_{x} f_{\xi} w\right\|
$$

Then for $p=(\varsigma, \xi, x) \in Y$ and $w \in E^{u}(p)$, with $b=\left(1-e^{-\epsilon_{0}}\right)^{-1 / 2}$ we have

$$
\begin{aligned}
\|w\|_{\epsilon_{0},-, p}^{u} & :=\left(\|w\| \|_{\epsilon_{0},-,(\xi, x)}^{u}\right)^{1-\varsigma}\left(\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-, F(\xi, x)}^{u}\right)^{\varsigma} \\
& \leq(L(\xi, x) b\|w\|)^{1-\varsigma}\left(L(\xi, x) e^{\epsilon_{0}} b\left\|D_{x} f_{\xi} w\right\|\right)^{\varsigma} \\
& =(L(\xi, x) b\|w\|)^{1-\varsigma}\left(L(\xi, x) e^{\epsilon_{0}} b D(\xi)\|w\|\right)^{\varsigma} \\
& \leq L(\xi, x) b e^{\epsilon_{0}} D(\xi)\|w\| .
\end{aligned}
$$

Declaring that $E^{u}(p)$ and $E^{s}(p)$ are orthogonal, we may extend the definitions of both the two-side and one-side Lyapunov norms to all of $T_{p} Y$. When clear from context, we will drop the majority of sub- and superscripts from the Lyapunov norms.
9.3. The time changed flow. It is convenient to work with a flow $\Psi^{s}$ that is a time change of $\Phi^{t}$ and for which the norm of the restriction of $D \Psi^{s}$ to the unstable spaces grows at a constant rate (with respect to the one-sided norm $\left\|\|\cdot\|_{\epsilon_{0},-}^{u}\right.$ ).

For $p \in Y$ and $t \in \mathbb{R}$, define

$$
\begin{equation*}
\mathscr{S}_{p}(t)=\log \left(\left\|D \Phi^{t} \Gamma_{E^{u}(p)}\right\|_{\epsilon_{0},-}^{u}\right) \tag{9.5}
\end{equation*}
$$

It follows from construction and Lemma 9.3 that, for $\omega$-a.e. $p \in Y$, the function $\mathscr{S}_{p}: \mathbb{R} \rightarrow \mathbb{R}$ is an orientation-preserving homeomorphism. Moreover, as $E^{u}(\xi, x)$ is one-dimensional, the map $Y \times \mathbb{R} \rightarrow \mathbb{R}$ given by $(p, t) \rightarrow \mathscr{S}_{p}(t)$ satisfies the cocycle equation $\mathscr{S}_{p}\left(t_{1}+t_{2}\right)=\mathscr{S}_{\Phi^{t_{2}}(p)}\left(t_{1}\right)+\mathscr{S}_{p}\left(t_{2}\right)$. It follows that $\Psi^{s}: Y \rightarrow Y$ given by

$$
\Psi^{s}(p)=\Phi^{\mathscr{S}_{p}^{-1}(s)}(p)
$$

defines a measurable flow on $Y$ that is a time change of $\Phi^{t}$.
Given $p=(\varsigma, \xi, x) \in Y$ define

$$
\begin{equation*}
h(p)=h(\xi, x)=\log \left(\| \| D_{x} f_{\xi} \upharpoonright_{E_{\xi}^{u}(x)} \|_{\epsilon_{0},-,(\xi, x)}^{u}\right) \tag{9.6}
\end{equation*}
$$

We note that for $-\varsigma \leq t<1-\varsigma$

$$
\log \left|\left|D \Phi^{t} \Gamma_{E^{u}(p)}\right| \|_{\epsilon_{0},-, p}=\operatorname{th}(p)\right.
$$

In particular, if $0 \leq \varsigma+s / h(p)<1$, then $\Psi^{s}(p)=(\varsigma+s / h(p), \xi, x)$; that is, $h(p)^{-1}$ is the local change of speed of the original flow $\Phi^{t}$. It follows that

$$
\mathscr{S}_{p}(t)=\int_{0}^{t} h\left(\Phi^{s}(p)\right) d s
$$

By (9.4), Lemma 6.2 and the fact that $h(p) \geq \lambda^{u}-\epsilon_{0}$ for almost all $p$, we have for any $t \geq 0$ that

$$
\begin{equation*}
\left(\lambda^{u}-\epsilon_{0}\right) t \leq \mathscr{S}_{p}(t) \leq a(p)+b_{0}(t+1) \tag{9.7}
\end{equation*}
$$

where

$$
a(\varsigma, \xi, x)=\log \left(L(\xi, x)^{2} D(\xi) c_{1}\right), \quad b_{0}=\lambda^{u}+3 \epsilon_{0}
$$

We claim the following.
Claim 9.5. $\int h(\xi, x) d \mu(\xi, x)<\infty$.
Proof. Consider $0 \neq w \in E_{\xi}^{u}(x)$. We have

$$
\begin{aligned}
\left(\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-, F(\xi, x)}^{u}\right)^{2} & =\left\|D f_{\xi} w\right\|^{2}+\sum_{n \leq-1}\left\|D f_{\xi}^{n+1} w\right\|^{2} e^{-2 \lambda^{u} n-2 \epsilon_{0}|n|} \\
& =\left\|D_{x} f_{\xi} w\right\|^{2}+e^{2 \lambda^{u}-2 \epsilon_{0}}\left(\sum_{\ell \leq 0}\left\|D f_{\xi}^{\ell} w\right\|^{2} e^{-2 \lambda^{u} \ell-2 \epsilon_{0}|\ell|}\right) \\
& =\left\|D_{x} f_{\xi} w\right\|^{2}+e^{2 \lambda^{u}-2 \epsilon_{0}}\left(\|w\|_{\epsilon_{0},-,(\xi, x)}^{u}\right)^{2}
\end{aligned}
$$

and since $\|w\|^{2} \leq\left(\|w\|_{\epsilon_{0},-,(\xi, x)}^{u}\right)^{2}$, we have

$$
\frac{\left(\left\|D_{x} f_{\xi} w\right\|_{\epsilon_{0},-, F(\xi, x)}^{u}\right)^{2}}{\left(\|w\|_{\epsilon_{0},-,(\xi, x)}^{u}\right)^{2}} \leq\left\|D f_{\xi} \upharpoonright_{E_{\xi}^{u}(x)}\right\|^{2}+e^{2 \lambda^{u}-2 \epsilon_{0}} \leq\left\|D f_{\xi}\right\|^{2}+e^{2 \lambda^{u}-2 \epsilon_{0}}
$$

Recall $\int \log ^{+}\left(\left|f_{\xi}\right|_{C^{1}}\right) d \nu(\xi)<\infty$ by hypothesis (IC). The claim follows as

$$
\int \log \left(\left|f_{\xi}\right|_{C_{1}}^{2}+e^{2 \lambda^{u}-2 \epsilon_{0}}\right) d \nu(\xi)<\infty
$$

From Claim 9.5 it follows that $\Psi^{s}: Y \rightarrow Y$ preserves a probability measure $\hat{\omega}$ given by

$$
d \hat{\omega}(\varsigma, \xi, x)=\frac{1}{\int h(\xi, x) d \mu(\xi, x)} h(\xi, x) d \mu(\xi, x) d \varsigma .
$$

Observe that $\hat{\omega}$ and $\omega$ are equivalent measures. Furthermore, since the $\sigma$-algebras of $\Phi^{t}$ - and $\Psi^{s}$-invariant sets coincide, it follows that $\Psi^{s}$ is $\hat{\omega}$-ergodic.
9.4. Decreasing subalgebras, conditional measures, and the martingale convergence argument. We write $\mathcal{S} \subset \mathcal{B}_{Y}$ and $\hat{\mathcal{S}} \subset \mathcal{B}_{[0,1) \times \Omega}$, respectively, for the completions of $\mathcal{B}_{[0,1)} \otimes \mathcal{F}$ and $\mathcal{B}_{[0,1)} \otimes \hat{\mathcal{F}}$. Note that we have

$$
\Phi^{t}(\mathcal{S}) \subset \mathcal{S}, \quad \Theta^{t}(\hat{\mathcal{S}}) \subset \hat{\mathcal{S}}
$$

for all $t \geq 0$ whence $\mathcal{S}$ and $\hat{\mathcal{S}}$ are decreasing $\sigma$-algebras for the respective flows. In particular, the map

$$
\begin{equation*}
Y \times[0, \infty) \rightarrow(Y, \mathcal{S}, \omega), \quad(p, t) \mapsto \Phi^{-t}(p) \tag{9.8}
\end{equation*}
$$

is $\mathcal{S} \otimes \mathcal{B}_{[0, \infty)}$-measurable. Thus the backwards flow $\Phi^{-t}, t \geq 0$ induces a measurable semi-flow on the factor space $(Y, \mathcal{S}, \omega)$.

As discussed in Remark 9.1 the past dynamics $\xi \mapsto f_{\xi}^{n}, n \leq 0$ is $\hat{\mathcal{F}}$-measurable, and thus the unstable spaces $E_{\xi}^{u}(x)$ and family of one-sided norms $\|\cdot\| \|_{\epsilon_{0},-,(\xi, x)}^{u}$ are
$\mathcal{F}$-measurable. Furthermore, as the extension of the norms $\|\cdot\|_{\epsilon_{0},-,(\xi, x)}^{u}$ to $Y$ in (9.3) involves only the past dynamics and a single future iterate $f_{\xi}$, it follows that the family of one-sided norms $\|\cdot\| \|_{\epsilon_{0},-, p}^{u}$ on $Y$ are $\mathcal{S}$-measurable. It follows that, restricted to the past, the cocycle defining the time change

$$
Y \times[0, \infty) \rightarrow[0, \infty), \quad(p, t) \mapsto \mathscr{S}_{p}(-t)
$$

is $\mathcal{S} \otimes \mathcal{B}_{[0, \infty)}$-measurable. Thus, the backwards time-changed flow $\Psi^{-s}, s \geq 0$, given by $\Psi^{-s}(p)=\Phi^{\mathscr{S}_{p}^{-1}(-s)}(p)$, induces a measurable semi-flow on $(Y, \mathcal{S}, \omega)$. In particular, $\Psi^{s}(\mathcal{S}) \subset \mathcal{S}$ for $s \geq 0$.

Given $m \in \mathbb{R}$ define the sub- $\sigma$-algebra on $Y$ by

$$
\mathcal{S}^{m}:=\Psi^{m}(\mathcal{S})=\left\{\Psi^{m}(C): C \in \mathcal{S}\right\}
$$

From the above discussion, we have the following.
Claim 9.6. For $m \leq \ell$ we have $\mathcal{S}^{\ell} \subset \mathcal{S}^{m}$. In particular, $\left\{\mathcal{S}^{m}\right\}_{m \geq 0}$ defines a decreasing filtration on $(Y, \hat{\omega})$.

As usual, we write $\mathcal{S}^{\infty}=\bigcap_{m=0}^{\infty} \mathcal{S}^{m}$.
9.4.1. Families of conditional measures. We fix once and for all a measurable partition ${ }^{2}$ of $\Omega$ into atoms of $\hat{\mathcal{F}}$ and an induced family of conditional probabilities $\left\{\nu_{\xi}^{\hat{\mathcal{F}}}\right\}_{\xi \in \Omega}$. Since in Theorem 4.10 we assume the map $\xi \mapsto \mu_{\xi}$ is $\hat{\mathcal{F}}$-measurable, defining a family of measures $\left\{\mu_{(\xi, x)}^{\mathcal{F}}\right\}_{(\xi, x) \in X}$ by

$$
d \mu_{(\xi, x)}^{\mathcal{F}}(\eta, y):=d \nu_{\xi}^{\hat{\mathcal{F}}}(\eta) \delta_{x}(y),
$$

it follows that $\left\{\mu_{(\xi, x)}^{\mathcal{F}}\right\}$ defines a family of conditional measures induced by $\mathcal{F}$. For $p=(\varsigma, \xi, x) \in Y$ we write $\omega_{p}^{\mathcal{S}}=\delta_{\varsigma} \times \mu_{(\xi, x)}^{\mathcal{F}}$. Then $\left\{\omega_{p}^{\mathcal{S}}\right\}_{p \in Y}$ is a family of conditional measures of $\omega$ induced by $\mathcal{S}$.

By a slight abuse of notation, for $p=(\varsigma, \xi, x)$ we may consider $\omega_{p}^{\mathcal{S}}$ as measures on $\Omega$ by declaring

$$
d \omega_{p}^{\mathcal{S}}(\eta)=d \omega_{p}^{\mathcal{S}}(\varsigma, \eta, x)
$$

This identifies $\omega_{p}^{\mathcal{S}}(\eta)$ with $\nu_{\xi}^{\hat{\mathcal{F}}}$. In particular, if $q=(\varsigma, \xi, y)$, then under this identification we have $\omega_{p}^{\mathcal{S}}=\omega_{q}^{\mathcal{S}}$.

Recall that $\omega$ and $\hat{\omega}$ are equivalent measures; moreover, $\frac{d \hat{\omega}}{d \omega}(\varsigma, \xi, x)=\frac{1}{\int h d \omega}$ $h(\varsigma, \xi, x)$ where $h$ is the speed change in (9.6). Thus, defining

$$
d \hat{\omega}_{p}^{\mathcal{S}}(q):=\frac{h(q)}{\int h d \omega_{p}^{\mathcal{S}}} d \omega_{p}^{\mathcal{S}}(q)
$$

it follows that $\left\{\hat{\omega}_{p}^{\mathcal{S}}\right\}_{p \in Y}$ defines a family of conditional measures for $\hat{\omega}$ induced by $\mathcal{S}$. As $h: Y \rightarrow \mathbb{R}$ is $\mathcal{S}$-measurable, we may take

$$
\hat{\omega}_{p}^{\mathcal{S}}=\omega_{p}^{\mathcal{S}}
$$

[^2]9.4.2. Martingale convergence argument. Consider any bounded measurable $g: Y \rightarrow$ $\mathbb{R}$. As $\hat{\omega}$ is $\Psi^{s}$-invariant, we have for $m \in \mathbb{R}$
\[

$$
\begin{align*}
\int g\left(\Psi^{m}(q)\right) d\left(\omega_{\Psi-m}^{\mathcal{S}}(p)\right)(q) & =\int g\left(\Psi^{m}(q)\right) d\left(\hat{\omega}_{\Psi^{-m}(p)}^{\mathcal{S}}\right)(q)  \tag{9.9}\\
& =\int g\left(q^{\prime}\right) d\left(\left(\Psi^{m}\right)_{*} \hat{\omega}_{\Psi^{-m}(p)}^{\mathcal{S}}\right)\left(q^{\prime}\right)  \tag{9.10}\\
& \doteq \mathbb{E}_{\hat{\omega}}\left(g \mid \mathcal{S}^{m}\right)(p), \tag{9.11}
\end{align*}
$$
\]

where the first two equalities hold everywhere by definition and the last equality holds as almost-everywhere defined functions.

The right-hand side of (9.11) defines a reverse martingale with respect to the decreasing filtration $\mathcal{S}^{m}$ on $(Y, \hat{\omega})$. By the convergence theorem for reverse martingales, along any discrete subsequence of $m_{j} \in[0, \infty)$ we have, almost surely, that

$$
\mathbb{E}\left(g \mid \mathcal{S}^{m_{j}}\right)(p) \rightarrow \mathbb{E}\left(g \mid \mathcal{S}^{\infty}\right)(p)
$$

On the other hand, given any $m$ and any $p \in Y$, writing $\Psi^{-m}(p)=(\varsigma, \xi, x)$, for all $\varepsilon<\left(\lambda^{u}-\epsilon_{0}\right)^{-1}(1-\varsigma)$

$$
\left(\Psi^{\varepsilon}\right)_{*} \hat{\omega}_{\Psi-m}^{\mathcal{S}}(p)=\hat{\omega}_{\Psi-m+\varepsilon(p)}^{\mathcal{S}}
$$

It follows that the sample paths defined by (9.10) are constant on half-open intervals whose lengths are at least $\left(\lambda^{u}-\epsilon_{0}\right)^{-1}$. Taking a discrete subgroup with gaps less than $\left(\lambda^{u}-\epsilon_{0}\right)^{-1}$ it follows that for almost every $p \in Y$, the left-hand side of (9.9) converges to $\mathbb{E}\left(g \mid \mathcal{S}^{\infty}\right)(p)$ as $m \rightarrow \infty$.
9.5. Stopping times and bi-Lipschitz estimates. Given $p=(\varsigma, \xi, x) \in Y$, $\delta>0, \varepsilon>0$, and $m \in \mathbb{R}$ define

$$
\tau_{p, \delta, \varepsilon}(m):=\sup \left\{\ell \in \mathbb{R}:\left\|D \Phi^{m} \Gamma_{E^{s}(p)}\right\|_{\epsilon_{0}, \pm, p}^{s} \cdot\left\|D \Phi^{\ell} \Gamma_{E^{u}\left(\Phi^{m}(p)\right)}\right\|_{\epsilon_{0}, \pm, \Phi^{m}(p)}^{u} \delta \leq \varepsilon\right\}
$$

and

$$
L_{p, \delta, \varepsilon}(m)=m+\tau_{p, \delta, \varepsilon}(m)
$$

Note that $\tau_{p, \delta, \varepsilon}: \mathbb{R} \rightarrow \mathbb{R}$ and $L_{p, \delta, \varepsilon}: \mathbb{R} \rightarrow \mathbb{R}$ are increasing homeomorphisms. In fact we have the following.

Lemma 9.7. $L_{p, \delta, \varepsilon}$ and $\tau_{p, \delta, \varepsilon}$ are bi-Lipschitz with constants uniform in $p, \delta, \varepsilon$. In particular, for $\ell \geq 0$

$$
\begin{gather*}
\frac{-\lambda^{s}-3 \epsilon_{0}}{\lambda^{u}+\epsilon_{0}} \ell \leq \tau_{p, \delta, \varepsilon}(m+\ell)-\tau_{p, \delta, \varepsilon}(m) \leq \frac{-\lambda^{s}+3 \epsilon_{0}}{\lambda^{u}-\epsilon_{0}} \ell  \tag{9.12}\\
\frac{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}{\lambda^{u}+\epsilon_{0}} \ell \leq L_{p, \delta, \varepsilon}(m+\ell)-L_{p, \delta, \varepsilon}(m) \leq \frac{\lambda^{u}-\lambda^{s}+2 \epsilon_{0}}{\lambda^{u}-\epsilon_{0}} \ell . \tag{9.13}
\end{gather*}
$$

Proof. Write $\tau_{p}=\tau_{p, \delta, \varepsilon}$. By definition we have

$$
\begin{align*}
& \left\|\left\|D \Phi^{\tau_{p}(m+\ell)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\|\right\|_{\epsilon_{0}, \pm} \cdot\| \| D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\| \|_{\epsilon_{0}, \pm}^{-1}  \tag{9.14}\\
& \quad \cdot\left\|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\|\left\|_{\epsilon_{0}, \pm} \cdot\right\| D \Phi^{m+\ell} \upharpoonright_{E^{s}(p)}\| \|_{\epsilon_{0}, \pm} \cdot \delta=\varepsilon .
\end{align*}
$$

As $\tau_{p}(m+\ell) \geq \tau_{p}(m)$, we bound the product of the first two terms of the left-hand side of (9.14) by

$$
\begin{aligned}
& \exp \left(\left(\lambda^{u}-\epsilon_{0}\right)\left(\tau_{p}(m+\ell)-\tau_{p}(m)\right)\right) \\
& \leq\left\|D \Phi^{\tau_{p}(m+\ell)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\|\left\|_{\epsilon_{0}, \pm} \cdot\right\| D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)} \|_{\epsilon_{0}, \pm}^{-1} \\
& \leq \exp \left(\left(\lambda^{u}+\epsilon_{0}\right)\left(\tau_{p}(m+\ell)-\tau_{p}(m)\right)\right) .
\end{aligned}
$$

To bound the remaining terms of (9.14) first note that

$$
\begin{aligned}
& \left\|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\| \|_{\epsilon_{0}, \pm} \\
& =\| \| D \Phi^{\tau_{p}(m)+\ell} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)} \mid\left\|_{\epsilon_{0}, \pm} \cdot\right\| D D \Phi^{\ell} \Gamma_{E^{u}\left(\Phi^{m}(p)\right)}\| \|_{\epsilon_{0}, \pm}^{-1} \\
& =\| \| D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)}\| \|_{\epsilon_{0}, \pm} \cdot\left\|D \Phi^{\ell} \upharpoonright_{E^{u}\left(\Phi^{m+\tau_{p}(m)}(p)\right)}\right\|\left\|_{\epsilon_{0}, \pm} \cdot\right\| D \Phi^{\ell} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)} \|_{\epsilon_{0}, \pm}^{-1} .
\end{aligned}
$$

We have

$$
e^{-2 \epsilon_{0} \ell} \leq\| \| D \Phi^{\ell} \Gamma_{E^{u}\left(\Phi^{m+\tau_{p}(m)}(p)\right)}\| \|_{\epsilon_{0}, \pm} \cdot\left\|D \Phi^{\ell} \Gamma_{E^{u}\left(\Phi^{m}(p)\right)}\right\|_{\epsilon_{0}, \pm}^{-1} \leq e^{2 \epsilon_{0} \ell}
$$

whence it follows that

$$
e^{-2 \epsilon_{0} \ell} \leq \frac{\left\|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\| \|_{\epsilon_{0}, \pm}}{\left\|\left|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)}\right|\right\|_{\epsilon_{0}, \pm}} \leq e^{2 \epsilon_{0} \ell} .
$$

As

$$
\begin{aligned}
& \exp \left(\left(\lambda^{s}-\epsilon_{0}\right) \ell\right) \varepsilon \\
& \leq\left\|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)} \mid\right\|\left\|_{\epsilon_{0}, \pm} \cdot\right\| D \Phi^{m+\ell} \upharpoonright_{E^{s}(p)} \|_{\epsilon_{0}, \pm} \cdot \delta \\
& \leq \exp \left(\left(\lambda^{s}+\epsilon_{0}\right) \ell\right) \varepsilon,
\end{aligned}
$$

we have

$$
\begin{aligned}
& \exp \left(\left(\lambda^{s}-3 \epsilon_{0}\right) \ell\right) \varepsilon \\
& \leq\left\|D \Phi^{\tau_{p}(m)} \upharpoonright_{E^{u}\left(\Phi^{m+\ell}(p)\right)}\right\|\left\|_{\epsilon_{0}, \pm} \cdot\right\| D D \Phi^{m+\ell} \Gamma_{E^{s}(p)}\| \|_{\epsilon_{0}, \pm} \cdot \delta \\
& \leq \exp \left(\left(\lambda^{s}+3 \epsilon_{0}\right) \ell\right) \varepsilon .
\end{aligned}
$$

Reassembling (9.14) we have

$$
\exp \left(\left(\lambda^{u}-\epsilon_{0}\right)\left(\tau_{p}(m+\ell)-\tau_{p}(m)\right)\right) \exp \left(\left(\lambda^{s}-3 \epsilon_{0}\right) \ell\right) \varepsilon \leq \varepsilon
$$

and

$$
\exp \left(\left(\lambda^{u}+\epsilon_{0}\right)\left(\tau_{p}(m+\ell)-\tau_{p}(m)\right)\right) \exp \left(\left(\lambda^{s}+3 \epsilon_{0}\right) \ell\right) \varepsilon \geq \varepsilon
$$

hence

$$
\frac{-\lambda^{s}-3 \epsilon_{0}}{\lambda^{u}+\epsilon_{0}} \ell \leq \tau_{p}(m+\ell)-\tau_{p}(m) \leq \frac{-\lambda^{s}+3 \epsilon_{0}}{\lambda^{u}-\epsilon_{0}} \ell
$$

proving (9.12).
We derive (9.13) from (9.12) noting

$$
\frac{-\lambda^{s}-3 \epsilon_{0}}{\lambda^{u}+\epsilon_{0}} \ell+\ell \leq L(m+\ell)-L(m) \leq \frac{-\lambda^{s}+3 \epsilon_{0}}{\lambda^{u}-\epsilon_{0}} \ell+\ell .
$$

Let Leb denote the Lebesgue measure on $\mathbb{R}$. We have the following fact.

Claim 9.8. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a bi-Lipschitz homeomorphism with

$$
a|y-x| \leq|g(y)-g(x)| \leq b|y-x| .
$$

Then $g_{*}$ Leb $\ll$ Leb $\ll g_{*}$ Leb and $a \leq \frac{d \text { Leb }}{d g_{*} \text { Leb }} \leq b$.
9.6. Dichotomy for invariant subspaces. For this and the following subsection, we return to the skew product $F: X \rightarrow X$. In this section we establish the following dichotomy for $D F$-invariant subbundles of $T X$. Let $F: X \rightarrow X$ and $\mu$ be as in Theorem 4.10 Consider a $\mu$-measurable line field $\mathcal{V} \subset T X$. Write $V_{\xi}(x) \subset T_{x} M$ for the family of subspaces with

$$
\mathcal{V}(\xi, x)=\{\xi\} \times V_{\xi}(x)
$$

The measurability of $\mathcal{V}$ with respect to a sub- $\sigma$-algebra of $X$ is the measurability of the function $(\xi, x) \mapsto V_{\xi}(x)$ with the standard Borel structure on $T M$. We say $\mathcal{V}$ is $D F$-invariant if for $\mu$-a.e. $(\xi, x) \in X$

$$
D F_{(\xi, x)} \mathcal{V}(\xi, x)=\mathcal{V}(F(\xi, x)) \text { or } D_{x} f_{\xi} V_{\xi}(x)=V_{\theta(\xi)}\left(f_{\xi}(x)\right) .
$$

Recall that $\mathcal{F}$ in Theorem 4.10 is a decreasing sub- $\sigma$-algebra; that is, $F(\mathcal{F}) \subset \mathcal{F}$. We write $\mathcal{F}_{\infty}$ for the smallest $\sigma$-algebra containing $\bigcup_{n \geq 0} F^{-n}(\mathcal{F})$. We similarly define $\hat{\mathcal{F}}_{\infty}$. (We remark that in the case that $\hat{\mathcal{F}}$ is the $\sigma$-algebra of local unstable sets in Section $4.3 \hat{\mathcal{F}}_{\infty}$ and $\mathcal{F}_{\infty}$ are, respectively, the completions of the Borel algebras on $\Sigma$ and $\Sigma \times M$.)

Recall we write $\left\{\nu_{\xi}^{\hat{\mathcal{F}}}\right\}_{\xi \in \Omega}$ for a family of conditional probabilities induced by $\hat{\mathcal{F}}$.
Lemma 9.9. Let $\mu$ and $\mathscr{F}$ be as in Theorem 4.10. Then
(1) the line field $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is $\mathcal{F}_{\infty}$-measurable;
(2) for any DF-invariant, $\mathcal{F}_{\infty}$-measurable line field $\mathcal{V} \subset T X$ either $(\xi, x) \mapsto$ $V_{\xi}(x)$ is $\mathcal{F}$-measurable or

$$
\begin{equation*}
\text { for } \nu \text {-a.e. } \xi, \mu_{\xi} \text {-a.e. } x, \text { and } \nu_{\xi}^{\hat{\mathcal{F}}} \text {-a.e. } \eta, V_{\xi}(x) \neq V_{\eta}(x) . \tag{9.15}
\end{equation*}
$$

Recall the family of conditional measures $\left\{\mu_{(\xi, x)}^{\mathcal{F}}\right\}$ induced by $\mathcal{F}$ is defined by $d \mu_{(\xi, x)}^{\mathcal{F}}(\eta, y)=d \nu_{\xi}^{\hat{\mathcal{F}}}(\eta) \times \delta_{x}(y)$. Thus, for $\nu$-a.e. $\xi$, and $\mu_{\xi}$-a.e. $x, V_{\eta}(x)$ is defined for $\nu_{\xi}^{\hat{\mathcal{F}}}$ a.e. $\eta$, and the comparison (9.15) is well defined a.e.

Proof. To see (1) we recall that $\xi \mapsto f_{\xi}^{-n}$ is $\hat{\mathcal{F}}$-measurable for all $n \geq 0$. Then

$$
\xi \mapsto f_{\xi}^{n}=\left(f_{\theta^{n}(\xi)}^{-n}\right)^{-1}
$$

is $\theta^{-n}(\hat{\mathcal{F}})$-measurable. It follows that $\xi \mapsto f_{\xi}^{n}$ is $\hat{\mathcal{F}}_{\infty}$-measurable for all $n \geq 0$. Since $E_{\xi}^{s}(x)$ depends only on $f_{\xi}^{n}$ for $n \geq 0$, we have

$$
(\xi, x) \mapsto E_{\xi}^{s}(x)=\left\{\left.v \in T_{x} M\left|\lim _{n \rightarrow \infty} \frac{1}{n}\right| D f_{\xi}^{n}(v) \right\rvert\,<0\right\}
$$

is $\mathcal{F}_{\infty}$-measurable.

To prove (2) let $\mathcal{P}$ denote the measurable partition of $X$ into level sets of $(\xi, x) \mapsto$ $V_{\xi}(x)$. We assume (9.15) fails, (9.16)

$$
\mu\left\{(\xi, x) \mid \mu_{(\xi, x)}^{\mathcal{F}}(\mathcal{P}(\xi, x))>0\right\}=\mu\left\{(\xi, x) \mid \nu_{\xi}^{\hat{\mathcal{F}}}\left\{\eta \mid V_{\xi}(x)=V_{\eta}(x)\right\}>0\right\}>0
$$

From (9.16) we will deduce $\mathcal{F}$-measurability of $(\xi, x) \mapsto V_{\xi}(x)$.
Let $\mathcal{F}_{n}:=F^{-n}(\mathcal{F})$, and write $\left\{\mu_{(\xi, x)}^{\mathcal{F}_{n}}\right\}$ for a corresponding family of conditional measures. Also write $\hat{\mathcal{F}}_{n}:=\theta^{-n}(\hat{\mathcal{F}})$.

For each $(\xi, x) \in X$ define

$$
\phi_{n}(\xi, x):=\mu_{(\xi, x)}^{\mathcal{F}_{n}}(\mathcal{P}(\xi, x)) .
$$

We have

$$
\phi_{n}(\xi, x)=\mathbb{E}_{\mu_{(\xi, x)}^{\mathcal{F}}}\left(1_{\mathcal{P}(\xi, x)}(\cdot) \mid \mathcal{F}_{n}\right)(\xi, x)=\mathbb{E}_{\nu_{\xi}^{\hat{\xi}}}\left(1_{\mathcal{P}(\xi, x)}(\cdot, x) \mid \hat{\mathcal{F}}_{n}\right)(\xi)
$$

Consider any $(\xi, x)$ with $\mu_{(\xi, x)}^{\mathcal{F}}(\mathcal{P}(\xi, x))>0$ and such that $\mathcal{V}$ is $\mathcal{F}_{\infty}$-measurable modulo $\mu_{(\xi, x)}^{\mathcal{F}}$. For $\eta \in \Omega$ define

$$
\psi_{n}(\eta):=\mathbb{E}_{\nu_{\xi}^{\hat{\xi}}}\left(1_{\mathcal{P}(\xi, x)}(\cdot, x) \mid \hat{\mathcal{F}}_{n}\right)(\eta) .
$$

Then $\psi_{n}(\eta)$ is a martingale (with filtration $\hat{\mathcal{F}}_{n}$ on the measure space $\left(\Omega, \mathcal{B}_{\Omega}, \nu_{\xi}^{\hat{\mathcal{F}}}\right)$ ), whence (using the $\mathcal{F}_{\infty}$-measurability of $\mathcal{V}$ )

$$
\psi_{n}(\eta) \rightarrow \mathbb{E}_{\nu \hat{\xi}}\left(1_{\mathcal{P}(\xi, x)}(\cdot, x) \mid \hat{\mathcal{F}}_{\infty}\right)(\eta)=1_{\mathcal{P}(\xi, x)}(\eta, x)
$$

$\nu_{\xi}^{\hat{\mathcal{F}}}$-a.s. as $n \rightarrow \infty$. In particular, for $\mu_{(\xi, x)}^{\mathcal{F}}$-a.e. $(\eta, x) \in \mathcal{P}(\xi, x)$

$$
\phi_{n}(\eta, x) \rightarrow 1
$$

as $n \rightarrow \infty$. It follows from (9.16) that

$$
\begin{equation*}
\mu\left\{(\xi, x) \in X \mid \phi_{n}(\xi, x) \mapsto 1 \text { as } n \rightarrow \infty\right\}>0 \tag{9.17}
\end{equation*}
$$

The $\mathcal{F}$-measurability of $(\xi, x) \mapsto V_{\xi}(x)$ is equivalent to the assertion that

$$
\mu\left\{(\xi, x) \mid \phi_{0}(\xi, x)=1\right\}=1
$$

Since $F_{*}^{n}\left(\mu_{(\xi, x)}^{\mathcal{F}_{n}}\right)=\mu_{F^{n}(\xi, x)}^{\mathcal{F}}, \mathcal{V}$ is $D F$-invariant, and $(\xi, x) \mapsto D_{x} f_{\xi}^{n}$ is $\mathcal{F}_{n^{-}}$ measurable, we have that $\phi_{0}\left(F^{n}(\xi, x)\right)=\phi_{n}(\xi, x)$. The ergodicity and $F$-invariance of $\mu$ and (9.17) then imply that $\phi_{0} \equiv 1$ on a set of full measure completing the proof.
9.7. Sets of good angles, geometry of intersections, and bounds on distortion. We remark that in this section, all estimates are with respect to the background Riemannian metric on $M$. Let $X_{1} \subset X$ denote the full $\mu$-measure subset such that $E_{\xi}^{u / s}(x)$ is defined and $W_{\xi}^{u / s}(x)$ is an injectively immersed curve tangent to $E_{\xi}^{u / s}(x)$. Furthermore, assume the affine parameters and corresponding parametrizations $\mathcal{I}^{u / s}$ in (6.4) are defined on $W_{\xi}^{u / s}(x)$ for every $(\xi, x) \in X_{1}$. Given $\gamma_{1}>0$, let $\Lambda\left(\gamma_{1}\right) \subset X_{1}$ denote the set of points where

$$
\angle\left(E_{\xi}^{s}(x), E_{\xi}^{u}(x)\right)>\gamma_{1} .
$$

Given $0<\gamma_{2}<\gamma_{1} / 2$ and $(\xi, x) \in \Lambda\left(\gamma_{1}\right)$ define $A_{\gamma_{2}}(\xi, x)$ to be the set of $\eta \in \Omega$ with
(1) $(\eta, x) \in X_{1}$,
(2) $\angle\left(E_{\xi}^{s}(x), E_{\eta}^{s}(x)\right)>\gamma_{2}$, and
(3) $\angle\left(E_{\xi}^{u}(x), E_{\eta}^{s}(x)\right)>\gamma_{2}$.

As $\mu\left(X_{1}\right)=1$, as remarked in the previous section, for almost every $(\xi, x)$ we have $(\eta, x) \in X_{1}$ for $\nu_{\xi}^{\hat{\mathcal{F}}}$-a.e. $\eta$. For $0<a<1$ we define the set $\mathcal{A}_{\gamma_{1}, \gamma_{2}, a} \subset \Lambda\left(\gamma_{1}\right)$ by

$$
\mathcal{A}_{\gamma_{1}, \gamma_{2}, a}:=\left\{(\xi, x) \in \Lambda\left(\gamma_{1}\right) \mid \nu_{\xi}^{\hat{\mathcal{F}}}\left(A_{\gamma_{2}}(\xi, x)\right)>a\right\} .
$$

From Lemma 9.9 we obtain the following.
Lemma 9.10. Assume that $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable. Then for any $\alpha>0$ and $0<a<1$ there exists $\gamma_{1}>0$ and $\gamma_{2}>0$ with

$$
\mu\left(\mathcal{A}_{\gamma_{1}, \gamma_{2}, a}\right)>1-\alpha .
$$

Fix a uniform $\rho_{0}>0$ to be smaller than the injectivity radius of $M$, and given $x \in M$ let

$$
\exp _{x}: B\left(\rho_{0}\right) \subset T_{x} M \rightarrow M
$$

denote the exponential map. We recall that for every $(\xi, x) \in X_{1}$ we have selected $v^{u}(\xi, x) \in E_{\xi}^{u}(x)$ and $v^{s}(\xi, x) \in E_{\xi}^{s}(x)$ such that $(\xi, x) \mapsto v^{u}(\xi, x)$ is $\mathcal{F}$-measurable and $(\xi, x) \mapsto v^{s}(\xi, x)$ is $\mu$-measurable.

By Lusin's theorem, there is a compact subset $\Lambda_{2} \subset X_{1}$, of measure arbitrarily close to 1 , on which the family of parametrized stable and unstable manifolds

$$
(\xi, x) \mapsto \mathcal{I}_{(\xi, x)}^{\sigma}
$$

vary continuously in the $C^{1}$ topology on the space of embeddings $C^{1}([-r, r], M)$ for $\sigma=\{s, u\}$ and all $0<r<1$.

Given $x \in M$, a subspace $V \subset T_{x} M$, and $0<\gamma<\pi$ we denote by $\mathcal{C}_{\gamma}(V)$ the open cone of angle $\gamma$ around the subspace $V$. We have the following.

Lemma 9.11. Given any $\gamma>0$, there exist $\hat{r}_{1}, \hat{r}_{0}>0$ such that for all $(\xi, x) \in \Lambda_{2}$ and all $(\xi, y) \in \Lambda_{2}$ with $d(x, y)<\hat{r}_{0}$,
(1) $\exp _{x}^{-1}\left(W_{\xi, \hat{r}_{1}}^{u}(x)\right) \subset \mathcal{C}_{\gamma}\left(E_{\xi}^{u}(x)\right)$;
(2) $\exp _{x}^{-1}\left(W_{\xi, \hat{r}_{1}}^{s}(x)\right) \subset \mathcal{C}_{\gamma}\left(E_{\xi}^{s}(x)\right)$;
(3) $\exp _{x}^{-1}\left(W_{\xi, \hat{r}_{1}}^{s}(y)\right) \subset \mathcal{C}_{\gamma}\left(E_{\xi}^{s}(x)\right)+\exp _{x}^{-1}(y)$.

Fix $\epsilon_{1}=\epsilon_{0} / 10$. Fix a family of Lyapunov charts $\phi(\xi, x)$ with corresponding function $\ell: X \rightarrow[1, \infty)$, and retain all related notation from Section 6.1.2, Let $\Lambda_{3} \subset \Lambda_{2}$ be a set on which $\ell$ is bounded above by $\ell_{0}$ and such that there exist $0<\tilde{r}_{0}$ and $0<\tilde{r}_{1}$ such that for $(\xi, x) \in \Lambda_{3}$,
(1) $W_{\xi, \tilde{r}_{1}}^{s}(x) \subset V^{s}(\xi, x)$ where $V^{s}(\xi, x)$ is the local stable manifold built in Theorem 6.4
(2) the diameters of $W_{\xi, \tilde{r}_{1}}^{s}(x)$ and $W_{\xi, \tilde{r}_{1}}^{u}(x)$ are less than $\frac{\ell_{0}^{-3} e^{-\lambda_{0}-\epsilon_{1}}}{10 k_{0}}$;
(3) if $(\xi, x),(\xi, y) \in \Lambda_{0}$ with $d(x, y) \leq \tilde{r}_{0}$, then

$$
\phi(\xi, x)\left(W_{\xi, \tilde{r}_{1}}^{s}(y)\right)
$$

is the graph of a 1-Lipschitz function $g_{x, y}: D \subset \mathbb{R}^{s} \rightarrow \mathbb{R}^{u}\left(\ell_{0}^{-1} e^{-\lambda_{0}-\epsilon_{1}}\right)$ for some $D \subset \mathbb{R}^{s}\left(\ell_{0}^{-1} e^{-\lambda_{0}-\epsilon_{1}}\right)$.
Note, in particular, for $(\xi, x)$ and $(\xi, y)$ above that $W_{\xi, \tilde{r}_{1}}^{s}(y)$ is in the domain of the chart $\phi(\xi, x)$. We may take $\mu\left(\Lambda_{3}\right)$ arbitrarily close to $\mu\left(\Lambda_{2}\right)$.


Figure 1. Lemma 9.12 ,

Appealing repeatedly to Lusin's theorem, and standard estimates in the construction of stable and unstable manifolds, we may choose parameters satisfying the following. See Figure (1) (Note that in our application of Figure 1 we have $\left.W_{\xi}^{u}(x)=W_{\eta}^{u}(x).\right)$

Lemma 9.12. For every $0<\gamma_{1}, 0<\gamma_{2}<\gamma_{1} / 2$, and $\Lambda_{3} \subset \Lambda_{2} \subset \Lambda\left(\gamma_{1}\right)$ as above there exist a subset $\Lambda^{\prime} \subset \Lambda_{3}$ with $\mu\left(\Lambda^{\prime}\right)$ arbitrarily close to $\mu\left(\Lambda_{3}\right)$, positive constants $r_{0}<\tilde{r}_{0}, r_{1}<\tilde{r}_{1}$, and constants $C_{1}, C_{2}, C_{3}, D_{1}>1$, with the following properties.

For $(\xi, x) \in \Lambda^{\prime}$ we have
(a) $\frac{1}{C_{2}} d(x, w) \leq\left\|H_{(\xi, x)}^{u}(w)\right\| \leq C_{2} d(x, w)$ for all $w \in W_{\xi, r_{1}}^{u}(x)$.
(b) $\frac{1}{C_{2}} d\left(x, w^{\prime}\right) \leq\left\|H_{(\xi, x)}^{s}\left(w^{\prime}\right)\right\| \leq C_{2} d\left(x, w^{\prime}\right)$ for all $w^{\prime} \in W_{\xi, r_{1}}^{s}(x)$.

For $(\xi, x),(\xi, y) \in \Lambda^{\prime}$ with $d(x, y)<r_{0}$
(c) $W_{\xi, r_{1}}^{s}(x) \cap W_{\xi, r_{1}}^{u}(y)$ is a singleton $\{z\}$ and the intersection is uniformly transverse;
furthermore, if $\eta \in A_{\gamma_{2}}(\xi, x)$ and $(\eta, y),(\eta, x) \in \Lambda^{\prime}$,
(d) $W_{\xi, r_{1}}^{u}(x) \cap W_{\eta, r_{1}}^{s}(y)$ is a singleton $\{v\}$ and the intersection is uniformly transverse, and
(e) if $D_{1} \cdot\left\|H_{(\xi, y)}^{u}(z)\right\| \leq\left\|H_{(\xi, x)}^{s}(z)\right\|$, then

$$
\frac{1}{C_{3}}\left\|H_{(\xi, x)}^{s}(z)\right\| \leq\left\|H_{(\xi, x)}^{u}(v)\right\| \leq C_{3}\left\|H_{(\xi, x)}^{s}(z)\right\| .
$$

Additionally, we have a uniform bound $C_{1}$ so that for $(\xi, x) \in \Lambda^{\prime}$ and $w \in W_{\xi, r_{1}}^{u}(x)$
(f) $\frac{1}{C_{1}} \leq\left\|D_{x} f_{\xi}^{-n} \mid{ }_{T_{x} W_{\xi, r_{1}}^{u}(x)}\right\| \cdot\left\|D_{w} f_{\xi}^{-n} \Gamma_{T_{w} W_{\xi, r_{1}}^{u}(x)}\right\|^{-1} \leq C_{1}$ for all $n \geq 0$, and for $(\xi, y) \in \Lambda^{\prime}$ with $d(x, y)<r_{0}$ and $z$ as in (c)
(g) $\frac{1}{C_{1}} \leq\left\|D_{x} f_{\xi}^{n} \upharpoonright_{T_{x} W_{\xi, r_{1}}^{u}(x)}\right\| \cdot\left\|D_{z} f_{\xi}^{n} \upharpoonright_{T_{z} W_{\xi, r_{1}}^{u}(y)}\right\|^{-1} \leq C_{1}$ for all $n \geq 0$.

Proof. Conclusions (a) (d) follow simply from the $C^{1}$ topology and Luzin's theorem.

For (e), we work in the exponential chart at $x$. By the law of sines, given fixed $\gamma_{1}$ and $\gamma_{2}$, we pick a sufficiently small $\gamma>0$ so that if $\hat{y} \in C_{\gamma}\left(E_{\xi}^{s}(x)\right)$, then there exists $\hat{C}>1$ with

$$
\frac{d(0, \hat{y})}{\hat{C}} \leq \max \left\{d(0, v): v \in C_{\gamma}\left(E_{\xi}^{u}(x)\right) \cap C_{\gamma}\left(E_{\eta}^{s}(x)\right)+\hat{y}\right\}<\hat{C} d(0, \hat{y})
$$

We then obtain (e) from the uniform Lipschitz bounds on the exponential map and the affine parameters. See Figure 1

The estimates (f) and (g) follow from the fact that the pairs $f_{\xi}^{-n}(x)$ and $f_{\xi}^{-n}(w)$, $f_{\xi}^{n}(x)$ and $f_{\xi}^{n}(z)$, and $D_{x} f_{\xi}^{n}\left(T_{x} W_{\xi, r_{1}}^{u}(x)\right)$ and $D_{z} f_{\xi}^{n}\left(T_{z} W_{\xi, r_{1}}^{u}(y)\right)$ are exponentially asymptotic while $\left|f_{\xi}\right|_{C^{1}}, \operatorname{Lip}\left(D f_{\xi}^{n}\right)$, and the Lipschitz constants for the variation of the tangent spaces to $f_{\xi}^{-n}\left(W_{\xi, r}^{u}(x)\right)$ grow sub-exponentially for $\xi \in \Omega_{0}$ and ( $\xi, x$ ) as in the the proof of Proposition 6.5.

The following lemma will be needed in Claim 10.3 below.
Lemma 9.13. Take $(\xi, x) \in \Lambda^{\prime}$ and $(\xi, y) \in \Lambda^{\prime}$ with $d(x, y)<r_{0}$, and set $z=$ $W_{\xi, r_{1}}^{s}(x) \cap W_{\xi, r_{1}}^{u}(y)$ and $w=W_{\xi, r_{1}}^{s}(y) \cap W_{\xi, r_{1}}^{u}(x)$. Let $\Gamma \subset W_{\xi, r_{1}}^{s}(y)$ be the curve with end points $w$ and $y$. Let $n \geq 0$ be such that

$$
\left|\phi\left(F^{-j}(\xi, x)\right)\left(f_{\xi}^{-j}(z)\right)\right| \leq \frac{e^{-\lambda_{0}-\epsilon_{1}} \ell_{0}^{-1} e^{-\epsilon_{1} j}}{10}
$$

for all $0 \leq j \leq n$. Then $f_{\xi}^{-n}(\Gamma)$ is in the domain of $\phi\left(F^{-n}(\xi, x)\right)$, and

$$
\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(\Gamma)\right)
$$

is the graph of a 1-Lipschitz function

$$
g: \hat{D} \subset \mathbb{R}^{s} \rightarrow \mathbb{R}^{u}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-n}(\xi, x)\right)^{-1}\right)
$$

for some $\hat{D} \subset \mathbb{R}^{s}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-n}(\xi, x)\right)^{-1}\right)$.
Proof. We prove by induction on $n$. For $n=0$ the conclusion follows from hypotheses and the choice of $\Lambda_{3}$. For $n \geq 1$, assume

$$
\phi\left(F^{-(n-1)}(\xi, x)\right)\left(f_{\xi}^{-(n-1)}(\Gamma)\right)
$$

is the graph of a 1-Lipschitz function $g: D \subset \mathbb{R}^{s} \rightarrow \mathbb{R}^{u}\left(\ell\left(F^{-(n-1)}(\xi, x)\right)^{-1} e^{-\lambda_{0}-\epsilon_{1}}\right)$ for some $D \subset \mathbb{R}^{s}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-(n-1)}(\xi, x)\right)^{-1}\right)$. From Lemma 6.3 it follows that $f_{\xi}^{-n}(\Gamma)$ is in the domain of $\phi\left(F^{-n}(\xi, x)\right)$ and

$$
\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(\Gamma)\right)
$$

is the graph of a 1-Lipschitz function $\hat{g}: \hat{D} \subset \mathbb{R}^{s} \rightarrow \mathbb{R}^{u}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-n}(\xi, x)\right)^{-1}\right)$ for some $\hat{D} \subset \mathbb{R}^{s}\left(\ell\left(F^{-n}(\xi, x)\right)^{-1}\right)$. By the hypothesis, we have that $f_{\xi}^{-n}(z)$ is contained in the domain of $\phi\left(F^{-n}(\xi, x)\right)$. We have

$$
\begin{aligned}
\left.d\left(f_{\xi}^{-n}(y)\right),\left(f_{\xi}^{-n}(z)\right)\right) & \leq \ell_{0} k_{0} e^{n\left(-\lambda^{u}+2 \epsilon_{1}\right)} \frac{\ell_{0}^{-3} e^{-\lambda_{0}-\epsilon_{1}}}{10 k_{0}} \\
& \leq e^{n\left(-\lambda^{u}+2 \epsilon_{1}\right)} \frac{1}{10} \ell_{0}^{-2} e^{-\lambda_{0}-\epsilon_{1}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left|\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(y)\right)-\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(z)\right)\right| & \leq \ell_{0} e^{n \epsilon_{1}} e^{n\left(-\lambda^{u}+2 \epsilon_{1}\right)} \frac{1}{10} \ell_{0}^{-2} e^{-\lambda_{0}-\epsilon_{1}} \\
& \leq \frac{1}{10} \ell\left(F^{-n}(\xi, x)\right)^{-1} e^{-\lambda_{0}-\epsilon_{1}} ;
\end{aligned}
$$

hence,

$$
\left|\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(x)\right)-\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(y)\right)\right| \leq \frac{2}{10} \ell\left(F^{-n}(\xi, x)\right)^{-1} e^{-\lambda_{0}-\epsilon_{1}}
$$

Similarly,

$$
\left|\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(x)\right)-\phi\left(F^{-n}(\xi, x)\right)\left(f_{\xi}^{-n}(w)\right)\right| \leq \frac{1}{10} \ell\left(F^{-n}(\xi, x)\right)^{-1} e^{-\lambda_{0}-\epsilon_{1}}
$$

hence $\hat{D} \subset \mathbb{R}^{s}\left(e^{-\lambda_{0}-\epsilon_{1}} \ell\left(F^{-n}(\xi, x)\right)^{-1}\right)$.
Under the hypotheses of Lemma 9.13, if $F^{-n}(\xi, x) \in \Lambda^{\prime}$ and $F^{-n}(\xi, y) \in \Lambda^{\prime}$, then $\phi\left(F^{-n}(\xi, x)\right)(\Gamma)$ and

$$
\phi\left(F^{-n}(\xi, x)\right)\left(W_{\theta^{-n}(\xi), r_{1}}^{u}\left(f_{\xi}^{-n}(x)\right)\right)
$$

have at most one point of intersection, thus we have the following corollary.
Corollary 9.14. For $n$ satisfying the hypotheses of Lemma 9.13, if $F^{-n}(\xi, x) \in \Lambda^{\prime}$, $F^{-n}(\xi, y) \in \Lambda^{\prime}$, and $d\left(f_{\xi}^{-n}(x), f_{\xi}^{-n}(y)\right) \leq r_{0}$, then

$$
W_{\theta^{-n}(\xi), r_{1}}^{s}\left(f_{\xi}^{-n}(y)\right) \cap W_{\theta^{-n}(\xi), r_{1}}^{u}\left(f_{\xi}^{-n}(x)\right)=f_{\xi}^{-n}(w) .
$$

Remark 9.15. In the case that $\Omega=\Sigma$, recall that the stable line fields, stable manifolds, and corresponding affine parameters are defined using only the forwards itinerary. As $f_{\eta}^{n}=f_{\xi}^{n}$ for all $\eta \in \Sigma_{\text {loc }}^{-}(\xi)$ and $n \geq 0$, it follows that we may choose $v^{s}(\xi, x)$ and the corresponding parametrizations $\mathcal{I}_{(\xi, x)}^{s}$ of the stable manifolds to be constant on sets of the form $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ and thus induce corresponding objects $v^{s}(\omega, x)$ and $\mathcal{I}_{(\omega, x)}^{s}$ for the one-sided skew product on $\Sigma_{+} \times M$. We may then take $\hat{\Lambda}^{s} \subset \Sigma_{+} \times M$ of $\left(\hat{\nu}^{\mathbb{N}} \times \hat{\mu}\right)$-measure arbitrarily close to 1 so that on $\hat{\Lambda}^{s}$ the parametrized stable manifolds

$$
(\omega, x) \mapsto\left(t \mapsto \mathcal{I}_{(\omega, x)}^{s}\left(t v^{s}(\omega, x)\right)\right)
$$

vary continuously in the space of $C^{1}$-embeddings $[-r, r] \rightarrow M$ for all sufficiently small $0<r<1$.

Let $\pi_{+}: \Sigma \rightarrow \Sigma_{+}$be the natural projection. We may modify parts (c) $($(e) (g) of Lemma 9.12 as follows: Choose $\Lambda^{\prime}$ in Lemma 9.12 so that if $(\xi, x) \in \Lambda^{\prime}$, then $\left(\pi_{+}(\xi), x\right) \in \hat{\Lambda}^{s}$. Then the constants can be chosen for the following lemma.

Lemma 9.12. For $(\xi, x) \in \Lambda^{\prime},(\zeta, y) \in \Lambda^{\prime}$ with $\pi_{+}(\zeta)=\pi_{+}(\xi)$ and $d(x, y)<r_{0}$
(c') $W_{\xi, r_{1}}^{s}(x) \cap W_{\zeta, r_{1}}^{u}(y)$ is a singleton $\{z\}$ and the intersection is uniformly
transverse;
furthermore, if $\eta \in A_{\gamma_{2}}(\xi, x), \eta^{\prime} \in \Sigma$ is such that $\pi_{+}\left(\eta^{\prime}\right)=\pi_{+}(\eta)$ and $\left(\eta^{\prime}, y\right),(\eta, x) \in$ $\Lambda^{\prime}$, then
(d') $W_{\xi, r_{1}}^{u}(x) \cap W_{\eta^{\prime}, r_{1}}^{s}(y)$ is a singleton $\{v\}$ and the intersection is uniformly transverse, and
(e') if $D_{1} \cdot\left\|H_{(\zeta, y)}^{u}(z)\right\| \leq\left\|H_{(\xi, x)}^{s}(z)\right\|$, then

$$
\frac{1}{C_{3}}\left\|H_{(\xi, x)}^{s}(z)\right\| \leq\left\|H_{(\xi, x)}^{u}(v)\right\| \leq C_{3}\left\|H_{(\xi, x)}^{s}(z)\right\|
$$

Additionally, for $x, y, z$ as above we have uniform bounds

$$
\text { (g') } \frac{1}{C_{1}} \leq\left\|D_{x} f_{\xi}^{n} \upharpoonright_{T_{x} W_{\xi, r_{1}}^{u}(x)}\right\| \cdot\left\|D_{z} f_{\zeta}^{n} \upharpoonright_{T_{z} W_{\zeta, r_{1}}^{u}(y)}\right\|^{-1} \leq C_{1} \text { for all } n \geq 0
$$

Note that in (g'), we have $f_{\xi}^{n}=f_{\zeta}^{n}$ for all $n \geq 0$.

## 10. Proof of Proposition 7.1

Given the skew product $F: X \rightarrow X$ and $\mu$ satisfying (IC), we have $\lambda^{s}<0<\lambda^{u}$ given by the hyperbolicity of $D F$ and $\epsilon_{0}<\min \left\{1, \lambda^{u} / 200,-\lambda^{s} / 200\right\}$ fixed in Section 6.1. We recall the family $\bar{\mu}_{(\xi, x)}$ introduced in Section 6.5 Recall that our goal is to prove for such measures that the measurable sets $G_{\varepsilon}$ and $G$ in Proposition 7.1 have positive $\mu$-measure (for some fixed $M$ and all sufficiently small $\varepsilon$ ). This will be shown in Section 10.4 .

We define $X_{0} \subset X_{1} \subset X$ to be the full $\mu$-measure, $F$-invariant subset of $\Omega_{0} \times M$ where all propositions from Section 6 hold and such that the stable and unstable manifolds, Lyapunov norms, affine parameters, and the parametrizations $\mathcal{I}^{u / s}$ are defined. We also assume that for $(\xi, x) \in X_{0}$ the measures $\mu_{\xi}, \mu_{(\xi, x)}^{u}, \mu_{(\xi, x)}^{s}$, and $\bar{\mu}_{(\xi, x)}$ are defined and satisfy $F_{*} \mu_{\xi}=\mu_{\theta(\xi)}, F_{*} \mu_{(\xi, x)}^{u / s} \simeq \mu_{F(\xi, x)}^{u / s}$, and $\left(\mathcal{I}_{F(\xi, x)}^{u}\right)^{-1} \circ$ $F \circ \mathcal{I}_{(\xi, x)}^{u}\left(\bar{\mu}_{(\xi, x)}\right) \simeq \bar{\mu}_{F(\xi, x)}$. We further assume $\bar{\mu}_{(\xi, x)}$ contains 0 in its support. We further take $X_{0}$ so that for $(\xi, x) \in X_{0}$ and $\nu_{\xi}^{\hat{\mathcal{F}}}$-a.e. $\eta \in \Omega$ we have $f_{\xi}^{n}=f_{\eta}^{n}$ for all $n \leq 0$; for such $\eta$ we have $W_{\xi}^{u}(x)=W_{\eta}^{u}(x)$, and corresponding equality of affine parameters, parametrizations $\mathcal{I}^{u}$, and measures $\bar{\mu}_{(\xi, x)}=\bar{\mu}_{(\eta, x)}$. Finally, we assume that if $(\xi, x) \in X_{0}$, then $\mu_{\xi}\left(X_{0}\right)=1$ and $\mu_{(\xi, x)}^{\mathcal{F}}\left(X_{0}\right)=\nu_{\xi}^{\hat{\mathcal{F}}}\left(\left\{\eta:(\eta, x) \in X_{0}\right\}\right)=1$.

Write $Y_{0}=[0,1) \times X_{0}$.
10.1. Choice of parameters and sets. We pick any
(A.) $0<\beta<1$ such that $\frac{1+\beta}{1-\beta}<\frac{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}{-\lambda^{s}+\epsilon_{0}}$;
and fix the following constants for the remainder
(B.) $\kappa_{1}=\frac{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}{\lambda^{u}+\epsilon_{0}} ; \kappa_{2}=\frac{\lambda^{u}-\lambda^{s}+2 \epsilon_{0}}{\lambda^{u}-\epsilon_{0}}$;
(C.) $\alpha_{0}=\frac{1}{2}-\frac{1}{2} \frac{(1+\beta)\left(-\lambda^{s}+\epsilon_{0}\right)}{(1-\beta)\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right)}$;
(D.) $\alpha=\left(\frac{\kappa_{1} \alpha_{0}}{5\left(\kappa_{1}+\kappa_{2}\right)}\right)$.

Note $\alpha_{0}>0$ by the choice of $\beta$.
Recall that the measures $\omega$ and $\hat{\omega}$ are equivalent. We select
(E.) $N_{0}$ such that $\omega\left\{p: \frac{1}{N_{0}} \leq \frac{d \hat{\omega}}{d \omega}(p) \leq N_{0}\right\}>1-\alpha / 2$.
(F.) By Lusin's theorem, we may choose a compact subset $K_{0} \subset Y_{0} \subset Y$ with $\omega$ - and $\hat{\omega}$-measure sufficiently close to 1 on which
i) the frames for the stable and unstable subbundles $p \mapsto v_{p}^{s}, p \mapsto v_{p}^{s}$ defined in Section 6.2.1.
ii) the stable and unstable manifolds parametrized by (6.4);
iii) all Lyapunov norms defined in Section 9.2.1
iv) the families of measures $\bar{\omega}_{p}$
vary continuously.
We may also assume the functions $a(p)$ in (9.7) and $\hat{L}(p)$ in Proposition 9.4 are bounded on $K_{0}$, respectively, by $a_{0}$ and $\hat{L}$. Finally, we may assume there is a $L_{1}>1$ so that for $p=(\varsigma, \xi, x) \in K_{0}$ and $y \in W_{(\varsigma, \xi), 1}^{s}(x)$, writing $\left(\varsigma_{t}, \xi_{t}, x_{t}\right)=\Phi^{t}(p)$, and $\left(\varsigma_{t}, \xi_{t}, y_{t}\right)=\Phi^{t}(\varsigma, \xi, y)$, for any $t \geq 0$

$$
d\left(x_{t}, y_{t}\right) \leq L_{1} e^{t\left(\lambda^{s}+\epsilon_{0}\right)} d(x, y)
$$

(G.) Let $M_{0}>1$ denote the maximal ratio of all Lyapunov and Riemannian norms on $K_{0}$ :

$$
M_{0}=\sup _{p=\in K_{0}} \sup _{0 \neq v \in T_{x} M}\left\{\left(\frac{\|v\|_{\epsilon_{0}, \pm, p}}{\|v\|_{\epsilon_{0},-, p}}\right)^{ \pm 1},\left(\frac{\|v\|_{\epsilon_{0}, \pm, p}}{\|v\|}\right)^{ \pm 1},\left(\frac{\|v\|_{\epsilon_{0},-, p}}{\|v\|}\right)^{ \pm 1}\right\}
$$

(H.) As discussed in Section 9.7, fix $\epsilon_{1}<\epsilon_{0} / 10$, a function $\ell: X \rightarrow[1, \infty)$, and a family of Lyapunov charts $\phi(\xi, x)$. Let $\ell_{0}>1$ be such that $\ell(\xi, x) \leq \ell_{0}$ for $(\xi, x) \in \Lambda_{3}$ where $\Lambda_{3}$ is as in Section 9.7
(I.) By Lemma 9.10, we pick $\gamma_{1}, \gamma_{2}$ so that $\mu\left(\mathcal{A}_{\gamma_{1}, \gamma_{2}, 0.9}\right)>1-\alpha$. We fix $C_{1}, C_{2}, C_{3}>1,0<r_{0}, r_{1}<1, D_{1}>1$, and $\Lambda^{\prime} \subset \Lambda_{3}$ with measure sufficiently close to 1 satisfying Lemma 9.12 . We write $\mathscr{A}=[0,1) \times \mathcal{A}_{\gamma_{1}, \gamma_{2}, 0.9}$ and for $p=(\varsigma, \xi, x) \in Y_{0}, A_{\gamma_{2}}(p)=A_{\gamma_{2}}(\xi, x)$.
(J.) Take $\hat{r}=\min \left\{r_{0} /\left(2 C_{2}\right), r_{1}\right\}$.
(K.) Set $\hat{T}=\log \left(\hat{L} M_{0}^{2} C_{1}^{3}\right) /\left(\lambda^{u}-\epsilon_{0}\right)$.

Fix a compact set $\Lambda^{\prime \prime} \subset Y_{0}$ and $D_{0}>0$ such that for $p \in \Lambda^{\prime \prime}$ and $t \in[-\hat{T}, \hat{T}]$

$$
D_{0}^{-1} \leq\left\|D \Phi^{t} \upharpoonright_{E^{u}(p)}\right\| \leq D_{0}
$$

(L.) We fix a compact $\Omega^{\prime} \subset \Omega$ with $\nu\left(\Omega^{\prime}\right)$ sufficiently close to 1 and such that for all $j \in \mathbb{Z}$ with $|j| \leq \hat{T}+1$ we have $\xi \mapsto \theta^{j}(\xi)$ and $\xi \mapsto f_{\xi}^{j}$ are continuous when restricted to $\Omega^{\prime}$. Consider $q_{n} \in[0,1) \times \Omega^{\prime} \times M$ converging to $q \in$ $[0,1) \times \Omega^{\prime} \times M$, and let $t_{n}$ be a sequence with $\left|t_{n}\right| \leq \hat{T}, t_{n} \rightarrow t$, and $\Phi^{t_{n}}\left(q_{n}\right) \in[0,1-a] \times \Omega \times M$ for all $n$ and some $a>0$. It then follows that $\Phi^{t_{n}}\left(q_{n}\right)$ converges to $\Phi^{t}(q)$.
(M.) The above choices can be made so that setting

$$
K:=K_{0} \cap \Lambda^{\prime \prime} \cap\left([0,1) \times \Lambda^{\prime}\right) \cap\left([0,1) \times \Omega^{\prime} \times M\right)
$$

we may ensure

$$
\begin{equation*}
\omega(K)>1-\frac{\alpha}{10} \text { and } \hat{\omega}(K)>1-\frac{\alpha}{40 N_{0}} . \tag{10.1}
\end{equation*}
$$

(N.) Fix $\hat{\alpha}=\frac{\alpha}{40 N_{0}}$. Let $U \subset Y$ be any open set to be specified later with $\hat{\omega}(U)<\hat{\alpha}$. We have $\hat{\omega}(K \backslash U)>1-\frac{\alpha}{20 N_{0}}$.
Recall if $\psi: Y \rightarrow[0,1]$ is $\omega$-measurable with $\int \psi d \omega>1-a b$, then

$$
\begin{equation*}
\omega\{p: \psi(p)>1-a\}>1-b . \tag{10.2}
\end{equation*}
$$

Recall that we have the filtration $\left\{\mathcal{S}^{m}: m \in \mathbb{R}\right\}$ on $(Y, \hat{\omega})$ decreasing to $\mathcal{S}^{\infty}=$ $\bigcap_{m \in \mathbb{R}} \mathcal{S}^{m}$. From (10.2) and (E) we claim
Claim 10.1. With $N_{0}, K$, and $U$ as above
(a) $\omega\left\{p: \mathbb{E}_{\omega}\left(1_{K} \mid \mathcal{S}\right)(p)>0.9\right\}>1-\alpha$;
(b) $\hat{\omega}\left\{p: \mathbb{E}_{\hat{\omega}}\left(1_{K \backslash U} \mid \mathcal{S}^{\infty}\right)(p)>0.9\right\}>1-\alpha /\left(2 N_{0}\right)$;
(c) $\omega\left\{p: \mathbb{E}_{\hat{\omega}}\left(1_{K \backslash U} \mid \mathcal{S}^{\infty}\right)(p)>0.9\right\}>1-\alpha$.
(O.) Let $S_{0}:=\left\{p \in Y \mid \omega_{p}^{\mathcal{S}}(K)>0.9\right\}$.

As discussed in Section 9.4.2, taking $g(p)=1_{K \backslash U}(p)$ we have

$$
\omega_{\Psi^{-m}(p)}^{\mathcal{S}}\left(\Psi^{-m}(K \backslash U)\right)=\mathbb{E}_{\hat{\omega}}\left(1_{K \backslash U} \mid \mathcal{S}^{m}\right)(p) \rightarrow \mathbb{E}_{\hat{\omega}}\left(1_{K \backslash U} \mid \mathcal{S}^{\infty}\right)(p)
$$

as $m \rightarrow \infty$ for $\omega$-a.e. $p \in Y$. Given $M>0$ let

$$
S_{M}=\left\{p \in Y \mid \omega_{\Psi-m}^{\mathcal{S}}(p)\left(\Psi^{-m}(K \backslash U)\right)>0.9 \text { for all } m \geq M\right\}
$$

(P.) Fix $\hat{M}$ so that $\omega\left(S_{\hat{M}}\right)>1-\alpha$.

Given $T>0$, define $\mathcal{R}(T) \subset K$ to be the set of $p \in K$ such that for $B=$ $K, \mathscr{A}, S_{\hat{M}}$, or $S_{0}$ and any $T^{\prime}, T^{\prime \prime} \geq T$
i) $\frac{1}{T_{1}^{\prime}} \operatorname{Leb}\left(\left\{t \in\left[0, T^{\prime}\right]: \Phi^{t}(p) \in B\right\}\right)>1-\alpha ;$
ii) $\frac{1}{T^{\prime \prime}} \operatorname{Leb}\left(\left\{t \in\left[-T^{\prime \prime}, 0\right]: \Phi^{t}(p) \in B\right\}\right)>1-\alpha$; and thus $\frac{1}{T^{\prime}+T^{\prime \prime}} \operatorname{Leb}\left(\left\{t \in\left[-T^{\prime \prime}, T^{\prime}\right]: \Phi^{t}(p) \in B\right\}\right)>1-\alpha$.
(Q.) By the pointwise ergodic theorem, fix $T_{0}$ with $\omega\left(\mathcal{R}\left(T_{0}\right)\right)>0$.
(R.) Finally, set $\varepsilon_{0}=\min \left\{\frac{\hat{r}}{M_{0}^{4}}, \frac{r_{1}}{2 C_{3} M_{0}^{6}}, \frac{e^{\lambda^{s}-\epsilon_{0}} e^{-\lambda_{0}-\epsilon_{1}} \ell_{0}^{-2}}{10 C_{2}}\right\}$.
10.2. Choice of time intervals. Consider a fixed $\varepsilon<\varepsilon_{0}$. This $\varepsilon$ will be as in Proposition 7.1 Given $0<\delta<1$ we define
(1) $m_{\delta}=\frac{(1+\beta) \log \delta-\log \left(M_{0}^{4}\right)}{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}$,
(2) $M_{\delta}=\frac{(1-\beta) \log \delta-\log \varepsilon}{-\lambda^{s}+\epsilon_{0}}$.

Note that for all sufficiently small $0<\delta<1$ we have $M_{\delta}<m_{\delta}<0$. For $0<\delta<\varepsilon$ consider any $\ell$ with

$$
\frac{\log \delta-\log \varepsilon}{-\lambda^{s}+\epsilon_{0}} \leq \ell \leq 0
$$

By the definition of $\tau_{p, \delta, \varepsilon}$ (see Section 9.5) we have $\tau_{p, \delta, \varepsilon}(\ell) \geq 0$ and

$$
e^{\tau_{p, \delta, \varepsilon}(\ell)\left(\lambda^{u}-\epsilon_{0}\right)} e^{\ell\left(\lambda^{s}+\epsilon_{0}\right)} \delta \leq \varepsilon \leq e^{\tau_{p, \delta, \varepsilon}(\ell)\left(\lambda^{u}+\epsilon_{0}\right)} e^{\ell\left(\lambda^{s}-\epsilon_{0}\right)} \delta .
$$

It follows for such $\ell$ that

$$
\begin{equation*}
\frac{\log (\varepsilon / \delta)+\left(-\lambda^{s}+\epsilon_{0}\right) \ell}{\lambda^{u}+\epsilon_{0}} \leq \tau_{p, \delta, \varepsilon}(\ell) \leq \frac{\log (\varepsilon / \delta)+\left(-\lambda^{s}-\epsilon_{0}\right) \ell}{\lambda^{u}-\epsilon_{0}} . \tag{10.3}
\end{equation*}
$$

In particular, for any $M_{\delta} \leq \ell \leq m_{\delta}<0$, (10.3) holds and $\tau_{p, \delta, \varepsilon}(\ell)>0$.
From (10.3) we obtain the following asymptotic behavior.
Claim 10.2. For fixed $\varepsilon>0$ we have that
(a) $\tau_{p, \delta, \varepsilon}(0)=L_{p, \delta, \varepsilon}(0) \rightarrow \infty$,
(b) $\tau_{p, \delta, \varepsilon}\left(M_{\delta}\right) \rightarrow \infty$, and
(c) $L_{p, \delta, \varepsilon}\left(M_{\delta}\right) \rightarrow-\infty$
as $\delta \rightarrow 0$; furthermore, the divergence is uniform in $p \in Y_{0}$.
Proof. Conclusions (a) and (b) follow from (10.3).
For (c) we have

$$
\begin{aligned}
L_{p, \delta, \varepsilon}\left(M_{\delta}\right) & \leq \frac{\log (\varepsilon / \delta)+\left(-\lambda^{s}-\epsilon_{0}\right) M_{\delta}}{\lambda^{u}-\epsilon_{0}}+M_{\delta} \\
& =\frac{\log \varepsilon+\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right) M_{\delta}-\log \delta}{\lambda^{u}-\epsilon_{0}} \\
& =\frac{\log \varepsilon}{\lambda^{u}-\epsilon_{0}}-\frac{\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right) \log \varepsilon}{\left(\lambda^{u}-\epsilon_{0}\right)\left(-\lambda^{s}+\epsilon_{0}\right)}+\left[\frac{(1-\beta)\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right)}{-\lambda^{s}+\epsilon_{0}}-1\right] \log \delta,
\end{aligned}
$$

and the limit follows from our choice of $\beta$ as from the fact that

$$
\frac{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}{-\lambda^{s}+\epsilon_{0}}>\frac{1+\beta}{1-\beta}>\frac{1}{1-\beta} .
$$

The choice of $m_{\delta}$ above guarantees that, for $(\varsigma, \xi, x)$ and $(\varsigma, \xi, y)$ in $K$ with $x$ and $y$ sufficiently close in $M$, the image of $y$ under the backwards flow $\Phi^{t}$ is in general position with respect to the hyperbolic splitting $T_{\Phi^{t}(\varsigma, \xi, x)} M=E^{u}\left(\Phi^{t}(\varsigma, \xi, x)\right) \oplus$ $E^{s}\left(\Phi^{t}(\varsigma, \xi, x)\right)$ for $t<m_{\delta} . M_{\delta}$ is chosen so that, in addition to the properties in Claim 10.2, the images of $x$ and $y$ do not drift too far apart under the backwards flow. We make this precise in the following claim.

Recall $\hat{r}$ is chosen in $(\mathrm{J})$ and that $r_{1}$ and $r_{0}$ were fixed in (I) to be as in Lemma 9.12. See Figure 2 .

Claim 10.3. Let $p=(\varsigma, \xi, x)$ and $q=(\varsigma, \xi, y)$ be in $K$ with $d(x, y)<r_{0}$. Let

$$
z=W_{\xi, r_{1}}^{u}(y) \cap W_{\xi, r_{1}}^{s}(x), \quad w=W_{\xi, r_{1}}^{s}(y) \cap W_{\xi, r_{1}}^{u}(x) .
$$

Assume $z \neq x$ and set $\delta=\left\|H_{p}^{s}(z)\right\|$. For any $m$ with

$$
M_{\delta} \leq m \leq m_{\delta}<0
$$

set
i) $\hat{p}_{m}=\left(\hat{\varsigma}_{m}, \hat{\xi}_{m}, \hat{x}_{m}\right)=\Phi^{m}(p), \hat{q}_{m}=\left(\hat{\varsigma}_{m}, \hat{\xi}_{m}, \hat{y}_{m}\right)=\Phi^{m}(q)$,
ii) $\left(\hat{\varsigma}_{m}, \hat{\xi}_{m}, \hat{z}_{m}\right)=\Phi^{m}(\varsigma, \xi, z),\left(\hat{\varsigma}_{m}, \hat{\xi}_{m}, \hat{w}_{m}\right)=\Phi^{m}(\varsigma, \xi, w)$,


Figure 2. Claim 10.3
iii) $\tilde{p}_{m}=\left(\tilde{\varsigma}_{m}, \tilde{\xi}_{m}, \tilde{x}_{m}\right)=\Phi^{L_{p, \delta, \varepsilon}(m)}(p), \tilde{q}_{m}=\left(\tilde{\varsigma}_{m}, \tilde{\xi}_{m}, \tilde{y}_{m}\right)=\Phi^{L_{p, \delta, \varepsilon}(m)}(q)$,
iv) $\left(\tilde{\varsigma}_{m}, \tilde{\xi}_{m}, \tilde{z}_{m}\right)=\Phi^{L_{p, \delta, \varepsilon}(m)}(\varsigma, \xi, z),\left(\tilde{\varsigma}_{m}, \tilde{\xi}_{m}, \tilde{w}_{m}\right)=\Phi^{L_{p, \delta, \varepsilon}(m)}(\varsigma, \xi, w)$.

Then, if $\hat{p}_{m}, \hat{q}_{m}, \tilde{p}_{m}, \tilde{q}_{m} \in K$, we have
(a) $\delta^{-\beta} \cdot\left\|H_{\hat{q}_{m}}^{u}\left(\hat{z}_{m}\right)\right\| \leq\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\| \leq \hat{r} \delta^{\beta}$,
(b) $\delta^{-\beta} \cdot\left\|H_{\hat{p}_{m}}^{u}\left(\hat{w}_{m}\right)\right\| \leq\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\| \leq \hat{r} \delta^{\beta}$,
(c) $d\left(\hat{x}_{m}, \hat{y}_{m}\right)<r_{0}$,
(d) $\left\|H_{\tilde{p}_{m}}^{u}\left(\tilde{w}_{m}\right)\right\| \leq \hat{r} \delta^{\beta}$,
(e) $d\left(\tilde{x}_{m}, \tilde{y}_{m}\right)<C_{2} \delta^{\beta} \hat{r}+C_{2} L_{1} e^{\tau_{p, \delta, \varepsilon}(m)\left(\lambda^{s}+\epsilon_{0}\right)} r_{1}$.

Proof. Note $0<\delta<1$. For (a), first observe that $\left\|H_{q}^{u}(z)\right\| \leq 1$. We then obtain the lower bound

$$
\begin{aligned}
\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\| \geq & \frac{1}{M_{0}}\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{s} \geq \frac{1}{M_{0}} e^{\left(\lambda^{s}+\epsilon_{0}\right) m}\left\|H_{p}^{s}(z)\right\|_{\epsilon_{0}, \pm, p}^{s} \\
\geq & \frac{1}{M_{0}^{2}} \delta e^{\left(\lambda^{s}+\epsilon_{0}\right) m} \geq \frac{1}{M_{0}^{2}} \delta e^{\left(\lambda^{s}+\epsilon_{0}\right) m_{\delta}} \\
= & \frac{1}{M_{0}^{2}} \exp \left[\left(\frac{\left(\lambda^{s}+\epsilon_{0}\right)\left((1+\beta) \log \delta-\log \left(M_{0}^{4}\right)\right)}{\lambda^{u}-\lambda^{s}-2 \epsilon_{0}}\right.\right. \\
& \left.\left.\quad+(1+\beta) \log \delta-\log \left(M_{0}^{4}\right)\right)-\beta \log \delta+\log \left(M_{0}^{4}\right)\right] \\
= & \frac{M_{0}^{4}}{M_{0}^{2}} e^{\left(\lambda^{u}-\epsilon_{0}\right) m_{\delta}} \delta^{-\beta} \geq \frac{M_{0}^{4}}{M_{0}^{3}} \delta^{-\beta}\left\|H_{\hat{q}_{m}}^{u}\left(\hat{z}_{m}\right)\right\| \|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{u} \\
\geq & \delta^{-\beta}\left\|H_{\hat{q}_{m}}^{u}\left(\hat{z}_{m}\right)\right\| .
\end{aligned}
$$

The lower bound in (b) is identical. The upper bound in (a) and (b) follows since

$$
\begin{aligned}
\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\| & \leq M_{0}\left\|H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right)\right\|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{s} \leq M_{0} e^{\left(-\lambda^{s}+\epsilon_{0}\right)|m|}\left\|H_{p}^{s}(z)\right\| \|_{\epsilon_{0}, \pm, p}^{s} \\
& \leq M_{0}^{2} e^{\left(-\lambda^{s}+\epsilon_{0}\right)|m|}\left\|H_{p}^{s}(z)\right\| \leq M_{0}^{2} e^{\left(-\lambda^{s}+\epsilon_{0}\right)\left|M_{\delta}\right|}\left\|H_{p}^{s}(z)\right\| \\
& \leq M_{0}^{2} \exp \left[\left(-\lambda^{s}+\epsilon_{0}\right) \frac{-(1-\beta) \log \delta+\log \varepsilon}{-\lambda^{s}+\epsilon_{0}}+\log \delta\right] \\
& \leq M_{0}^{2} \varepsilon \delta^{\beta} \leq \hat{r} \delta^{\beta}
\end{aligned}
$$

For (d) we have

$$
\begin{aligned}
\left\|H_{\tilde{p}_{m}}^{u}\left(\tilde{w}_{m}\right)\right\| & \leq M_{0}\| \| H_{\tilde{p}_{m}}^{u}\left(\tilde{w}_{m}\right)\| \|_{\epsilon_{0}, \pm, \tilde{p}_{m}}^{u} \\
& \leq M_{0}^{2}\| \| D \Phi^{\tau_{p, \delta, \varepsilon}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)}\| \|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{u}\left\|H_{\hat{p}_{m}}^{u}\left(\hat{w}_{m}\right)\right\| \\
& \leq \delta^{\beta} M_{0}^{2}\| \| D \Phi^{\tau_{p, \delta, \varepsilon}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)} \mid\left\|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{u}\right\| H_{\hat{p}_{m}}^{s}\left(\hat{z}_{m}\right) \| \\
& \leq \delta^{\beta} M_{0}^{4}\| \| D \Phi^{\tau_{p, \delta, \varepsilon}(m)} \upharpoonright_{E^{u}\left(\Phi^{m}(p)\right)} \mid\| \|_{\epsilon_{0}, \pm, \hat{p}_{m}}^{u}\left\|D \Phi^{m} \upharpoonright_{E^{s}(p)}\right\|_{\epsilon_{0}, \pm, p}^{s} \delta \\
& \leq \delta^{\beta} M_{0}^{4} \varepsilon \leq \delta^{\beta} \hat{r} .
\end{aligned}
$$

From (a) and Lemmas 9.12 (a) and 9.12 (b) we have the inequality $d\left(\hat{x}_{m}, \hat{y}_{m}\right) \leq 2 C_{2} \hat{r}$.
By the definition of $\hat{r}$ we obtain (c).
For (e), recall that $z$ is in the domain of the chart $\phi(\xi, x)$. We have

$$
|\phi(\xi, x)(z)| \leq C_{2} \ell_{0} \delta
$$

Then for $0 \leq j \leq\left\lfloor-M_{\delta}+1\right\rfloor$ we have

$$
\begin{aligned}
\left|\phi\left(F^{-j}(\xi, x)\right)\left(f_{\xi}^{-j}(z)\right)\right| & \leq e^{\left(-\lambda^{s}+2 \epsilon_{1}\right) j} C_{2} \ell_{0} \delta \\
& \leq e^{-8 \epsilon_{1} j} e^{\left(-\lambda^{s}+\epsilon_{0}\right) j} C_{2} \ell_{0} \delta \\
& \leq e^{-8 \epsilon_{1} j} e^{\left(-\lambda^{s}+\epsilon_{0}\right)\left(-M_{\delta}+1\right)} C_{2} \ell_{0} \delta \\
& \leq e^{-8 \epsilon_{1} j} e^{\left(-\lambda^{s}+\epsilon_{0}\right)} C_{2} \ell_{0} \delta^{\beta} \varepsilon
\end{aligned}
$$

Since $\varepsilon \leq \varepsilon_{0} \leq \frac{e^{\lambda^{s}-\epsilon_{0}} e^{-\lambda_{0}-\epsilon_{1}} \ell_{0}^{-2}}{10 C_{2}}$ and $d\left(\hat{y}_{m}, \hat{x}_{m}\right)<r_{0}$, it follows from Corollary 9.14 that $\hat{w}_{m} \in W_{\hat{\xi}_{m}, r_{1}}^{s}\left(\hat{y}_{m}\right)$. Conclusion (e) follows as

$$
\begin{aligned}
d\left(\tilde{x}_{m}, \tilde{y}_{m}\right) & \leq d\left(\tilde{x}_{m}, \tilde{w}_{m}\right)+d\left(\tilde{w}_{m}, \tilde{y}_{m}\right) \\
& \leq C_{2}\left\|H_{\tilde{p}_{m}}^{u}\left(\tilde{w}_{m}\right)\right\|+L_{1} e^{\tau_{p, \delta, \varepsilon}(m)\left(\lambda^{s}+\epsilon_{0}\right)} d\left(\hat{y}_{m}, \hat{w}_{m}\right) \\
& \leq C_{2} \delta^{\beta} \hat{r}+L_{1} e^{\tau_{p, \delta, \varepsilon}(m)\left(\lambda^{s}+\epsilon_{0}\right)}\left(C_{2} r_{1}\right)
\end{aligned}
$$

We observe that for

$$
g(\delta)=\frac{M_{\delta}-m_{\delta}}{M_{\delta}}=1-\frac{\left((1+\beta) \log \delta-\log \left(M_{0}^{4}\right)\right)\left(-\lambda^{s}+\epsilon_{0}\right)}{((1-\beta) \log \delta-\log \varepsilon)\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right)}
$$

we have

$$
\lim _{\delta \rightarrow 0} g(\delta)=1-\frac{(1+\beta)\left(-\lambda^{s}+\epsilon_{0}\right)}{(1-\beta)\left(\lambda^{u}-\lambda^{s}-2 \epsilon_{0}\right)}=2 \alpha_{0}
$$

We define one final parameter in addition to those from Section 10.1 .
(S.) For $T_{0}>0$ and $\varepsilon_{0}$ fixed in (Q) and (R) given any $0<\varepsilon<\varepsilon_{0}$ we define $0<\delta_{0}\left(T_{0}, \varepsilon\right)<1$ so that for all $0<\delta<\delta_{0}\left(T_{0}, \varepsilon\right)$ we have
(1) $M_{\delta}<m_{\delta}<0$,
(2) $\frac{M_{\delta}-m_{\delta}}{M_{\delta}} \geq \alpha_{0}$,
(3) $M_{\delta}<-T_{0}$,
(4) $L_{p, \delta, \varepsilon}\left(M_{\delta}\right)<-T_{0}$ for all $p \in Y_{0}$,
(5) $L_{p, \delta, \varepsilon}(0)>T_{0}$ for all $p \in Y_{0}$,
(6) $\tau_{p, \delta, \varepsilon}\left(M_{\delta}\right)>\max \left\{\hat{M}, \frac{\hat{M}}{\lambda^{u}-\epsilon_{0}}\right\}$ where $\hat{M}$ was fixed in (P)
(7) $\delta^{-\beta}>D_{1}$.
10.3. Key lemma. We have the following lemma, whose proof follows from the above choices.

Lemma 10.4. Given $0<\varepsilon<\varepsilon_{0}$ and any open $U \subset Y$ with $\hat{\omega}(U)<\hat{\alpha}$, there exist sequences
i) $\tilde{p}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{x}_{j}\right)$,
ii) $\tilde{q}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{y}_{j}\right)$,
iii) $p_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, x_{j}^{\prime}\right)$,
iv) $q_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, y_{j}^{\prime}\right)$,
v) $q_{j}^{\prime \prime}=\left(\varsigma_{j}^{\prime \prime}, \xi_{j}^{\prime \prime}, y_{j}^{\prime \prime}\right)$,
such that for all $j$
(a) $p_{j}^{\prime}, \tilde{p}_{j}, q_{j}^{\prime \prime}, \tilde{q}_{j} \in K$;
(b) $p_{j}^{\prime} \notin U$;
(c) $q_{j}^{\prime}=\Phi^{t_{j}}\left(q_{j}^{\prime \prime}\right)$ for some $\left|t_{j}\right| \leq \hat{T}$ where $\hat{T}$ is as in (K);
(d) there is $v_{j}^{\prime} \in W_{\xi_{j}^{\prime}, r_{1}}^{u}\left(x_{j}^{\prime}\right)$ with $\frac{1}{C_{3} M_{0}^{6}} \varepsilon \leq\left\|H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\right\| \leq C_{3} M_{0}^{6} \varepsilon$ and $d\left(y_{j}^{\prime}, v_{j}^{\prime}\right) \rightarrow$ 0 as $j \rightarrow \infty$.
Moreover,
(e) $d\left(\tilde{x}_{j}, \tilde{y}_{j}\right) \rightarrow 0$ as $j \rightarrow \infty$ and
(f) for every $j$ there are $a_{j}$ and $b_{j}$ with $\left|a_{j}\right|,\left|b_{j}\right| \in\left[M_{0}^{-4}, M_{0}^{4}\right]$ with

$$
\bar{\omega}_{\tilde{p}_{j}} \simeq\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{p_{j}^{\prime}}, \quad \text { and } \quad \bar{\omega}_{\tilde{q}_{j}} \simeq\left(\lambda_{b_{j}}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}} .
$$

In (f) above, $\lambda_{a}: \mathbb{R} \rightarrow \mathbb{R}$ denotes the multiplication map $\lambda_{a}: t \mapsto a t$.
10.3.1. Construction of the sequences in Lemma 10.4. Let $0<\varepsilon<\varepsilon_{0}$ be fixed, and let $U \subset Y$ satisfy $\hat{\omega}(U)<\hat{\alpha}$. We take this to be the $U$ in (N). We construct the sequences in Lemma 10.4 through a sequence of claims and then show they have the desired properties.

Recall that we assume $\mu_{\xi}$ is non-atomic for $\nu$-a.e. $\xi$. It follows that $\mu_{\xi}$ is not locally supported on $W_{r}^{u}(\xi, x)$ for almost every $(\xi, x)$. Indeed, as $\mu_{\xi}$ is assumed non-atomic, if otherwise it would follow that $\mu_{(\xi, x)}^{u}$ was not atomic, whence $h_{\mu}(F \mid$ $\pi)>0$. But then $\mu_{(\xi, x)}^{s}$ would necessarily be non-atomic a.s. and thus $\mu_{\xi}$ could not be locally supported on $W_{r}^{u}(\xi, x)$. It follows that $\omega_{(\varsigma, \xi)}$ is not locally supported on $W_{r}^{u}(\varsigma, \xi, x)$.


Figure 3. Choices of points in Lemma 10.4 and proof of Lemma 10.8 .

Recall $\mathcal{R}\left(T_{0}\right)$ fixed in (Q). We fix $p=(\varsigma, \xi, x) \in \mathcal{R}\left(T_{0}\right) \subset K$ such that $p$ is a $\omega_{(\varsigma, \xi)}$-density point of $\mathcal{R}\left(\overline{T_{0}}\right)$ for our fixed $T_{0}>0$ and such that $\omega_{(\varsigma, \xi)}$ is not locally supported on $W_{r}^{u}(\varsigma, \xi, x)$. It follows that there exists a sequence of points $\left\{y_{j}\right\} \subset M$
such that $q_{j}=\left(\varsigma, \xi, y_{j}\right) \in \mathcal{R}\left(T_{0}\right) \subset K, d\left(x, y_{j}\right) \leq r_{0}$, and $y_{j} \notin W_{r_{1}}^{u}(\xi, x)$ for all $j$ and $d\left(x, y_{j}\right) \rightarrow 0$ as $j \rightarrow \infty$. For each $j>0$ set (cf. Figure (2)

- $z_{j}=W_{\xi, r_{1}}^{s}(x) \cap W_{\xi, r_{1}}^{u}\left(y_{j}\right)$;
- $w_{j}=W_{\xi, r_{1}}^{s}\left(y_{j}\right) \cap W_{\xi, r_{1}}^{u}(x)$;
- $\delta_{j}=\left\|H_{p}^{s}\left(z_{j}\right)\right\|$.

We have $\delta_{j}>0$ for all $j$ and $\delta_{j} \rightarrow 0$. By omission, we may assume $\delta_{j}<\delta_{0}\left(T_{0}, \varepsilon\right)$ for all $j$.

We select a sequence of times $\left\{m_{j}\right\}$ satisfying the following claim. Recall $\mathscr{A}$ defined in (H)

Claim 10.5. Writing $\delta=\delta_{j}$, there exists an $m \in\left[M_{\delta}, m_{\delta}\right]$ with
(1) $\Phi^{m}(p) \in \mathscr{A} \cap K \cap S_{0}$ and $\Phi^{m}\left(q_{j}\right) \in K \cap S_{0}$;
(2) $\Phi^{L_{p, \delta, \varepsilon}(m)}(p) \in S_{\hat{M}} \cap K$ and $\Phi^{L_{p, \delta, \varepsilon}(m)}\left(q_{j}\right) \in S_{\hat{M}} \cap K$.

Proof of Claim 10.5, Let
(1) $F_{1}=\left\{t \in \mathbb{R}: \Phi^{t}(p) \in K \cap \mathscr{A} \cap S_{0}\right\}$;
(2) $F_{2}=\left\{t \in \mathbb{R}: \Phi^{t}\left(q_{j}\right) \in K \cap S_{0}\right\}$;
(3) $F_{3}=\left\{t \in \mathbb{R}: \Phi^{t}(p) \in K \cap S_{\hat{M}}\right\}$;
(4) $F_{4}=\left\{t \in \mathbb{R}: \Phi^{t}\left(q_{j}\right) \in K \cap S_{\hat{M}}\right\}$.

Write $L=L_{p, \delta, \varepsilon}$. Since $p, q_{j} \in \mathcal{R}\left(T_{0}\right), M_{\delta} \leq-T_{0}$, and $\frac{M_{\delta}-m_{\delta}}{M_{\delta}}>\alpha_{0}$, we have

$$
\begin{equation*}
\operatorname{Leb}\left(\left[M_{\delta}, m_{\delta}\right] \cap F_{1} \cap F_{2}\right) \geq\left(\alpha_{0}-5 \alpha\right)\left|M_{\delta}\right| . \tag{10.4}
\end{equation*}
$$

Furthermore, as $\left[-T_{0}, T_{0}\right] \subset L\left(\left[M_{\delta}, 0\right]\right)$, we have

$$
\operatorname{Leb}\left(L\left(\left[M_{\delta}, 0\right]\right) \cap F_{3} \cap F_{4}\right) \geq(1-4 \alpha) \operatorname{Leb}\left(L\left(\left[M_{\delta}, 0\right]\right)\right) ;
$$

hence, by Lemma 9.7 and Claim 9.8

$$
\operatorname{Leb}\left(L\left(\left[M_{\delta}, 0\right]\right) \backslash\left(F_{3} \cap F_{4}\right)\right) \leq(4 \alpha) \operatorname{Leb}\left(L\left(\left[M_{\delta}, 0\right]\right)\right) \leq 4 \alpha \kappa_{2}\left|M_{\delta}\right| .
$$

Then,

$$
\operatorname{Leb}\left(\left[M_{\delta}, 0\right] \backslash L^{-1}\left(F_{3} \cap F_{4}\right)\right) \leq 4 \alpha \kappa_{1}^{-1} \kappa_{2}\left|M_{\delta}\right| .
$$

Thus,
$\operatorname{Leb}\left(\left[M_{\delta}, m_{\delta}\right] \cap F_{1} \cap F_{2} \cap L_{p, \delta, \varepsilon}^{-1}\left(F_{3}\right) \cap L_{p, \delta, \varepsilon}^{-1}\left(F_{4}\right)\right) \geq\left(\alpha_{0}-5 \alpha-4 \alpha \kappa_{1}^{-1} \kappa_{2}\right)\left|M_{\delta}\right|$.
Our choice of $\alpha$ ensures $\alpha_{0}-5 \alpha-4 \alpha \kappa_{1}^{-1} \kappa_{2}>0$.
For each $j$, select a $m_{j}<0$ satisfying Claim 10.5 We define $\tilde{p}_{j}$ and $\tilde{q}_{j}$ satisfying the conclusions in Lemma 10.4 by

- $\tilde{p}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{x}_{j}\right)=\Phi^{L_{p, \delta_{j}}\left(m_{j}\right)}(p)$;
- $\tilde{q}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{y}_{j}\right)=\Phi^{L_{p, \delta_{j}}\left(m_{j}\right)}\left(q_{j}\right)$.

We also define

- $\hat{p}_{j}=\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{x}_{j}\right)=\Phi^{m_{j}}(p) ; \hat{q}_{j}=\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{y}_{j}\right)=\Phi^{m_{j}}\left(q_{j}\right)$;
- $s_{j}^{\prime}=\mathscr{S}_{\hat{p}_{j}}\left(\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)\right) ; s_{j}^{\prime \prime}=\mathscr{S}_{\hat{q}_{j}}\left(\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)\right)$.

Then $\tilde{p}_{j}=\Psi^{s_{j}^{\prime}}\left(\hat{p}_{j}\right)$ and $\tilde{q}_{j}=\Psi^{s_{j}^{\prime \prime}}\left(\hat{q}_{j}\right)$.

With the above choices, for each $j$ we choose a $\hat{\eta}_{j}$ satisfying the following.
Claim 10.6. Given $p, q_{j}$, and $m_{j}$ as above, for each $j$ there exists $\eta \in \Omega$ with
(a) $\left(\hat{\varsigma}_{j}, \eta, \hat{x}_{j}\right) \in K \cap \Psi^{-s_{j}^{\prime}}(K \backslash U)$;
(b) $\left(\hat{\varsigma}_{j}, \eta, \hat{y}_{j}\right) \in K \cap \Psi^{-s_{j}^{\prime \prime}}(K)$;
(c) $\eta \in\left(A_{\gamma_{2}}\left(\hat{p}_{j}\right)\right)$.

Furthermore, we may choose $\eta$ so that
(d) $f_{\xi}^{n}=f_{\eta}^{n}$ for all $n \leq 0$;
(e) $W_{\eta}^{u}\left(\hat{x}_{j}\right)=W_{\hat{\xi}_{j}}^{u}\left(\hat{x}_{j}\right)$ and $\bar{\omega}_{\hat{p}_{j}}=\bar{\omega}_{\left(\hat{\kappa}_{j}, \eta, \hat{x}_{j}\right)}$;
(f) $W_{\eta}^{u}\left(\hat{y}_{j}\right)=W_{\hat{\xi}_{j}}^{u}\left(\hat{y}_{j}\right)$ and $\bar{\omega}_{\hat{q}_{j}}=\bar{\omega}_{\left(\hat{\varsigma}_{j}, \eta, \hat{y}_{j}\right)}$.

Proof of Claim 10.6. We have $\hat{p}_{j}=\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{x}_{j}\right)$ and $\hat{q}_{j}=\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{y}_{j}\right)$ in $K \subset Y_{0}$. Then (d) (f) hold for $\nu_{\hat{\xi}_{j}}^{\hat{\hat{\beta}}}$-a.e. $\eta$.

Recall for $p=(\varsigma, \xi, x)$ we have $\omega_{p}^{\mathcal{S}}=\delta_{\varsigma} \times \nu_{\xi}^{\hat{F}} \times \delta_{x}$ as discussed in Section 9.4.1. Since $\hat{p}_{j} \in S_{0} \cap \mathscr{A}, \hat{q}_{j} \in S_{0}$, we have
(1) $\omega_{\hat{p}_{j}}^{\mathcal{S}}(K) \geq 0.9$;
(2) $\omega_{\hat{q}_{j}}^{S}(K) \geq 0.9$;
(3) $\nu_{\hat{\xi}_{j}}^{\hat{F}}\left(A_{\gamma_{2}}\left(\hat{p}_{j}\right)\right) \geq 0.9$.

Furthermore, since $\tilde{p}_{j}, \tilde{q}_{j} \in S_{\hat{M}}$, and since

$$
s_{j}^{\prime} \geq\left(\lambda^{u}-\epsilon_{0}\right) \tau_{p, \delta, \varepsilon}\left(m_{j}\right) \geq\left(\lambda^{u}-\epsilon_{0}\right) \tau_{p, \delta, \varepsilon}\left(M_{\delta_{j}}\right) \geq \hat{M}
$$

and similarly $s_{j}^{\prime \prime}>\hat{M}$, we have by (P)
(4) $\omega_{\hat{p}_{j}}^{\mathcal{S}}\left(\Psi^{-s_{j}^{\prime}}(K \backslash U)\right) \geq 0.9$;
(5) $\omega_{\hat{q}_{j}}^{\mathcal{S}}\left(\Psi^{-s_{j}^{\prime \prime}}(K)\right) \geq 0.9$.

From the natural identification of $\omega_{\hat{\mathcal{p}}_{j}}^{\mathcal{S}}$ and $\omega_{\tilde{q}_{j}}^{\mathcal{S}}$ with $\nu_{\hat{\mathcal{F}}_{j}}^{\hat{\hat{R}}}$, it follows that the set of $\eta$ satisfying the conclusions of the claim has $\nu_{\hat{\xi}_{j}}^{\hat{\hat{\beta}}}$-measure at least 0.5.
10.3.2. Proof of Lemma 10.4, Having selected $p, y_{j}, m_{j}$, and $\hat{\eta}_{j}$ above, we define

- $\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{z}_{j}\right)=\Phi^{L_{p, \delta, \varepsilon}\left(m_{j}\right)}\left(\varsigma, \xi, z_{j}\right)$, and $\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{w}_{j}\right)=\Phi^{L_{p, \delta, \varepsilon}\left(m_{j}\right)}\left(\varsigma, \xi, w_{j}\right)$;
- $\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{z}_{j}\right)=\Phi^{m_{j}}\left(\varsigma, \xi, z_{j}\right)$, and ( $\left.\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{w}_{j}\right)=\Phi^{m_{j}}\left(\varsigma, \xi, w_{j}\right)$;
- $\bar{p}_{j}=\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{x}_{j}\right)$, and $\bar{q}_{j}=\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{y}_{j}\right)$;
- $t_{j}^{\prime}=\mathscr{S}_{\bar{p}_{j}}^{-1}\left(s_{j}^{\prime}\right)$, and $t_{j}^{\prime \prime}=\mathscr{S}_{\bar{q}_{j}}^{-1}\left(s_{j}^{\prime \prime}\right)$.

We show Lemma 10.4 holds with $\tilde{p}_{j}, \tilde{q}_{j}$ defined above and

- $p_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, x_{j}^{\prime}\right):=\Psi^{s_{j}^{\prime}}\left(\bar{p}_{j}\right)=\Phi^{t_{j}^{\prime}}\left(\bar{p}_{j}\right) ;$
- $q_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, y_{j}^{\prime}\right):=\Phi^{t_{j}^{\prime}}\left(\bar{q}_{j}\right)$;
- $q_{j}^{\prime \prime}=\left(\varsigma_{j}^{\prime \prime}, \xi_{j}^{\prime \prime}, y_{j}^{\prime \prime}\right):=\Psi^{s_{j}^{\prime \prime}}\left(\bar{q}_{j}\right)=\Phi^{t_{j}^{\prime \prime}}\left(\bar{q}_{j}\right)$.

Proof of Lemma 10.4. Part (a) of Lemma 10.4 follows from the selection procedure in the above claims. Part (b) follows from Claim 10.4(a). Part (e) follows immediately from Claim 10.3(e) since as $j \rightarrow \infty, \delta_{j} \rightarrow 0$ and $\tau_{p, \delta, \varepsilon}\left(m_{j}\right) \geq \tau_{p, \delta, \varepsilon}\left(M_{\delta_{j}}\right) \rightarrow \infty$.

By Claim 10.3(c) we have $d\left(\hat{y}_{j}, \hat{x}_{j}\right)<r_{0}$. By Lemma 9.12, and the fact that $\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{x}_{j}\right)$ and $\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{y}_{j}\right)$ are in $K \subset[0,1) \times \Lambda^{\prime}$, we define $\hat{v}_{j}$ to be the point of
intersection

$$
\hat{v}_{j}=W_{\hat{\xi}_{j}, r_{1}}^{u}\left(\hat{x}_{j}\right) \cap W_{\tilde{\eta}_{j}, r_{1}}^{s}\left(\hat{y}_{j}\right) .
$$

From Lemma 9.12. Claim 10.3(a), and the fact that $\delta_{j}^{-\beta} \geq D_{1}$, for each $j$ we have

$$
\frac{1}{C_{3}}\left\|H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\right\| \leq\left\|H_{\hat{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| \leq C_{3}\left\|H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\right\|
$$

Recall that $W_{\hat{\xi}_{j}, r_{1}}^{u}\left(\hat{x}_{j}\right)=W_{\hat{\eta}_{j}, r_{1}}^{u}\left(\hat{x}_{j}\right)$ and

$$
\left\|H_{\hat{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\|=\left\|H_{\bar{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| .
$$

We define $v_{j}^{\prime}$ in Lemma 10.4(d) by

$$
\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, v_{j}^{\prime}\right)=\Phi^{t_{j}^{\prime}}\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{v}_{j}\right)
$$

We claim
Claim 10.7. $\frac{1}{C_{3} M_{0}^{6}} \varepsilon \leq\left\|H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\right\| \leq C_{3} M_{0}^{6} \varepsilon$.
Proof of Claim 10.7. We have the upper bound

$$
\begin{aligned}
\left\|H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\right\| & \leq M_{0}\| \| H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\| \|_{\epsilon_{0},-} \\
& =M_{0}\| \| H_{\bar{p}_{j}}^{u}\left(\hat{v}_{j}\right)\| \|_{\epsilon_{0},-}\left\|D \Phi^{t_{j}} \upharpoonright_{E^{u}\left(\bar{p}_{j}\right)}\right\|\left\|_{\epsilon_{0},-}=M_{0}\right\|\left\|_{\bar{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| \|_{\epsilon_{0},-} e^{s_{j}^{\prime}} \\
& \leq M_{0}^{2}\left\|H_{\bar{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| e^{s_{j}^{\prime}}=M_{0}^{2}\left\|H_{\hat{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| e^{s_{j}^{\prime}} \leq M_{0}^{2} C_{3}\left\|H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\right\| e^{s_{j}^{\prime}} \\
& \leq M_{0}^{3} C_{3}\| \| H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\| \|_{\epsilon_{0}, \pm} e^{s_{j}^{\prime}} \\
& =M_{0}^{3} C_{3}\| \| H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\| \|_{\epsilon_{0}, \pm}\| \| D \Phi^{\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)} \upharpoonright_{E^{u}\left(\hat{p}_{j}\right)}\| \|_{\epsilon_{0},-} \\
& \leq M_{0}^{5} C_{3}\| \| H_{\hat{p}_{j}}^{s}\left(\hat{z}_{j}\right)\| \|_{\epsilon_{0}, \pm}\left\|D \Phi^{\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)} \upharpoonright_{E^{u}\left(\hat{p}_{j}\right)}\right\| \|_{\epsilon_{0}, \pm} \\
& \leq M_{0}^{6} C_{3}\| \| D \Phi^{m_{j}}\left\lceil_{E^{s}(p)}\| \|_{\epsilon_{0}, \pm} \delta_{j}\left\|D \Phi^{\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)} \upharpoonright_{E^{u}\left(\hat{p}_{j}\right)}\right\| \|_{\epsilon_{0}, \pm}\right. \\
& =M_{0}^{6} C_{3} \varepsilon .
\end{aligned}
$$

The lower bound is identical.
As $\bar{q}_{j} \in K$ we have

$$
\begin{aligned}
d\left(y_{j}^{\prime}, v_{j}^{\prime}\right) \leq & L_{1} e^{t_{j}^{\prime}\left(\lambda^{s}+\epsilon_{0}\right)} d\left(\hat{y}_{j}, \hat{v}_{j}\right) \\
& \leq C_{2} L_{1} e^{t_{j}^{\prime}\left(\lambda^{s}+\epsilon_{0}\right)} r_{1}
\end{aligned}
$$

Since $\bar{q}_{j} \in K$ for each $j$ and since $s_{j}^{\prime}, \rightarrow \infty$, by the upper bound in (9.7) and the fact that $a$ in (9.7) is bounded on $K$ we have $t_{j}^{\prime} \rightarrow \infty$ as $j \rightarrow \infty$. Thus, $d\left(y_{j}^{\prime}, v_{j}^{\prime}\right) \rightarrow 0$ as $j \rightarrow \infty$, completing the proof of Lemma 10.4)(d)

To derive the bound in Lemma 10.4(c), first consider the case $t_{j}^{\prime} \geq t_{j}^{\prime \prime}$. As $\bar{p}_{j} \in K$, by the lower bound in (9.4) we have

$$
\|\mid\| \Phi^{t_{j}^{\prime \prime}}{ }_{E^{u}\left(\bar{p}_{j}\right)}\| \|_{\epsilon_{0},-} \geq \hat{L}^{-1}\left\|D \Phi^{t_{j}^{\prime \prime}}{ }_{E^{u}\left(\bar{p}_{j}\right)}\right\| .
$$

Moreover, as $\bar{q}_{j}$ and $q_{j}^{\prime \prime}=\Phi^{t_{j}^{\prime \prime}}\left(\bar{q}_{j}\right)$ are in $K$,

$$
\| D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)}\left\|\geq \frac{1}{M_{0}^{2}}\right\| \| D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)}\| \|_{\epsilon_{0},-} .\right.\right.
$$

Write $p_{j}^{\prime \prime}=\left(\varsigma_{j}^{\prime \prime}, \xi_{j}^{\prime \prime}, x_{j}^{\prime \prime}\right):=\Phi^{t_{j}^{\prime \prime}}\left(\bar{p}_{j}\right)$ and $\left(\varsigma_{j}^{\prime \prime}, \xi_{j}^{\prime \prime}, v_{j}^{\prime \prime}\right):=\Phi^{t_{j}^{\prime \prime}}\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{v}_{j}\right)$. For $n^{\prime}=$ $\left\lfloor\hat{\varsigma}_{j}+t_{j}^{\prime}\right\rfloor \geq n^{\prime \prime}=\left\lfloor\hat{\varsigma}_{j}+t_{j}^{\prime \prime}\right\rfloor \geq 0$ we have

$$
\begin{aligned}
& =\frac{\left\|D f_{\hat{\eta}_{j}}^{n^{\prime \prime}} \upharpoonright_{T_{\hat{y}_{j}} W_{\tilde{\eta}_{j}}^{u}\left(\hat{y}_{j}\right)}\right\|}{\left\|D f_{\hat{\eta}_{j}}^{n_{j}^{\prime \prime}} \upharpoonright_{T_{\hat{v}_{j}} W_{\tilde{\eta}_{j}}^{u}\left(\hat{x}_{j}\right)}\right\|} \cdot \frac{\left\|D f_{\xi_{j}^{\prime \prime}}^{-n^{\prime \prime}} \upharpoonright_{T_{x_{j}^{\prime \prime}} W_{\xi_{j}^{\prime \prime}}^{u}\left(x_{j}^{\prime \prime}\right)}\right\|}{\left\|D f_{\xi_{j}^{\prime \prime}}^{-n^{\prime \prime}} \upharpoonright_{T_{v_{j}^{\prime \prime}} W_{\xi_{j}^{\prime \prime}}^{u}\left(x_{j}^{\prime \prime}\right)}\right\|} \\
& =\frac{\left\|D f_{\hat{\eta}_{j}}^{n^{\prime \prime}} \upharpoonright_{T_{\hat{y}_{j}} W_{\tilde{\eta}_{j}}^{u}\left(\hat{y}_{j}\right)}\right\|}{\left\|D f_{\hat{\eta}_{j}}^{n_{j}^{\prime \prime}} \upharpoonright_{T_{\hat{v}_{j}} W_{\tilde{\eta}_{j}}^{u}\left(\hat{x}_{j}\right)}\right\|} \cdot \frac{\left\|D f_{\xi_{j}^{\prime}}^{-n^{\prime}} \upharpoonright_{T_{x_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|}{\left\|D f_{\xi_{j}^{\prime}}^{-n^{\prime}} \upharpoonright_{T_{v_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|} \cdot \frac{\left\|D f_{\xi_{j}^{\prime}}^{-\left(n^{\prime}-n^{\prime \prime}\right)} \upharpoonright_{T_{v_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|}{\left\|D f_{\xi_{j}^{\prime}}^{-\left(n^{\prime}-n^{\prime \prime}\right)} \upharpoonright_{T_{x_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|} .
\end{aligned}
$$

As $p_{j}^{\prime}, \bar{q}_{j} \in K$ we have $\left(\hat{\eta}_{j}, \hat{y}_{j}\right) \in \Lambda^{\prime}$ and $\left(\xi_{j}^{\prime}, x_{j}^{\prime}\right) \in \Lambda^{\prime}$. Moreover, as $\left\|H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\right\| \leq$ $r_{1}$ from Lemmas 9.12](f) and 9.12](g) we have that (10.5) is bounded above by $C_{1}^{3}$. Thus

$$
\left\|\left|D \Phi^{t_{j}^{\prime \prime}}\right|_{E^{u}\left(\bar{p}_{j}\right)}\right\|\left\|_{\epsilon_{0},-} \geq \frac{1}{C_{1}^{3} M_{0}^{2} \hat{L}}\right\| \| D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)}\| \|_{\epsilon_{0},-} .\right.
$$

As

$$
\begin{aligned}
\left\|\left\|D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)}\right)\right\|_{\epsilon_{0},-}\right. & =\mid \| D \Phi^{t_{j}^{\prime}}\left\lceil_{E^{u}\left(\bar{p}_{j}\right)}\| \|_{\epsilon_{0},-} \geq e^{\left(\lambda^{u}-\epsilon_{0}\right)\left(t_{j}^{\prime}-t_{j}^{\prime \prime}\right)}\| \| D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{p}_{j}\right)}\right) \|_{\epsilon_{0},-}\right. \\
& \geq e^{\left(\lambda^{u}-\epsilon_{0}\right)\left(t_{j}^{\prime}-t_{j}^{\prime \prime}\right)} \frac{1}{C_{1}^{3} M_{0}^{2} \hat{L}}\left\|D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)}\right)\right\| \|_{\epsilon_{0},-}
\end{aligned}
$$

it follows that

$$
t_{j}^{\prime}-t_{j}^{\prime \prime} \leq \frac{\log \left(C_{1}^{3} M_{0}^{2} \hat{L}\right)}{\lambda^{u}-\epsilon_{0}}=\hat{T}
$$

If $t_{j}^{\prime \prime} \geq t_{j}^{\prime}$ we similarly have

Then with $n^{\prime}=\left\lfloor\hat{\varsigma}_{j}+t_{j}^{\prime}\right\rfloor \geq 0$ we have
and, as above, by Lemmas 9.12(f) and 9.12(g) the expression in (10.6) is bounded below by $\frac{1}{C_{1}^{2}}$. Thus

$$
\left\|\| D \Phi^{t_{j}^{\prime}}\left\lceil_ { E ^ { u } ( \overline { q } _ { j } ) } \left|\| _ { \epsilon _ { 0 } , - } \geq \frac { 1 } { C _ { 1 } ^ { 2 } M _ { 0 } ^ { 2 } \hat { L } } | \| D \Phi ^ { t _ { j } ^ { \prime } } \left\lceil_{E^{u}\left(\bar{p}_{j}\right)} \mid \|_{\epsilon_{0},-}\right.\right.\right.\right.
$$

and the same analysis as above gives

$$
t_{j}^{\prime \prime}-t_{j}^{\prime} \leq \frac{\log \left(C_{1}^{2} M_{0}^{2} \hat{L}\right)}{\lambda^{u}-\epsilon_{0}} \leq \hat{T}
$$

Finally, for Lemma 10.4 f) we have

$$
\bar{\omega}_{\tilde{p}_{j}} \simeq\left(\left.\lambda_{ \pm \| D \Phi^{\tau_{p, \delta_{j}}, \varepsilon}\left(m_{j}\right)}\right|_{E^{u}\left(\hat{p}_{j}\right)} \|\right)_{*} \bar{\omega}_{\hat{p}_{j}}=\left(\lambda_{ \pm\left\|D \Psi^{s_{j}^{\prime}} \upharpoonright_{E^{u}\left(\hat{p}_{j}\right)}\right\|}\right)_{*} \bar{\omega}_{\hat{p}_{j}},
$$

where the sign depends on whether $D \Phi^{\tau_{p, \delta_{j}, \varepsilon}\left(m_{j}\right)} \upharpoonright_{E^{u}\left(\hat{p}_{j}\right)}: E^{u}\left(\hat{p}_{j}\right) \rightarrow E^{u}\left(\tilde{p}_{j}\right)$ preserves orientation. We similarly have

$$
\bar{\omega}_{p_{j}^{\prime}} \simeq\left(\lambda_{ \pm \| D \Phi^{t_{j}^{\prime}}\left\lceil_{\sum^{u}\left(\bar{p}_{j}\right)}\right)}\right)_{*} \bar{\omega}_{\bar{p}_{j}}=\left(\lambda_{ \pm \| D \Psi^{s_{j}^{\prime}}\left\lceil_{E^{u}\left(\bar{P}_{j}\right)} \|\right.}\right)_{*} \bar{\omega}_{\bar{p}_{j}} .
$$

Since $\bar{\omega}_{\bar{p}_{j}}=\bar{\omega}_{\hat{p}_{j}}$ we have

$$
\bar{\omega}_{p_{j}^{\prime}} \simeq\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{\tilde{p}_{j}}
$$

where

$$
\left|a_{j}\right|=\frac{\| D \Psi^{s_{j}^{\prime}}\left\lceil_{E^{u}\left(\bar{p}_{j}\right)} \|\right.}{\| D \Psi^{s_{j}^{\prime}}\left\lceil_{E^{u}\left(\hat{p}_{j}\right)} \|\right.} \leq M_{0}^{4} \frac{\left\|D \Psi^{s_{j}^{\prime}}{ }_{{ }_{E^{u}}\left(\bar{p}_{j}\right)}\right\| \|_{\epsilon_{0},-}}{\| D \Psi^{s_{j}^{\prime}}\left\lceil_{E^{u}\left(\hat{p}_{j}\right)}\| \|_{\epsilon_{0},-}\right.}=M_{0}^{4}
$$

proving the upper bound in Lemma 10.4 (f). The lower bound on $\left|a_{j}\right|$ and the existence of $b_{j}$ and its bounds are similar.
10.4. Proof of Proposition 7.1. We show Proposition 7.1 follows with

$$
\begin{equation*}
M:=C_{1} M_{0}^{14} C_{3} D_{0} \tag{10.7}
\end{equation*}
$$

where $M_{0}$ is as in (G), $C_{1}$ and $C_{3}$ are as in (I), and $D_{0}$ is as in (K) For $\varepsilon<\varepsilon_{0}$ define the set $G_{\varepsilon}$ as in Remark 7.2. Set $\tilde{G}_{\varepsilon}=[0,1) \times G_{\varepsilon}$. Consider $G_{\varepsilon} \cap K$. Were $\hat{\omega}\left(\tilde{G}_{\varepsilon} \cap K\right)<\hat{\alpha}$ there would exist an open $U \supset\left(\tilde{G}_{\varepsilon} \cap K\right)$ with $\hat{\omega}(U)<\hat{\alpha}$. With such a $U$ we obtain a sequence $p_{j}^{\prime} \in K$ satisfying the conclusions of Lemma 10.4 We have the following.
Lemma 10.8. Let $p$ be an accumulation point of $\left\{p_{j}^{\prime}\right\}$. Then $p \in \tilde{G}_{\varepsilon}$.
On the other hand, as $p_{j}^{\prime} \notin U$ for every $j$, we have $p \notin U$. This yields a contradiction showing $\hat{\omega}\left(\tilde{G}_{\varepsilon} \cap K\right) \geq \hat{\alpha}$ for all $\varepsilon<\varepsilon_{0}$. Then for $G$ defined as in Proposition 7.1 and for

$$
G \times[0,1)=\left\{p \mid p \in \tilde{G}_{1 / N} \text { for infinitely many } N\right\}=: \tilde{G}
$$

we have that $\hat{\omega}(\tilde{G}) \geq \hat{\alpha}$ and hence, as $\omega$ and $\hat{\omega}$ are equivalent measures, $\omega(\tilde{G})>0$. Then $\mu(G)>0$ and Proposition 7.1 follows.

We prove the lemma, concluding the proof of Proposition 7.1.

Proof of Lemma 10.8. With $U$ as above, we recall all notation from Lemma 10.4 , We have that each $p_{j}^{\prime}$ and $q_{j}^{\prime \prime}$ is contained in the compact set $K$. Let $p_{0} \in G_{\varepsilon}$ be an accumulation point of $\left\{p_{j}^{\prime}\right\}$. We may restrict to an infinite subset $B \subset \mathbb{N}_{0}$ such that $\lim _{j \in B \rightarrow \infty} p_{j}^{\prime}=p_{0}=\left(\varsigma_{0}, \xi_{0}, x_{0}\right)$. Further restricting $B$ we may assume that the sequence $\left(q_{j}^{\prime \prime}\right)_{j \in B}$ converges. Let $q_{1}=\lim _{j \in B \rightarrow \infty} q_{j}^{\prime \prime}$.

Recall that $\Phi^{t_{j}}\left(q_{j}^{\prime \prime}\right)=q_{j}^{\prime}$ for some $\left|t_{j}\right| \leq \hat{T}$. We may assume $\left(t_{j}\right)_{j \in B}$ converges. Note that $q_{j}^{\prime}$ is not assumed to be contained in $K$. However, as $p_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, x_{j}^{\prime}\right)_{j \in B}$ converges we have $\varsigma_{j}^{\prime} \in[0,1-a]$ for some $a>0$ and all $j \in B$. As $q_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, y_{j}^{\prime}\right)$, from (L) we have that $q_{j}^{\prime}=\Phi^{t_{j}}\left(q_{j}^{\prime \prime}\right)$ converges to $q_{0}=\left(\varsigma_{0}, \xi_{0}, y_{0}\right)=\Phi^{\hat{t}}\left(q_{1}\right)$ for some $|\hat{t}| \leq \hat{T}$. See Figure 3 .

Note that $q_{1} \in K$, and by Lemma 10.4(d) $q_{0} \in W_{r_{1}}^{u}\left(p_{0}\right)$ and

$$
\frac{1}{C_{3} M_{0}^{6}} \varepsilon \leq\left\|H_{p_{0}}^{u}\left(y_{0}\right)\right\| \leq C_{3} M_{0}^{6} \varepsilon
$$

We need not have $q_{0} \in K$. However-as $q_{1} \in K \subset Y_{0}, q_{0}=\Phi^{\hat{t}}\left(q_{1}\right)$, and $Y_{0}$ is $\Phi^{t}$ invariant - we have $q_{0} \in Y_{0}$. Thus, the unstable line field $E^{u}\left(q_{0}\right)$, unstable manifold $W^{u}\left(q_{0}\right)=W^{u}\left(p_{0}\right)$, trivialization $\mathcal{I}_{q_{0}}^{u}$, affine parameters $H_{q_{0}}^{u}$, and measure $\bar{\omega}_{q_{0}}$ are defined at $q_{0}$.

Fix $\gamma:=d\left(\left(\mathcal{I}_{q_{0}}^{u}\right)^{-1} \circ \mathcal{I}_{p_{0}}^{u}(t)\right) / d t(0)$, and let $v:=\left(\mathcal{I}_{p_{0}}^{u}\right)^{-1}\left(y_{0}\right)$ where $\mathcal{I}_{p}^{u}$ is defined in (6.4). As $p_{0} \in K$, by Proposition 6.5 and (6.3) (applied to unstable manifolds), and Lemma 9.12 (f) we have $C_{1}^{-1} \leq|\gamma| \leq C_{1}$. We also have $\left(C_{3} M_{0}^{6}\right)^{-1} \varepsilon \leq|v| \leq C_{3} M_{0}^{6} \varepsilon$. Define the map $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
\phi: t \mapsto \gamma(t-v) .
$$

By construction, we have

$$
\begin{equation*}
\phi_{*} \bar{\omega}_{p_{0}} \simeq \bar{\omega}_{q_{0}} \tag{10.8}
\end{equation*}
$$

Recall that given $\alpha \in \mathbb{R}$ we write $\lambda_{\alpha}: \mathbb{R} \rightarrow \mathbb{R}$ for the linear map $\lambda_{\alpha}: x \mapsto \alpha x$. Let $\beta:=\left\|D \Phi^{\hat{t}}{ }_{E^{u}\left(q_{1}\right)}\right\|$. As $q_{1} \in K$ we have $\frac{1}{D_{0}} \leq \beta \leq D_{0}$. Also, $\bar{\omega}_{q_{0}} \simeq\left(\lambda_{ \pm \beta}\right)_{*} \bar{\omega}_{q_{1}}$ where the sign depends on whether $\left.D \Phi^{\hat{t}}\right|_{E^{u}\left(q_{1}\right)}: E^{u}\left(q_{1}\right) \rightarrow E^{u}\left(q_{0}\right)$ preserves orientation. It remains to relate the measures $\bar{\omega}_{p_{0}}$ and $\bar{\omega}_{q_{1}}$.

Let $a_{j}$ and $b_{j}$ be as in Lemma 10.4(f). We further restrict the set $B \subset \mathbb{N}$ so that the limits

$$
\lim _{j \in B \rightarrow \infty} a_{j}=a, \quad \lim _{j \in B \rightarrow \infty} b_{j}=b
$$

are defined.
We claim that $\left(\lambda_{a}\right)_{*} \bar{\omega}_{p_{0}} \simeq\left(\lambda_{b}\right)_{*} \bar{\omega}_{q_{1}}$. Indeed, for all $j$ we have

$$
\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{p_{j}^{\prime}} \simeq \bar{\omega}_{\tilde{p}_{j}}, \quad\left(\lambda_{b_{j}}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}} \simeq \bar{\omega}_{\tilde{q}_{j}}
$$

We introduce normalization factors

$$
c_{j}:=\bar{\omega}_{p_{j}^{\prime}}\left(\left[-a_{j}^{-1}, a_{j}^{-1}\right]\right)^{-1}, \quad d_{j}:=\bar{\omega}_{q_{j}^{\prime \prime}}\left(\left[-b_{j}^{-1}, b_{j}^{-1}\right]\right)^{-1}
$$

and

$$
c:=\bar{\omega}_{p_{0}}\left(\left[-a^{-1}, a^{-1}\right]\right)^{-1}, \quad d:=\bar{\omega}_{q_{0}}\left(\left[-b^{-1}, b^{-1}\right]\right)^{-1}
$$

We remark that for $q \in Y_{0}$, the measure $\bar{\omega}_{q}$ has at most one atom which by assumption is at 0 . It follows that non-trivial intervals centered at 0 are continuity sets for each $\bar{\omega}_{q}$ and thus $c_{j} \rightarrow c$ and $d_{j} \rightarrow d$. Let $f$ be a continuous, compactly
supported function $f: \mathbb{R} \rightarrow \mathbb{R}$. We note that $q \mapsto \bar{\omega}_{q}(f)$ is uniformly continuous on $K$ and that

$$
\left|\left(\lambda_{a}\right)_{*} \bar{\omega}_{q}(f)-\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{q}(f)\right|=\left|\int f(a t)-f\left(a_{j} t\right) d \bar{\omega}_{q}(t)\right|
$$

approaches zero uniformly in $q$ as $j \in B \rightarrow \infty$. Thus for any $\kappa>0$ and for all sufficiently large $j \in B$ we have

- $\left|c\left(\lambda_{a}\right)_{*} \bar{\omega}_{p_{0}}(f)-c\left(\lambda_{a}\right)_{*} \bar{\omega}_{p_{j}^{\prime}}(f)\right| \leq \kappa$,
- $\left|c\left(\lambda_{a}\right)_{*} \bar{\omega}_{p_{j}^{\prime}}(f)-c_{j}\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{p_{j}^{\prime}}(f)\right| \leq \kappa$,
- $\left|d\left(\lambda_{b}\right)_{*} \bar{\omega}_{q_{1}}(f)-d\left(\lambda_{b}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}}(f)\right| \leq \kappa$,
- $\left|d\left(\lambda_{b}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}}(f)-d_{j}\left(\lambda_{b_{j}}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}}(f)\right| \leq \kappa$,
- $\left|\bar{\omega}_{\tilde{p}_{j}}(f)-\bar{\omega}_{\tilde{q}_{j}}(f)\right| \leq \kappa$,
where the final estimate follows since $\tilde{p}_{j}$ and $\tilde{q}_{j}$ become arbitrarily close in $K \subset Y$ as $j \in B \rightarrow \infty$ by Lemma 10.4)(e).

Since

$$
c_{j}\left(\lambda_{a_{j}}\right)_{*} \bar{\omega}_{p_{j}^{\prime}}(f)=\bar{\omega}_{\tilde{p}_{j}}(f), \quad d_{j}\left(\lambda_{b_{j}}\right)_{*} \bar{\omega}_{q_{j}^{\prime \prime}}(f)=\bar{\omega}_{\tilde{q}_{j}}(f),
$$

we conclude $c\left(\lambda_{a}\right)_{*} \bar{\omega}_{p_{0}}=d\left(\lambda_{b}\right)_{*} \bar{\omega}_{q_{1}}$, or

$$
\bar{\omega}_{p_{0}} \simeq\left(\lambda_{b / a}\right)_{*} \bar{\omega}_{q_{1}} .
$$

Combining the above with (10.8), it follows that map (with the appropriate sign discussed above)

$$
\psi=\left(\lambda_{b / a}\right) \circ \lambda_{ \pm \beta^{-1}} \circ \phi: t \mapsto \pm \frac{b \gamma}{\beta a}(t-v)
$$

satisfies

$$
\psi_{*} \bar{\omega}_{p_{0}} \simeq \bar{\omega}_{p_{0}} .
$$

It follows that $p_{0} \in \tilde{G}_{\varepsilon}$.

## 11. Geometry of the stable support of stationary measures

In this and the following sections, we return to the special case where $\Omega=\Sigma$ to prove Theorem 4.8. Recall the measure $\mu$ constructed by Proposition 4.2. We show that if the fiber-wise measures $\mu_{\xi}$ are finitely supported and if $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable, then the stationary measure $\hat{\mu}$ is finitely supported and hence $\hat{\nu}$-a.s. invariant. This result is analogous to [BQ1, Lemmas 3.10 and 3.11] but our methods of proof are completely different.

In Section 12, assuming $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable, we show that the fiber-wise measures are non-atomic under the additional assumption that the conditional measures along total stable sets (in both the $\Sigma_{\text {loc }}^{-}$and fiber-wise stable directions) satisfy a certain geometric criterion. In this section we consider the case in which the geometric criterion mentioned above fails. This degenerate case forces some rigidity of the measure $\mu$ which implies that the stationary measure $\hat{\mu}$ is $\hat{\nu}$-a.s. invariant.

We remark, however, that in this section we do not use the fact that $M$ is a surface though we still require that the stationary measure $\hat{\mu}$ be hyperbolic to obtain Lemma 11.2 below. Thus, for this section alone, take $M$ to be any closed manifold, take $\hat{\nu}$ as a measure on $\operatorname{Diff}^{2}(M)$ satisfying (困, and take $\hat{\mu}$ to be an ergodic, hyperbolic, $\hat{\nu}$-stationary measure. $\mu$ is as in Proposition 4.2. We note that if $\hat{\mu}$ has only positive exponents, then, by the invariance principle in AV, $\hat{\mu}$ is $\hat{\nu}$-a.s.
invariant and $\mu=\hat{\nu}^{\mathbb{Z}} \times \hat{\mu}$. We thus also assume $\hat{\mu}$ has one negative exponent. If all exponents of $\hat{\mu}$ are negative, the analysis and conclusions in this section are still valid.

Consider $\mathcal{P}$ a $\mu$-measurable partition of $\Sigma \times M$ with the property that for $\mu$-a.e. $(\xi, x) \in X$, there is an $r(\xi, x)$ with

$$
\begin{equation*}
\Sigma_{\mathrm{loc}}^{-}(\xi) \times W_{r(\xi, x)}^{s}(\xi, x) \subset \mathcal{P}(\xi, x) \subset \Sigma_{\mathrm{loc}}^{-}(\xi) \times W^{s}(\xi, x) \tag{11.1}
\end{equation*}
$$

Let $\left\{\mu_{(\xi, x)}^{\mathcal{P}}\right\}$ denote an associated family of conditional measures. We consider here the degenerate situation where $\mu_{(\xi, x)}^{\mathcal{P}}$ is supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ for $\mu$-a.e. $(\xi, x)$. Note that hyperbolicity and recurrence imply that for any other partition $\mathcal{P}^{\prime}$ satisfying (11.1) we have that $\mu_{(\xi, x)}^{\mathcal{P}^{\prime}}$ is supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ for $\mu$-a.e. $(\xi, x)$ and that

$$
\mu_{(\xi, x)}^{\mathcal{P}}=\mu_{(\xi, x)}^{\mathcal{P}^{\prime}} .
$$

In particular, the hypothesis that $\mu_{(\xi, x)}^{\mathcal{P}}$ is supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ for $\mu$-a.e. ( $\xi, x$ ) implies that the partition $\mathcal{P}^{\prime}$ given by $\mathcal{P}^{\prime}=\left\{\Sigma_{\text {loc }}^{-}(\xi) \times W^{s}(\xi, x)\right\}$ is measurable.

The purpose of this section is to prove the following proposition.
Proposition 11.1. Assume for some partition $\mathcal{P}$ as above that the measures $\mu_{(\xi, x)}^{\mathcal{P}}$ are supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ for $\mu$-a.e. $(\xi, x)$. Then $\mu=\hat{\nu}^{\mathbb{Z}} \times \hat{\mu}$ and $\hat{\mu}$ is $\hat{\nu}$-a.s. invariant.

The idea behind the proof of Proposition 11.1 is that if, for $\mathcal{P}$ as in (11.1), the conditional measures $\mu_{(\xi, x)}^{\mathcal{P}}$ are supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$, then the entropy of the skew product $F:(X, \mu) \rightarrow(X, \mu)$ has no fiber-wise entropy and thus the $\mu$-entropy of $F$ equals the entropy of the shift $\sigma:\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right) \rightarrow\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right)$. As $F$ is hyperbolic, the entropy of $F:(X, \mu) \rightarrow(X, \mu)$ should be captured by the mean conditional entropy $H_{\mu}(F \mathcal{P} \mid \mathcal{P})$ for any (decreasing) partition $\mathcal{P}$ subordinated to the stable sets of $F$ in $X$ (a partition $\mathcal{P}$ as in (11.1) will be such a partition under the assumptions on the support of $\left.\mu_{(\xi, x)}^{\mathcal{P}}\right)$. Let $\beta$ denote the partition on $X$ given by $\beta(\xi, x)=\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$. Then $\beta$ is equivalent to $\mathcal{P} \bmod \mu$, and we have $H_{\mu}(F \beta \mid \beta)=h_{\hat{\nu}^{Z}}(\sigma)$. Using Jensen's inequality in a manner analogous to the proof of Led1, Theorem 3.4] (see also [Y1, (6.1)] for the argument in English) one could show that the conditional measures $\mu_{(\xi, x)}^{\beta}$ are canonically identified with $\nu^{\mathbb{N}}$ almost everywhere. This would complete the proof.

However, the main technical obstruction in implementing the above procedure is that $h_{\hat{\nu}^{\mathbb{Z}}}(\sigma)$ is not assumed to be finite. Thus extra care is needed to approximate differences of the form $\infty-\infty$ arising from the outline above.
11.1. Proof of Proposition 11.1. Before presenting the proof of Proposition 11.1 we recall some facts about mean conditional entropy. A primary reference is Rok. Let $(X, \mu)$ be a Lebesgue probability space. Given measurable partitions $\alpha, \beta$ of ( $X, \mu$ ) (which may be uncountable) we define the mean conditional entropy of $\alpha$ relative to $\beta$ to be

$$
H_{\mu}(\alpha \mid \beta)=-\int \log \left(\mu_{x}^{\beta}(\alpha(x))\right) d \mu(x)
$$

where $\left\{\mu_{x}^{\beta}\right\}$ is a family of conditional measures relative to the partition $\beta$. The entropy of $\alpha$ is $H_{\mu}(\alpha)=H_{\mu}(\alpha \mid\{\varnothing, X\})$. Note that if $H_{\mu}(\alpha)<\infty$, then $\alpha$ is
necessarily countable. Given measurable partitions $\alpha, \beta, \gamma$ of $(X, \mu)$ we have
(1) $H_{\mu}(\alpha \vee \gamma \mid \beta)=H_{\mu}(\alpha \mid \beta)+H_{\mu}(\gamma \mid \alpha \vee \beta)$;
(2) If $\alpha \geq \beta$, then $H_{\mu}(\alpha \mid \gamma) \geq H_{\mu}(\beta \mid \gamma)$ and $H_{\mu}(\gamma \mid \alpha) \leq H_{\mu}(\gamma \mid \beta)$;
(3) If $\gamma_{n} \nearrow \gamma$ and if $H_{\mu}\left(\alpha \mid \gamma_{1}\right)<\infty$, then $H_{\mu}\left(\alpha \mid \gamma_{n}\right) \searrow H_{\mu}(\alpha \mid \gamma)$.

We proceed with the proof of Proposition 11.1.
Proof of Proposition 11.1, Let $\beta$ denote the partition on $X$ given by $\beta(\xi, x)=$ $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$. As remarked above, the hypothesis that $\mu_{(\xi, x)}^{\mathcal{P}}$ is supported on $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ for $\mu$-a.e. ( $\left.\xi, x\right)$ for some partition $\mathcal{P}$ satisfying (11.1) implies that all such partitions are equivalent modulo $\mu$ and, furthermore, that any such partition $\mathcal{P}$ is equivalent to $\beta$ modulo $\mu$.

Given a measure $\lambda$ on $\Sigma \times M$ and a $\lambda$-measurable partition $\mathcal{Q}$ of $\Sigma \times M$ we write $\lambda \Gamma_{\mathcal{Q}}$ for the restriction of $\lambda$ to the sub- $\sigma$-algebra of $\mathcal{Q}$-saturated subsets and $\lambda \mathcal{V}_{(\xi, x)}^{\mathcal{Q}}$ for the conditional measure of $\lambda$ along the atom $\mathcal{Q}(\xi, x)$. As we explain below, the proposition follows if we can show the conditional measures $\mu_{(\xi, x)}^{\beta}$ take the form

$$
d \mu_{(\xi, x)}^{\beta}(\eta, y)=\delta_{x}(y) d \hat{\nu}^{\mathbb{N}}\left(\ldots, \eta_{-3}, \eta_{-2}, \eta_{-1}\right) \delta_{\xi_{0}}\left(\eta_{0}\right) \delta_{\xi_{1}}\left(\eta_{1}\right) \delta_{\xi_{2}}\left(\eta_{2}\right) \ldots
$$

To this end, define a measure $\lambda$ on $\Sigma \times M$ with $\lambda \upharpoonright_{\beta}=\mu \upharpoonright_{\beta}$ and define $\lambda_{(\xi, x)}^{\beta}$ by

$$
d \lambda_{(\xi, x)}^{\beta}(\eta, y)=\delta_{x}(y) d \hat{\nu}^{\mathbb{N}}\left(\ldots, \eta_{-3}, \eta_{-2}, \eta_{-1}\right) \delta_{\xi_{0}}\left(\eta_{0}\right) \delta_{\xi_{1}}\left(\eta_{1}\right) \delta_{\xi_{2}}\left(\eta_{2}\right) \ldots .
$$

In what follows we show-under the hypothesis that $\mu_{(\xi, x)}^{\mathcal{P}}$ is supported on $\Sigma_{\text {loc }}^{-}(\xi) \times$ $\{x\}-$ that $\mu=\lambda$.

Define the partition $\mathcal{Q}$ of $\Sigma \times M$ by

$$
\mathcal{Q}(\xi, x)=\Sigma_{\mathrm{loc}}^{-}(\xi) \times M
$$

Observe for any $k \geq 0$ that

$$
\begin{equation*}
F^{k} \beta=F^{k} \mathcal{Q} \vee \beta \tag{11.2}
\end{equation*}
$$

Given a partition $\alpha$ of $\Sigma \times M$ we write

$$
\alpha^{-}:=\bigvee_{i=0}^{\infty} f^{-i} \alpha .
$$

We need the following lemma whose proof we postpone until the next subsection.
Lemma 11.2. There exists a finite entropy partition $\alpha$ of $\Sigma \times M$ with $\alpha \leq \beta$ and

$$
\begin{equation*}
\alpha^{-} \vee \mathcal{Q} \doteq \beta . \tag{11.3}
\end{equation*}
$$

Our strategy below will be to show that $\mu=\lambda$ by showing that

$$
\begin{equation*}
\mu_{(\xi, x)}^{\alpha-} \upharpoonright_{F^{k}(\beta)}^{\alpha-}=\lambda_{(\xi, x)}^{\alpha^{-}} \upharpoonright_{F^{k}(\beta)} \tag{11.4}
\end{equation*}
$$

for a.e. $(\xi, x)$ and all $k \geq 0$. Note that the equality $\lambda \upharpoonright_{\alpha^{-}}=\mu \upharpoonright_{\alpha^{-}}$and the $k=0$ case

$$
\mu_{(\xi, x)}^{\alpha-} \upharpoonright_{\beta}=\lambda_{(\xi, x)}^{\alpha-} \upharpoonright_{\beta}
$$

follow from the construction of $\lambda$ and that $\alpha^{-} \leq \beta$. Thus, as $F^{k} \beta$ generates the point partition for $k \geq 0$, showing (11.4) for all $k \geq 1$ is sufficient to prove that $\mu=\lambda$.

As noted above, the maps $F:(\Sigma \times M, \mu) \rightarrow(\Sigma \times M, \mu)$ and $\sigma:\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right) \rightarrow$ $\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right)$ may have infinite entropy. Thus it is necessary in the below argument to
approximate $\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right)$ by a finite entropy subsystem. Fix an increasing family of partitions $\mathcal{A}_{n}, n \in \mathbb{N}$, of $\left(\operatorname{Diff}^{2}(M), \hat{\nu}\right)$ with the following properties:
(1) $\mathcal{A}_{n}$ contains $n$ elements;
(2) $\mathcal{A}_{n+1} \geq \mathcal{A}_{n}$;
(3) $\mathcal{A}_{n}$ increases to the point partition on $\left(\operatorname{Diff}^{2}(M), \hat{\nu}\right)$.

Let $\overline{\mathcal{A}}_{n}$ be the partition of $\left(\Sigma, \hat{\nu}^{\mathbb{Z}}\right)$ defined by $\overline{\mathcal{A}}_{n}(\xi)=\left\{\eta \mid \eta_{0} \in \mathcal{A}_{n}\left(\xi_{0}\right)\right\}$. Define the partition $\mathcal{Q}_{n}$ on $\Sigma \times M$ by

$$
\mathcal{Q}_{n}(\xi, x)=\left\{(\eta, y) \mid \eta_{k} \in \mathcal{A}_{n}\left(\xi_{k}\right) \text { for all } k \geq 0\right\}
$$

Continue to write $\pi: \Sigma \times M \rightarrow \Sigma$. Then $\mathcal{Q}_{n}=\left(\pi^{-1} \overline{\mathcal{A}}_{n}\right)^{-}$. We have

$$
h_{\hat{\mathcal{L}}^{\mathbb{Z}}}\left(\sigma, \overline{\mathcal{A}}_{n}\right)=h_{\mu}\left(F, \pi^{-1} \overline{\mathcal{A}}_{n}\right)=H_{\mu}\left(F \mathcal{Q}_{n} \mid \mathcal{Q}_{n}\right) \leq \log (n) .
$$

Given $i \leq j \in \mathbb{Z}$ and $n \in \mathbb{N}$ define a (finite) partition $\mathcal{R}_{n}^{[i, j]}$ of $\Sigma \times M$ by

$$
\mathcal{R}_{n}^{[i, j]}(\xi, x):=\left\{(\eta, y): \eta_{\ell} \in \mathcal{A}_{n}\left(\xi_{\ell}\right) \text { for all } i \leq \ell \leq j\right\} .
$$

We have $\mathcal{R}_{n}^{[-k, m]} \nearrow F^{k}\left(\mathcal{Q}_{n}\right) \nearrow F^{k}(\mathcal{Q})$, respectively, as $m \rightarrow \infty$ and $n \rightarrow \infty$.
For fixed ( $\xi, x$ ) and $k \geq 0$, consider the sequence

$$
\begin{equation*}
\frac{\lambda_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)} \tag{11.5}
\end{equation*}
$$

as $(\eta, y)$ varies over $\alpha^{-}(\xi, x)$. For fixed $k$, this forms a non-negative supermartingale (on $\left(\alpha^{-}(\xi, x), \mu_{(\xi, x)}^{\alpha^{-}}\right)$, indexed by $m$ ) and hence converges pointwise.

From (11.2), (11.3), and the fact that $\mathcal{Q} \leq F^{k} \mathcal{Q}$ we have

$$
\begin{equation*}
\alpha^{-} \vee F^{k} \mathcal{Q}=\alpha^{-} \vee F^{k} \mathcal{Q} \vee \mathcal{Q}=F^{k} \beta \tag{11.6}
\end{equation*}
$$

As the $\sigma$-algebras generated by $\mathcal{R}_{m}^{[-k, m]}$ increase to the algebra generated by $F^{k} \mathcal{Q}$ as $m \rightarrow \infty$, by a theorem of Andersen and Jessen ( AJ]; see also [Sch, Hor for statements) the pointwise limit of (11.5) is the Radon-Nikodym derivative

$$
\lim _{m \rightarrow \infty} \frac{\lambda_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}=\frac{d \lambda_{(\xi, x)}^{\alpha^{-}} \upharpoonright_{F^{k} \beta}}{d \mu_{(\xi, x)}^{\alpha-} \upharpoonright_{F^{k} \beta}}(\eta, y) .
$$

Note that $\mathcal{R}_{n}^{[0, m]} \leq \beta$ for all $m \geq 0$ and $n \geq 1$, and hence

$$
\lambda_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[0, m]}(\xi, x)\right)=\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[0, m]}(\xi, x)\right)
$$

for any $m \geq 0$. For $(\eta, y) \in \alpha^{-}(\xi, x) \cap \mathcal{R}_{n}^{[0, m]}(\xi, x)$ we have

$$
\frac{\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-\vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}=\frac{\lambda_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)} \cdot \frac{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[0, m]}(\xi, x)\right)}{\lambda_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[0, m]}(\xi, x)\right)} .
$$

Thus

$$
\frac{\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-\vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}=\frac{\lambda_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}
$$

For every $k, n, m$, and $(\xi, x)$ we have

$$
\left.\int_{\left(\alpha^{-} \vee \mathcal{R}\right.} \mathcal{R}_{n}^{[0, m]}\right)(\xi, x), \frac{\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-\vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)} d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) \leq 1
$$

Consider the expressions

$$
I_{1}(n, m)=\iint \log \left(\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)\right) d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) d \mu(\xi, x)
$$

and

$$
I_{2}(n, m)=\iint \log \left(\mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)\right) d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) d \mu(\xi, x)
$$

From the above inequality and Jensen's inequality, for every $k, n$, and $m$ we have that $I_{1}(n, m)-I_{2}(n, m) \leq 0$. From the explicit form of $\lambda_{(\xi, x)}^{\beta}$, for $(\eta, y) \in \alpha^{-}(\xi, x) \vee$ $\mathcal{R}_{n}^{[0, m]}(\xi, x)$ we have for $k \geq 1$

$$
\begin{aligned}
\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right) & =\prod_{i=1}^{k} \hat{\nu}\left(\mathcal{A}_{n}\left(\eta_{-i}\right)\right) \\
& =\mu_{(\eta, y)}^{\mathcal{Q}_{n}}\left(F^{k} \mathcal{Q}_{n}(\eta, y)\right),
\end{aligned}
$$

whence

$$
\begin{aligned}
I_{1}(n, m) & =\int\left(\log \mu_{(\eta, y)}^{\mathcal{Q}_{n}}\left(F^{k} \mathcal{Q}_{n}(\eta, y)\right)\right) d \mu(\eta, y)=-H\left(F^{k} \mathcal{Q}_{n} \mid \mathcal{Q}_{n}\right) \\
& =-h_{\mu}\left(F^{k}, \pi^{-1}\left(\overline{\mathcal{A}}_{n}\right)\right) .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
I_{2}(n, m) & =\iint \log \left(\mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)\right) d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) d \mu(\xi, x) \\
& =\iint \log \left(\mu_{(\eta, y)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)\right) d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) d \mu(\xi, x) \\
& =\int \log \left(\mu_{(\eta, y)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{n}^{[-k, m]}(\eta, y)\right)\right) d \mu(\eta, y) \\
& =-H_{\mu}\left(\mathcal{R}_{n}^{[-k, m]} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right) .
\end{aligned}
$$

Recall the facts about mean conditional entropy collected above. We have the formula

$$
\begin{gather*}
H_{\mu}\left(\mathcal{R}_{n}^{[-k, m]} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right)=H_{\mu}\left(\mathcal{R}_{n}^{[-k, m]} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right) \\
\quad+H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]} \vee \mathcal{R}_{n}^{[-k, m]}\right) . \tag{11.7}
\end{gather*}
$$

As $H_{\mu}(\alpha)<\infty$ we have $H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-}\right)<\infty$. In particular, as $\mathcal{R}_{n}^{[-k, m]}$ and $\mathcal{R}_{n}^{[0, m]}$ are finite partitions, both terms on the right-hand side of (11.7) are finite.

By (11.6) and the fact that $\mathcal{Q} \leq F^{k} \mathcal{Q}$, we have $H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee F^{k} \mathcal{Q}\right)=0$. As

$$
\begin{aligned}
H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[-k, m]}\right) & \underset{m \rightarrow \infty}{\searrow} H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee F^{k} \mathcal{Q}_{n}\right) \\
& \searrow H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee F^{k} \mathcal{Q}\right)
\end{aligned}
$$

given $\varepsilon>0$ we may select $m_{0}$ so that

$$
H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{m_{0}}^{\left[-k, m_{0}\right]}\right)<\varepsilon
$$

Furthermore for any $n>0$

$$
\begin{aligned}
H_{\mu}\left(\mathcal{R}_{n}^{[-k, m]} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right) & =H_{\mu}\left(\mathcal{R}_{n}^{[-k,-1]} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right) \\
& \searrow H_{\mu}\left(\mathcal{R}_{n}^{[-k,-1]} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{Q}_{n}\right) \\
& =H_{\mu}\left(F^{k} \mathcal{Q}_{n} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{Q}_{n}\right) .
\end{aligned}
$$

But, for any $n$

$$
H_{\mu}\left(F^{k} \mathcal{Q}_{n} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{Q}_{n}\right)=h_{\mu}\left(F^{k}, \pi^{-1}\left(\overline{\mathcal{A}}_{n}\right) \vee \alpha\right) \geq H_{\mu}\left(F^{k} \mathcal{Q}_{n} \mid \mathcal{Q}_{n}\right)
$$

Thus for $m_{0}$ above we have

$$
\begin{aligned}
& I_{1}\left(m_{0}, m_{0}\right)- I_{2}\left(m_{0}, m_{0}\right)=-H_{\mu}\left(F^{k} \mathcal{Q}_{m_{0}} \mid \mathcal{Q}_{m_{0}}\right)+H_{\mu}\left(\mathcal{R}_{m_{0}}^{\left[-k, m_{0}\right]} \mid \alpha^{-} \vee \mathcal{R}_{m_{0}}^{\left[0, m_{0}\right]}\right) \\
&=- H_{\mu}\left(F^{k} \mathcal{Q}_{m_{0}} \mid \mathcal{Q}_{m_{0}}\right)+H_{\mu}\left(\mathcal{R}_{n}^{[-k, m]} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}\right) \\
&-H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{m_{0}}^{\left[-k, m_{0}\right]}\right) \\
& \geq- H_{\mu}\left(F^{k} \mathcal{Q}_{m_{0}} \mid \mathcal{Q}_{m_{0}}\right)+H_{\mu}\left(F^{k} \mathcal{Q}_{m_{0}} \vee F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{Q}_{m_{0}}\right) \\
& \quad-H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{m_{0}}^{\left[-k, m_{0}\right]}\right) \\
& \geq- H_{\mu}\left(F^{k} \alpha^{-} \mid \alpha^{-} \vee \mathcal{R}_{m_{0}}^{\left[-k, m_{0}\right]}\right) \\
& \geq-\varepsilon .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \iint \log \left(\frac{\lambda_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}\right) d \mu_{(\xi, x)}^{\alpha^{-}}(\eta, y) d \mu(\xi, x) \\
& =\iint \log \left(\frac{\lambda_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-\vee \vee \mathcal{R}_{n}^{[0, m]}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}\right) d \mu_{(\xi, x)}^{\alpha^{-} \vee \mathcal{R}_{n}^{[0, m]}}(\eta, y) d \mu(\xi, x) \\
& =I_{1}(m, m)-I_{2}(m, m)
\end{aligned}
$$

approaches 0 as $m \rightarrow \infty$.
We have the following elementary claim.
Claim 11.3. Let $f_{n}$ be a sequence of positive, $\mu$-integrable functions. Assume $\int f_{n} d \mu \leq 1$ for every $n$ and that $\int \log f_{n} d \mu \rightarrow 0$ as $n \rightarrow \infty$. Then $f_{n}$ converges to 1 in measure.

Proof. Given $\delta>0$, there is a $c_{\delta}>0$ such that for all $x \in(0, \infty)$ with $|x-1|>\delta$ we have $\log x \leq x-1-c_{\delta}$. Then for every $n$,

$$
\begin{aligned}
\int \log f_{n} d \mu & \leq \int f_{n} d \mu-1-\mu\left(\left\{x:\left|f_{n}(x)-1\right|>\delta\right\}\right) c_{\delta} \\
& \leq-\mu\left(\left\{x:\left|f_{n}(x)-1\right|>\delta\right\}\right) c_{\delta} .
\end{aligned}
$$

As $\int \log f_{n} d \mu \rightarrow 0$ we have $\mu\left(\left\{x:\left|f_{n}(x)-1\right|>\delta\right\}\right) \rightarrow 0$ as $n \rightarrow \infty$.
As $\frac{\lambda_{(\xi, x)}^{\alpha^{-}}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)}{\mu_{(\xi, x)}^{\alpha-}\left(\mathcal{R}_{m}^{[-k, m]}(\eta, y)\right)} \rightarrow \frac{d \lambda_{(\xi, x)}^{\alpha^{-}} \upharpoonright^{k} \beta}{d \mu_{(\xi, x)}^{\alpha-} \upharpoonright^{k} \beta}(\eta, y)$ it follows from Claim 11.3 that

$$
\frac{d \lambda_{(\xi, x)}^{\alpha^{-}} \upharpoonright_{F^{k} \beta}}{d \mu_{(\xi, x)}^{\alpha-} \upharpoonright_{F^{k} \beta}}(\eta, y)=1
$$


Now consider an atom of $\mathcal{Q}(\xi, x)$. We have the canonical product representation $\mathcal{Q}(\xi, x)=\Sigma_{\text {loc }}^{-}(\xi) \times M$. Let $\bar{\mu}_{(\xi, x)}^{\mathcal{Q}}$ denote the projection of $\mu_{(\xi, x)}^{\mathcal{Q}}$ on $\Sigma_{\text {loc }}^{-}(\xi) \times M$ onto $M$. Using that $\mu=\lambda$, in these coordinates we have for $\eta \in \Sigma_{\text {loc }}^{-}(\xi)$ and $y \in M$ that

$$
d \mu \underset{(\xi, x)}{\mathcal{Q}}(\eta, y)=d \hat{\nu}\left(\eta_{-1}\right) d \hat{\nu}\left(\eta_{-2}\right) \cdots d \bar{\mu}_{(\xi, x)}^{\mathcal{Q}}(y)
$$

Then we have the natural identification $\mu_{\eta}=\mu_{\xi}=\bar{\mu}_{(\xi, x)}^{\mathcal{Q}}$ for $\hat{\nu}^{\mathbb{N}}$-a.e. $\eta \in \Sigma_{\text {loc }}^{-}(\xi)$. In particular, the function $\xi \mapsto \mu_{\xi}$ is a.s.-constant on almost every local stable set. As $\xi \mapsto \mu_{\xi}$ is a.s.-constant on almost every local unstable set in $\Sigma$, an argument similar to Proposition 4.6 shows that $\xi \mapsto \mu_{\xi}$ is a.s. constant on $\Sigma$.
11.2. Proof of Lemma 11.2, We remark that we continue to assume $M$ to be a compact, $d$-dimensional manifold. For $\hat{\nu}$ a measure on $\operatorname{Diff}^{2}(M)$ satisfying (*), we take $\hat{\mu}$ to be an ergodic, $\hat{\nu}$-stationary measure. We further assume that $\hat{\mu}$ is hyperbolic. Take $\kappa>0$ so that $\hat{\mu}$ has no exponents in the interval $[-\kappa, \kappa]$.
11.2.1. One-sided Lyapunov charts and stable manifolds as Lipschitz graphs. Let $k$ be the almost-surely constant value of $\operatorname{dim} E_{\omega}^{s}(x)$. Given $v \in \mathbb{R}^{d}=\mathbb{R}^{k} \times \mathbb{R}^{d-k}$, decompose $v=v_{1}+v_{2}$ and write $|v|_{i}=\left|v_{i}\right|$ and $|v|=\max \left\{|v|_{i}\right\}$. We will write $d_{\mathbb{R}^{d}}(\cdot, \cdot)$ for the induced metric on $\mathbb{R}^{d}$ and $d(\cdot, \cdot)$ for the metric on $M$. We use the notation $\mathbb{R}^{d}(r)$ to denote the ball of radius $r$ centered at 0 . To emphasize the one-sidedness of our constructions we work on $\Sigma_{+} \times M$. Recall the associated skew product $\hat{F}: \Sigma_{+} \times M \rightarrow \Sigma_{+} \times M$ and the corresponding $\hat{F}$-invariant measure $\hat{\nu}^{\mathbb{N}} \times \hat{\mu}$ on $\Sigma_{+} \times M$.

As outlined in [LY3, (4.1)], for every sufficiently small $\varepsilon>0$, there is a measurable function $l: \Sigma_{+} \times M \rightarrow[1, \infty)$ and a full measure set $\Lambda \subset \Sigma_{+} \times M$ such that
(1) for $(\omega, x) \in \Lambda$ and every $n \in \mathbb{N}$, there exists a diffeomorphism $\phi_{n}$ defined on a small neighborhood of $f_{\omega}^{n}(x)$ whose range is $\mathbb{R}^{d}\left(\ell(\omega, x)^{-1} e^{-n \varepsilon}\right)$ with
(a) $\phi_{0}(\omega, x)(x)=0$;
(b) $D \phi_{0}(\omega, x) E_{\omega}^{s}(x)=\mathbb{R}^{k} \times\{0\}$;
(c) $D \phi_{0}(\omega, x)\left(E_{\omega}^{s}(x)\right)^{\perp}=\{0\} \times \mathbb{R}^{d-k}$;
(2) for $n \geq 1$, writing $\tilde{f}_{n}(\omega, x)=\phi_{n+1}(\omega, x) \circ f_{\sigma^{n-1}(\omega)} \circ \phi_{n}(\omega, x)^{-1}$ where defined, for all $n \geq 0$ we have
(a) $\tilde{f}_{n}(\omega, x)(0)=0$;
(b) $D_{0} \tilde{f}_{n}(\omega, x)=\left(\begin{array}{cc}A_{n} & 0 \\ 0 & B_{n}\end{array}\right)$ where $A_{n} \in \mathrm{GL}(k, \mathbb{R}), B_{n} \in \mathrm{GL}(d-k, \mathbb{R})$ and $\left|A_{n} v\right| \leq e^{-\kappa+\varepsilon}|v|, v \in \mathbb{R}^{k}, e^{\kappa-\varepsilon}|v| \leq\left|B_{n} v\right|, v \in \mathbb{R}^{d-k}$;
(c) $\operatorname{Lip}\left(\tilde{f}_{n}(\omega, x)-D_{0} \tilde{f}_{n}(\omega, x)\right)<\varepsilon$,
where $\operatorname{Lip}(\cdot)$ denotes the Lipschitz constant of a map on its domain;
(3) $\ell(\omega, x)^{-1} e^{-n \varepsilon} \leq \operatorname{Lip}\left(\phi_{n}(\omega, x)\right) \leq \ell(\omega, x) e^{n \varepsilon}$.

Note that the domain of $\phi_{n}(\omega, x)$ contains a ball of radius $\ell(\omega, x)^{-2} e^{-2 n \varepsilon}$ centered at $f_{\omega}^{n}(x)$ in $M$. We remark that while the Lipschitz constant of $\tilde{f}_{n}$, norm of $B_{n}$, and conorm of $A_{n}$ need not be bounded, the hyperbolicity of $D_{0} \tilde{f}_{n}$ and the Lipschitz closeness of $\tilde{f}_{n}$ to $D_{0} \tilde{f}_{n}$ is uniform in $n$.

Relative to the charts $\phi_{n}(\omega, x)$, one may apply the Perron-Irwin method of constructing stable manifolds through each point of the orbit $\left\{f_{\omega}^{n}(x), n \geq 0\right\}$. See the proof of Theorem 3.1 in $\overline{L Q}$ or the similar proof of QXZ, Theorem V.4.2]. Choosing $\varepsilon>0$ above sufficiently small, the outcome is the following.

Proposition 11.4. For $(\omega, x) \in \Lambda$ and every $n \geq 0$ there is a Lipschitz function

$$
h_{n}(\omega, x): \mathbb{R}^{k}\left(\ell(\omega, x)^{-1} e^{-n \varepsilon}\right) \rightarrow \mathbb{R}^{d-k}
$$

with
(1) $h_{n}(\omega, x)(0)=0$;
(2) $\operatorname{Lip}\left(h_{n}(\omega, x)\right) \leq 1$;
(3) $\tilde{f}_{n}\left(\operatorname{graph}\left(h_{n}(\omega, x)\right)\right) \subset \operatorname{graph}\left(h_{n+1}(\omega, x)\right)$, and if $y, z \in \operatorname{graph}\left(h_{n}(\omega, x)\right)$, then

$$
\left|\tilde{f}_{n}(\omega, x)(y)-\tilde{f}_{n}(\omega, x)(z)\right| \leq\left(e^{-\kappa+\varepsilon}+\varepsilon\right)|y-z|
$$

Note that we have that $\operatorname{graph}\left(h_{n}(\omega, x)\right)$ is contained in the domain of $\tilde{f}_{n}$. We have that $\phi_{n}^{-1}\left(\operatorname{graph}\left(h_{n}(\omega, x)\right)\right.$ is an open subset of $W_{\sigma^{n}(\omega)}^{s}\left(f_{\omega}^{n}(x)\right)$.
11.2.2. Divergence from the stable manifold in local charts. We have the following claim.

Claim 11.5. Fix $(\omega, x) \in \Lambda$, and suppose $y \in \mathbb{R}^{d}\left(\ell(\omega, x)^{-1} e^{-n \varepsilon}\right)$ is in the domain of $\tilde{f}_{n}(\omega, x)$. Write $y=(u, v)$ and $f(y)=\left(u^{\prime}, v^{\prime}\right)$. Then

$$
\left|v^{\prime}-h_{n+1}(\omega, x)\left(u^{\prime}\right)\right|_{2} \geq\left(e^{\kappa-\varepsilon}-2 \varepsilon\right)\left|v-h_{n}(\omega, x)(u)\right|_{2}
$$

Proof. Write $z=\left(u, h_{n}(\omega, x)(u)\right)$ and $(\hat{u}, \hat{v})=\tilde{f}_{n}(\omega, x)(z)$. Then

$$
\left|v^{\prime}-h_{n+1}(\omega, x)\left(u^{\prime}\right)\right|_{2} \geq\left|v^{\prime}-\hat{v}\right|_{2}-\left|\hat{v}-h_{n+1}(\omega, x)\left(u^{\prime}\right)\right|_{2} .
$$

As

$$
\tilde{f}_{n}(\omega, x)(y)-\tilde{f}_{n}(\omega, x)(z)=D_{0} \tilde{f}_{n}(\omega, x)(y-z)+w
$$

where $|w| \leq \varepsilon|y-z|$, we have
(1) $\left|u^{\prime}-\hat{u}\right|_{1}=\left|\tilde{f}_{n}(\omega, x)(y)-\tilde{f}_{n}(\omega, x)(z)\right|_{1} \leq \varepsilon|y-z|$;
(2) $\left|v^{\prime}-\hat{v}\right|_{2}=\left|\tilde{f}_{n}(\omega, x)(y)-\tilde{f}_{n}(\omega, x)(z)\right|_{2}=\left|\tilde{f}_{n}(\omega, x)(y)-\tilde{f}_{n}(\omega, x)(z)\right| \geq$ $e^{\kappa-\varepsilon}|y-z|-\varepsilon|y-z|$.
As $\operatorname{Lip}\left(h_{n}(\omega, x)\right) \leq 1$ and as $\tilde{f}_{n}(\omega, x)(z) \in \operatorname{graph}\left(h_{n+1}(\omega, x)\right)$, we have

$$
\left|\hat{v}-h_{n+1}(\omega, x)\left(u^{\prime}\right)\right|_{2}=\left|h_{n+1}(\omega, x)(\hat{u})-h_{n+1}(\omega, x)\left(u^{\prime}\right)\right|_{2} \leq\left|u^{\prime}-\hat{u}\right|_{1} \leq \varepsilon|y-z| .
$$

As $|y-z|=\left|v-h_{n}(\omega, x)(u)\right|_{2}$, the claim follows.

Note that having taken $\varepsilon>0$ sufficiently small we can arrange that $e^{\kappa-\varepsilon}-2 \varepsilon \geq$ $e^{\kappa-3 \varepsilon}$.

We write $\widetilde{W}_{m}^{s}(\omega, x):=\operatorname{graph}\left(h_{m}(\omega, x)\right)$ for the remainder. Note that $\widetilde{W}_{m}^{s}(\omega, x)$ is the path-connected component of

$$
\left.\phi_{m}(\omega, x)\left(W_{\sigma^{m}(\omega)}^{s}\right)\left(f_{\omega}^{m}(x)\right)\right)
$$

in $\mathbb{R}^{d}\left(\ell(\omega, x)^{-1} e^{-m \varepsilon}\right)$ containing 0 .
11.2.3. Radius function and related estimates. Fix $K_{0} \subset \Sigma_{+} \times M$ with a positive measure on which the function $\ell(\omega, x)$ is bounded above by some $\ell>10$. Fix $m_{0} \in \mathbb{N}$ so that

$$
\chi:=\left(e^{-m_{0}(\kappa-4 \varepsilon)}\right) 2 \ell^{2}<1 .
$$

For $(\omega, x) \in K_{0}$ define $n(\omega, x)$ to be the $m_{0}$ th return of $(\omega, x)$ to $K_{0}$. We define $\rho: \Sigma \times M \rightarrow(0, \infty)$ as

$$
\rho(\omega, x)= \begin{cases}\frac{1}{4} \ell^{-4} e^{-2 \varepsilon n(\omega, x)}\left(\prod_{k=0}^{n(\omega, x)-1}\left(\left|f_{\sigma^{k}(\omega)}\right|_{C^{1}}\right)^{-1}\right) & (\omega, x) \in K_{0} \\ \ell^{-1} & (\omega, x) \notin K_{0}\end{cases}
$$

Consider $(\omega, x) \in K_{0}$ and $y \in M$ with $d(x, y)<\rho(\omega, x)$. Let $n=n(\omega, x)$, and for $0 \leq j \leq n$ write $x_{j}=f_{\omega}^{j}(x)$ and $y_{j}=f_{\omega}^{j}(y)$. For all $0 \leq j \leq n$ we have $d\left(x_{j}, y_{j}\right) \leq \frac{1}{4} \ell^{-4} e^{-2 \varepsilon n}$, and hence $y_{j}$ is in the domain of $\phi_{j}(\omega, x)$; it follows that for $0 \leq j \leq n-1$ we have that $\phi_{j}(\omega, x)\left(y_{j}\right)$ is in the domain of $\tilde{f}_{j}(\omega, x)$. We claim

$$
\begin{equation*}
d_{\mathbb{R}^{d}}\left(\phi_{0}(\omega, x)(y), \widetilde{W}_{0}^{s}(\omega, x)\right) \leq 2 e^{-n(\kappa-3 \varepsilon)} d_{\mathbb{R}^{d}}\left(\phi_{n}(\omega, x)\left(y_{n}\right), \widetilde{W}_{n}^{s}(\omega, x)\right) . \tag{11.8}
\end{equation*}
$$

Indeed write $\left(u_{j}, v_{j}\right)=\phi_{j}(\omega, x)\left(y_{j}\right)$. By Claim 11.5 and the fact that $\widetilde{W}_{n}^{s}(\omega, x)$ is a graph of the 1-Lipschitz function we have

$$
\begin{aligned}
2 d_{\mathbb{R}^{d}}\left(\phi_{n}(\omega, x)\left(y_{n}\right), \widetilde{W}_{n}^{s}(\omega, x)\right) & \geq\left|v_{n}-h_{n}(\omega, x)\left(u_{n}\right)\right|_{2} \\
& \geq e^{n(\kappa-3 \varepsilon)}\left|v_{0}-h_{0}(\omega, x)\left(u_{0}\right)\right|_{2} \\
& \geq e^{n(\kappa-3 \varepsilon)} d_{\mathbb{R}^{d}}\left(\phi_{0}(\omega, x)(y), \widetilde{W}_{0}^{s}(\omega, x)\right) .
\end{aligned}
$$

We now consider the transition between the charts $\phi_{n}(\omega, x)$ and $\phi_{0}\left(\hat{F}^{n}(\omega, x)\right)$. Recall $n=n(\omega, x)$ and write $\hat{x}=x_{n}, \hat{y}=y_{n}$, and $\hat{\omega}=\sigma^{n}(\omega)$. Recall that $(\hat{\omega}, \hat{x}) \in K_{0}$. As $d(\hat{x}, \hat{y}) \leq \frac{1}{4} \ell^{-4} e^{-2 \varepsilon n}$ we have that $\hat{y}$ is in the domain of $\phi_{0}(\hat{\omega}, \hat{x})$. Furthermore, as $\left|\phi_{0}(\hat{\omega}, \hat{x})(\hat{y})\right| \leq \ell^{-3} \leq 0.01 \ell^{-1}$, we can find $z \in W_{\hat{\omega}}^{s}(\hat{x})$ such that

$$
d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \widetilde{W}_{0}^{s}(\hat{\omega}, \hat{x})\right)=d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \phi_{0}(\hat{\omega}, \hat{x})(z)\right) .
$$

Let $\phi_{0}(\hat{\omega}, \hat{x})(z)=(u, v)=\left(u, h_{0}(\hat{\omega}, \hat{x})(u)\right)$. As $h_{0}(\hat{\omega}, \hat{x})$ has a Lipschitz constant less than 1 , for $t \in[0,1]$ we have

$$
\left|\left(t u, h_{0}(\hat{\omega}, \hat{x})(t u)\right)\right|=\left|\left(t u, h_{0}(\hat{\omega}, \hat{x})(t u)\right)\right|_{1} \leq|u|_{1}=|(u, v)| .
$$

Then for any $0 \leq t \leq 1$, writing

$$
z(t)=\phi_{0}(\hat{\omega}, \hat{x})^{-1}\left(t u, h_{0}(\hat{\omega}, \hat{x})(u)\right)
$$

we have

$$
\begin{aligned}
d(\hat{x}, z(t)) & \leq \ell\left|\phi_{0}(\hat{\omega}, \hat{x})(z)\right| \\
& \leq \ell\left(d_{\mathbb{R}^{d}}\left(0, \phi_{0}(\hat{\omega}, \hat{x})(\hat{y})\right)+d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \phi_{0}(\hat{\omega}, \hat{x})(z)\right)\right) \\
& \leq \ell 2\left|\phi_{0}(\hat{\omega}, \hat{x})(\hat{y})\right| \\
& \leq 2 \ell^{2} d(\hat{x}, \hat{y}) \\
& \leq \frac{1}{2} \ell^{-2} e^{-2 \varepsilon n} .
\end{aligned}
$$

Thus $z(t)$ is in the domain of $\phi_{n}(\omega, x)$ for all $0 \leq t \leq 1$ whence $\phi_{n}(\omega, x)(z(t)) \in$ $\widetilde{W}_{n}^{s}(\omega, x)$ for all $0 \leq t \leq 1$. It follows that

$$
\begin{aligned}
d_{\mathbb{R}^{d}}\left(\phi_{n}(\omega, x)(\hat{y}), \widetilde{W}_{n}^{s}(\omega, x)\right) & \leq d_{\mathbb{R}^{d}}\left(\phi_{n}(\omega, x)(\hat{y}), \phi_{n}(\omega, x)(z)\right) \\
& \leq \ell e^{n \varepsilon} d(\hat{y}, z) \\
& \leq \ell^{2} e^{n \varepsilon} d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \widetilde{W}_{0}^{s}(\hat{\omega}, \hat{x})\right) .
\end{aligned}
$$

Combining the above with (11.8) we have

$$
\begin{align*}
d_{\mathbb{R}^{d}} & \left(\phi_{0}(\omega, x)(y), \widetilde{W}_{0}^{s}(\omega, x)\right) \\
& \leq 2 e^{-n(\kappa-3 \varepsilon)} \ell^{2} e^{n \varepsilon} d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \widetilde{W}_{0}^{s}(\hat{\omega}, \hat{x})\right) \\
& \leq \chi d_{\mathbb{R}^{d}}\left(\phi_{0}(\hat{\omega}, \hat{x})(\hat{y}), \widetilde{W}_{0}^{s}(\hat{\omega}, \hat{x})\right) . \tag{11.9}
\end{align*}
$$

Now let $n_{j}$ denote the $\left(j m_{0}\right)$ th return of ( $\omega, x$ ) to $K_{0}$. Suppose for some $k$ that $d\left(f_{\omega}^{n_{j}}(x), f_{\omega}^{n_{j}}(y)\right) \leq \rho\left(\hat{F}^{n_{j}}(\omega, x)\right)$ for all $0 \leq j \leq k$. By induction on (11.9) we have that

$$
d_{\mathbb{R}^{d}}\left(\phi_{0}(\omega, x)(y), \widetilde{W}_{0}^{s}(\omega, x)\right) \leq \chi^{k} d_{\mathbb{R}^{d}}\left(\phi_{0}\left(\hat{F}^{n_{k}}(\omega, x)\right)\left(f_{\omega}^{n_{k}}(y)\right), \widetilde{W}_{0}^{s}\left(\hat{F}^{n_{k}}(\omega, x)\right)\right) .
$$

This establishes the following claim.
Claim 11.6. Let $(\omega, x) \in K_{0}$, and let $y \in M$ be such that $d\left(f_{\omega}^{n}(x), f_{\omega}^{n}(y)\right) \leq$ $\rho\left(\hat{F}^{n}(\omega, x)\right)$ for all $n \geq 0$. Then $y \in W_{\omega}^{s}(x)$.
11.2.4. Construction of the partition $\alpha$. Recall the integrability hypothesis (困). As $\omega \mapsto \log ^{+}\left|f_{\omega}\right|_{C^{2}}$ is integrable, it follows that

$$
\left.\int \mid \log \rho(\omega, x)\right) \mid d\left(\hat{\nu}^{\mathbb{Z}} \times \hat{\mu}\right)(\omega, x)<\infty .
$$

We adapt Mañ, Lemma 2] to our $\rho$ to produce a finite entropy partition $\hat{\alpha}$ of $\Sigma_{+} \times M$ such that $\operatorname{diam}\left(\hat{\alpha}(\omega, x) \cap M_{\omega}\right) \leq \rho(\omega, x)$ for almost every $(\omega, x)$. The only modification needed in the proof of Mañ, Lemma 2] is to replace, for each $r$, the family $\mathscr{P}_{r}$ at the top of page 97 with the partition $\overline{\mathscr{P}}_{r}=\left\{\Sigma \times P \mid P \in \mathscr{P}_{r}\right\}$.

Take $\alpha$ to be the preimage of $\hat{\alpha}$ under the natural projection $\bar{\pi}_{+}: \Sigma \times M \rightarrow$ $\Sigma_{+} \times M$. Then clearly $\alpha \leq \beta$. Furthermore, if $(\eta, y) \in \mathcal{Q} \vee \alpha^{-}(\xi, x)$, then
(1) there is an $\omega \in \Sigma_{+}$with $\bar{\pi}_{+}(\eta, y)=(\omega, y)$ and $\bar{\pi}_{+}(\xi, x)=(\omega, x)$, and
(2) $\hat{F}^{n}(\omega, y) \in \hat{\alpha}\left(\hat{F}^{n}(\omega, x)\right)$ for all $n \geq 0$.

If $(\omega, x) \in K_{0}$, then, by Claim 11.6 $y \in W_{\omega}^{s}(x)$. If $(\omega, x) \notin K_{0}$, then take $n$ so that $\hat{F}^{n}(\omega, x) \in K_{0}$. Then $f_{\omega}^{n}(y) \in W_{\sigma^{n}(\omega)}^{s}\left(f_{\omega}^{n}(x)\right)$ whence $y \in W_{\omega}^{s}(x)$. This completes the proof of Lemma 11.2

## 12. Proof of Theorem 4.8

We continue to work in the case $\Omega=\Sigma$. As remarked earlier, the $\mathcal{F}$-measurability of $(\xi, x) \mapsto E_{\xi}^{s}(s)$ holds trivially if all exponents of $\hat{\mu}$ are negative and the $\hat{\nu}$-a.s. invariance of $\hat{\mu}$ follows from the invariance principle of AV if all exponents of $\hat{\mu}$ are positive. We thus assume $\hat{\mu}$ has two exponents, one of each sign $\lambda^{s}<0<\lambda^{u}$. Moreover, assume that the $\operatorname{map}(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable.

As above, let $\mathcal{P}$ be a measurable partition of $\Sigma \times M$ satisfying (11.1). We show that if $\mu_{(\xi, x)}^{\mathcal{P}}$ is not supported on a set of the form $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$, then the measures $\mu_{\xi}$ are non-atomic. From this contradiction and Proposition 11.1 the finiteness and $\hat{\nu}$-a.s. invariance of $\hat{\mu}$ follows. The non-atomicity of the measures $\mu_{\xi}$ is established, under the above hypotheses, through a procedure similar to the proof of Proposition 7.1.

We introduce one piece of new notation in the specific case $\Omega=\Sigma$.
Definition 12.1. Given $\xi=\left(\ldots, \xi_{-1}, \xi_{0}, \xi_{1}, \xi_{2}, \ldots\right)$ and $\eta=\left(\ldots, \eta_{-1}, \eta_{0}, \eta_{1}, \eta_{2}, \ldots\right)$ in $\Sigma$ define

$$
[\xi, \eta]:=\left(\ldots, \xi_{-2}, \xi_{-1}, \eta_{0}, \eta_{1}, \eta_{2}, \ldots\right)
$$

Recall that in Section 9.1 we replaced the $\sigma$-algebra of $\Sigma_{\text {loc }}^{+}$-saturated sets with its preimage under $\sigma$. Let

$$
\Sigma_{\mathrm{loc},-1}^{+}(\xi)=\sigma^{-1}\left(\Sigma_{\mathrm{loc}}^{+}(\sigma(\xi))\right)=\left\{\eta \in \Sigma: \eta_{j}=\xi_{j} \text { for all } j \leq 0\right\}
$$

Then $\hat{\mathcal{F}}$ as modified in Section 9.1 is the sub- $\sigma$-algebra of $\Sigma_{\text {loc },-1}^{+}$-saturated sets.
The proof of Theorem 4.8 is a simplified version of the proof of Theorem 4.10 except our initial points $p$ and $q$ remain fixed and, as $p$ and $q$ are in the same total stable space, we use only positive times. In particular, the open set $U$, the choice of $M_{\delta}, m_{\delta}$, and the estimates in Section 10.2 are not used here.

Proof of Theorem 4.8. We assume in the setting of Theorem 4.8 that the map $(\xi, x) \mapsto E_{\xi}^{s}(x)$ is not $\mathcal{F}$-measurable.

We recall all constructions and notations from Sections 9 and 10 in the case that $\Omega=\Sigma$. In particular, we retain the notation $Y=(\mathbb{R} \times \Sigma \times M) / \sim$ equipped with the measurable parametrization, $\Phi^{t}$ the suspension flow, $\mathscr{S}_{p}(t)$ and $\Psi^{s}$ the time change and corresponding flow, $\omega$ and $\hat{\omega}$ the $\Phi^{t}$ - and $\Psi^{s}$-invariant measures, and $\tau_{p, \delta, \varepsilon}$, and $L_{p, \delta, \varepsilon}$ the stopping times.

Recall the choice of $\kappa_{1}, \kappa_{2}$ in Section 10.1 and take $\alpha=\left(\frac{\kappa_{1}}{5\left(\kappa_{1}+\kappa_{2}\right)}\right)$. Recall the choices of various parameters

$$
M_{0}, \hat{M}, \gamma_{1}, \gamma_{2}, r_{0}, r_{1}, \hat{r}, C_{1}, C_{2}, C_{3}, D_{0}, D_{1}, L_{1}, a_{0}, \hat{L}, \hat{T}, T_{0}
$$

in Section 10.1 as well as the sets $K_{0}, S_{0}, S_{\hat{M}}, \mathscr{A}, \mathcal{R}\left(T_{0}\right)$ and the $\sigma$-algebras $\mathcal{S}, \mathcal{S}^{m}$. In this section, the constants $r_{0}, r_{1}, C_{1}, C_{3}, D_{1}$ are chosen so that Lemma 9.12]holds and we take $\mathcal{R}\left(T_{0}\right) \subset K$ where $K$ is defined below.

We assume for the sake of contradiction that the measures $\mu_{\xi}$ are finitely supported $\hat{\nu}^{\mathbb{Z}}$-a.s. but that $\hat{\mu}$ is not $\hat{\nu}$-a.s. invariant. By ergodicity, each $\mu_{\xi}$ is supported on a finite set $F(\xi) \subset M$ with the same cardinality a.s. We fix a compact $\Lambda^{\prime \prime \prime} \subset \Sigma \times M$ such that $\mu_{\xi}$ has an atom at $(\xi, x)$ for every $(\xi, x) \in \Lambda^{\prime \prime \prime}$ and

$$
\min \left\{d(x, y) \mid(\xi, x) \in \Lambda^{\prime \prime \prime}, y \in F(\xi) \backslash\{x\}\right\}
$$

is bounded below by some $\varepsilon_{1}>0$.

By choosing the above parameters so that the associated sets have sufficiently large measures, we can take the compact set

$$
K=K_{0} \cap \Lambda^{\prime \prime} \cap\left([0,1) \times \Lambda^{\prime}\right) \cap\left([0,1) \times \Omega^{\prime} \times M\right) \cap\left([0,1) \times \Lambda^{\prime \prime \prime}\right)
$$

where $K_{0}, \Lambda^{\prime}, \Lambda^{\prime \prime}, \Omega^{\prime}$ are as in Section 10.1, to be such that

$$
\omega(K)>1-\frac{\alpha}{10} \quad \text { and } \quad \hat{\omega}(K)>1-\frac{\alpha}{20 N_{0}}
$$

We have the same estimates as in Claim 10.1 (with $U=\varnothing$ ).
As we assume the measure $\hat{\mu}$ is not $\hat{\nu}$-a.s. invariant, by Proposition 11.1, it follows that the measures $\left\{\mu_{\xi}^{\mathcal{P}}\right\}$ are not supported on sets of the form $\Sigma_{\text {loc }}^{-}(\xi) \times\{x\}$ where $\mathcal{P}$ is a partition of $\Sigma \times M$ satisfying (11.1). Recall the set $\mathcal{R}\left(T_{0}\right) \subset K$ in (Q). We may find $p=(\varsigma, \xi, x)$ and $q=(\varsigma, \zeta, y)$ in $Y$ with

- $p \in \mathcal{R}\left(T_{0}\right), q \in \mathcal{R}\left(T_{0}\right)$;
- $\zeta \in \Sigma_{\text {loc }}^{-}(\xi)$;
- $y \in W_{\xi, r_{1}}^{s}(x) \backslash\{x\}$.

Fix $\delta=\left\|H_{p}^{s}(y)\right\|>0$. We may assume $\delta<\varepsilon_{1} /\left(2 C_{2} C_{3} M_{0}^{6}\right)$.
As in Claim 10.5 we have the following. Note that unlike in Claim 10.5, $\ell_{j}>0$.
Claim 12.2. The exists a sequence $\left\{\ell_{j}\right\}$ with $\ell_{j} \rightarrow \infty$ such that
(a) $\Phi^{\ell_{j}}(p) \in K \cap S_{0} \cap \mathscr{A}$;
(b) $\Phi^{\ell_{j}}(q) \in K \cap S_{0}$;
(c) $\Phi^{L_{p, \delta, \delta}\left(\ell_{j}\right)}(p) \in K \cap S_{\hat{M}}$;
(d) $\Phi^{L_{p, \delta, \delta}\left(\ell_{j}\right)}(q) \in K \cap S_{\hat{M}}$.

Proof. Let $F_{k}$ be as in Claim 10.5 (with $q=q_{j}$ ). Then, as in the proof of Claim 10.5 , for our fixed $T_{0}$ and any $T>T_{0}$ with $L_{p, \delta, \delta}(T)>T_{0}$ we have

$$
\operatorname{Leb}\left([0, T] \cap F_{1} \cap F_{2}\right) \geq(1-5 \alpha) T
$$

and, as $L_{p, \delta, \delta}(0)=\tau_{p, \delta, \delta}(0)=0$,

$$
\operatorname{Leb}\left(L_{p, \delta, \delta}([0, T]) \backslash\left(F_{3} \cap F_{4}\right)\right) \leq(4 \alpha) \operatorname{Leb}\left(L_{p, \delta, \delta}([0, T])\right) \leq 4 \alpha \kappa_{2} T
$$

whence

$$
\operatorname{Leb}\left([0, T] \backslash L_{p, \delta, \delta}^{-1}\left(F_{3} \cap F_{4}\right)\right) \leq 4 \alpha \kappa_{1}^{-1} \kappa_{2} T
$$

Then

$$
\operatorname{Leb}\left([0, T] \cap F_{1} \cap F_{2} \cap L_{p, \delta, \delta}^{-1}\left(F_{3}\right) \cap L_{p, \delta, \delta}^{-1}\left(F_{4}\right)\right)>\left(1-5 \alpha-4 \alpha \kappa_{1}^{-1} \kappa_{2}\right) T
$$

The choice of $\alpha$ guarantees $\left(1-5 \alpha-4 \alpha \kappa_{1}^{-1} \kappa_{2}\right) T \rightarrow \infty$ as $T \rightarrow \infty$.
Let $\left\{\ell_{j}\right\}$ be a sequence of times satisfying Claim12.2, As in Section 10, for each $j$ write $\hat{p}_{j}=\left(\hat{\varsigma}_{j}, \hat{\xi}_{j}, \hat{x}_{j}\right)=\Phi^{\ell_{j}}(p), \hat{q}_{j}=\left(\hat{\varsigma}_{j}, \hat{\zeta}_{j}, \hat{y}_{j}\right)=\Phi^{\ell_{j}}(q), \tilde{p}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\xi}_{j}, \tilde{x}_{j}\right)=\Phi^{L_{p, \delta, \delta}\left(\ell_{j}\right)}$ $(p), \tilde{q}_{j}=\left(\tilde{\varsigma}_{j}, \tilde{\zeta}_{j}, \tilde{y}_{j}\right)=\Phi^{L_{p, \delta, \delta}\left(\ell_{j}\right)}(p), s_{j}^{\prime}=\mathscr{S}_{\hat{p}_{j}}\left(\tau_{p, \delta, \delta}\left(\ell_{j}\right)\right), s_{j}^{\prime \prime}=\mathscr{S}_{\hat{q}_{j}}\left(\tau_{p, \delta, \delta}\left(\ell_{j}\right)\right)$.

Note that $\tau_{p, \delta, \delta}\left(\ell_{j}\right) \rightarrow \infty$ as $\ell_{j} \rightarrow \infty$. Then for $\ell_{j}$ large enough,

$$
s_{j}^{\prime \prime}, s_{j}^{\prime} \geq\left(\lambda^{u}-\epsilon_{0}\right) \tau_{p, \delta, \delta}\left(\ell_{j}\right) \geq \hat{M}
$$

and, since $\tilde{p}_{j}, \tilde{q}_{j} \in S_{\hat{M}}$, it follows that

$$
\mathbb{E}_{\hat{\omega}}\left(1_{K} \mid \mathcal{S}^{s_{j}^{\prime}}\right)\left(\tilde{p}_{j}\right)>0.9, \quad \mathbb{E}_{\hat{\omega}}\left(1_{K} \mid \mathcal{S}^{s_{j}^{\prime \prime}}\right)\left(\tilde{q}_{j}\right)>0.9
$$

As in Section 10 we have $\hat{p}_{j}, \hat{q}_{j} \in K, \omega_{\hat{p}_{j}}^{\mathcal{S}}(K)>0.9, \hat{\nu}^{\mathbb{N}}\left(A_{\gamma_{2}}\left(\hat{p}_{j}\right)\right)>0.9, \omega_{\hat{p}_{j}}^{\mathcal{S}}\left(\Psi^{-s_{j}^{\prime}}\right.$ $(K))>0.9, \omega_{\hat{q}_{j}}^{\mathcal{S}}(K)>0.9$, and $\omega_{\hat{q}_{j}}^{\mathcal{S}}\left(\Phi^{-s_{j}^{\prime \prime}}(K)\right)>0.9$.

The measures $\omega_{\hat{p}_{j}}^{\mathcal{S}}$ and $\omega_{\tilde{q}_{j}}^{\mathcal{S}}$ are, respectively, canonically identified with $\hat{\nu}^{\mathbb{N}}$ (the $\hat{\mathcal{F}}^{\text {-conditional measure) }}$ on $\Sigma_{\text {loc, }-1}^{+}\left(\hat{\xi}_{j}\right)$ and $\Sigma_{\text {loc, }-1}^{+}\left(\hat{\zeta}_{j}\right)$. Furthermore, the natural identification

$$
\Sigma_{\text {loc },-1}^{+}\left(\hat{\xi}_{j}\right) \rightarrow \Sigma_{\text {loc },-1}^{+}\left(\hat{\zeta}_{j}\right), \quad \eta \mapsto \eta^{\prime}=\left[\hat{\zeta}_{j}, \eta\right]
$$

preserves the measure $\hat{\nu}^{\mathbb{N}}$. Thus the set of $\hat{\eta}_{j} \in \Sigma_{\text {loc, }-1}^{+}\left(\hat{\xi}_{j}\right)$ such that
(1) $\hat{\eta}_{j} \in A_{\gamma_{2}}\left(\hat{p}_{j}\right)$,
(2) $\bar{p}_{j}:=\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{x}_{j}\right) \in K \cap \Psi^{-s_{j}^{\prime}}(K)$,
(3) $\bar{q}_{j}:=\left(\hat{\varsigma}_{j}, \eta_{j}^{\prime}, \hat{y}_{j}\right) \in K \cap \Psi^{-s_{j}^{\prime \prime}}(K)$,
where $\eta_{j}^{\prime}=\left[\hat{\zeta}_{j}, \hat{\eta}_{j}\right]$, has $\hat{\nu}^{\mathbb{N}}$-measure at least $1 / 2$. For each $j$, fix such a pair $\hat{\eta}_{j}$ and $\eta_{j}^{\prime}=\left[\hat{\zeta}_{j}, \hat{\eta}_{j}\right]$.

As before, write $t_{j}^{\prime}=\mathscr{S}_{\bar{p}_{j}}^{-1}\left(s_{j}^{\prime}\right), t_{j}^{\prime \prime}=\mathscr{S}_{\bar{q}_{j}}^{-1}\left(s_{j}^{\prime \prime}\right)$, and define $p_{j}^{\prime}=\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, x_{j}^{\prime}\right):=$ $\Psi^{s_{j}^{\prime}}\left(\bar{p}_{j}\right)=\Phi^{t_{j}^{\prime}}\left(\bar{p}_{j}\right) \in K, q_{j}^{\prime \prime}=\left(\varsigma_{j}^{\prime \prime}, \zeta_{j}^{\prime \prime}, y_{j}^{\prime \prime}\right):=\Psi^{s_{j}^{\prime \prime}}\left(\bar{q}_{j}\right)=\Phi^{t_{j}^{\prime \prime}}\left(\bar{q}_{j}\right) \in K$, and $q_{j}^{\prime}=$ $\left(\varsigma_{j}^{\prime}, \zeta_{j}^{\prime}, y_{j}^{\prime}\right):=\Phi^{t_{j}^{\prime}}\left(\bar{q}_{j}\right)$. For $\ell_{j}$ sufficiently large we have $d\left(\hat{x}_{j}, \hat{y}_{j}\right)<r_{0}$. For such $\ell_{j}$, as $\bar{p}_{j}, \bar{q}_{j} \in K$ let

$$
\hat{v}_{j}=W_{\eta_{j}^{\prime}, r_{1}}^{s}\left(\hat{y}_{j}\right) \cap W_{\hat{\xi}_{j}, r_{1}}^{u}\left(\hat{x}_{j}\right) .
$$

Since $\hat{y}_{j} \in W_{\hat{\xi}_{j}, r_{1}}^{s}\left(\hat{x}_{j}\right)$, by (e') of Lemma 9.12] we have

$$
\frac{1}{C_{3}}\left\|H_{\hat{p}_{j}}^{s}\left(\hat{y}_{j}\right)\right\| \leq\left\|H_{\hat{p}_{j}}^{u}\left(\hat{v}_{j}\right)\right\| \leq C_{3}\left\|H_{\hat{p}_{j}}^{s}\left(\hat{y}_{j}\right)\right\| .
$$

Exactly as in Claim 10.7 we have

$$
\frac{1}{C_{3} M_{0}^{6}} \delta \leq\left\|H_{p_{j}^{\prime}}^{u}\left(v_{j}^{\prime}\right)\right\| \leq C_{3} M_{0}^{6} \delta,
$$

where $\left(\varsigma_{j}^{\prime}, \xi_{j}^{\prime}, v_{j}^{\prime}\right)=\Phi^{t_{j}^{\prime}}\left(\hat{\varsigma}_{j}, \hat{\eta}_{j}, \hat{v}_{j}\right)$. (We take $\hat{z}_{j}=\hat{y}_{j}$ in the proof.) Hence

$$
\frac{1}{C_{2} C_{3} M_{0}^{6}} \delta \leq d\left(x_{j}^{\prime}, v_{j}^{\prime}\right) \leq C_{2} C_{3} M_{0}^{6} \delta
$$

As in Lemma 10.4(c) we have $q_{j}^{\prime}=\Phi^{\hat{t}_{j}}\left(q_{j}^{\prime \prime}\right)$ for some $\left|\hat{t}_{j}\right| \leq \hat{T}$. To adapt the proof to the current setting, we replace the estimate (10.5) with

$$
\frac{\| D \Phi^{t_{j}^{\prime \prime}}\left\lceil_{E^{u}\left(\bar{q}_{j}\right)} \|\right.}{\left.\|\left. D \Phi^{t_{j}^{\prime \prime}}\right|_{E^{u}\left(\bar{p}_{j}\right)}\right)}=\frac{\|\left. D f_{\eta_{j}^{\prime \prime}}^{n^{\prime \prime}}\right|_{T_{\hat{y}_{j}}} W_{\eta_{j}^{\prime}}^{u}\left(\hat{y}_{j}\right)}{} \|
$$

$$
\begin{equation*}
=\frac{\left\|D f_{\eta_{j}^{\prime}}^{n^{\prime \prime}} \upharpoonright_{T_{\hat{\vartheta}_{j}}} W_{\eta_{j}^{\prime}\left(\hat{y}_{j}\right)}^{u}\right\|}{\| D f_{\hat{\eta}_{j}}^{n_{j}^{\prime \prime}} \upharpoonright_{T_{\hat{v}_{j}}} W_{\hat{\eta}_{j}}^{u}\left(\hat{x}_{j}\right)} \| \frac{\left\|D f_{\xi_{j}^{\prime}}^{-n^{\prime}} \upharpoonright_{T_{x_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|}{\left\|D f_{\xi_{j}^{\prime}}^{-n^{\prime}} \upharpoonright_{T_{v_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|} \cdot \frac{\left\|D f_{\xi_{j}^{\prime}}^{-\left(n^{\prime}-n^{\prime \prime}\right)} \upharpoonright_{T_{v_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|}{\left\|D f_{\xi_{j}^{\prime}}^{-\left(n^{\prime}-n^{\prime \prime}\right)} \upharpoonright_{T_{x_{j}^{\prime}} W_{\xi_{j}^{\prime}}^{u}\left(x_{j}^{\prime}\right)}\right\|} \tag{12.1}
\end{equation*}
$$

and similarly modify (10.6). Note that the bound on the first term of (12.1) now follows from Lemma 9.12 $\left(g^{\prime}\right)$ as $\pi_{+}\left(\eta_{j}^{\prime}\right)=\pi_{+}\left(\hat{\eta}_{j}\right)$.

Consider an accumulation point $p_{0}=\left(\varsigma_{0}, \xi_{0}, x_{0}\right)$ of $\left\{p_{j}^{\prime}\right\}$ and $B \subset \mathbb{N}$ such that $\lim _{j \in B \rightarrow \infty} p_{j}^{\prime}=p_{0}$. Then the measure $\omega_{\left(\varsigma_{0}, \xi_{0}\right)}$ has an atom at $p_{0}$.

Note that, as $\xi_{j_{k}}^{\prime} \rightarrow \xi_{0}$ for some subsequence $\left\{j_{k}\right\}$, we have $\zeta_{j_{k}}^{\prime} \rightarrow \xi_{0}$. Indeed, for any $j$ and any $n \in \mathbb{N}$ with $n \leq \ell_{j}$ we have that $\hat{\xi}_{j}$ and $\hat{\zeta}_{j}$, and hence $\hat{\eta}_{j}$ and $\eta_{j}^{\prime}$, agree in the $k$ th index for all $-n \leq k \leq \infty$. As $s_{j}^{\prime}>0, \xi_{j}^{\prime}$ and $\zeta_{j}^{\prime}$ agree in the $k$ th index for all $-n \leq k \leq \infty$.

Thus, as in the proof of Lemma 10.8 passing to subsequences of $B$ there are accumulation points $q_{0}=\left(\varsigma_{0}, \xi_{0}, y_{0}\right)$ of $\left\{q_{j}^{\prime}\right\}$ and $q_{1}=\left(\varsigma_{1}, \xi_{1}, y_{1}\right) \in K$ of $\left\{q_{j}^{\prime \prime}\right\}$ and a $\hat{t} \in[-\hat{T}, \hat{T}]$ such that $\Phi^{\hat{t}}\left(q_{1}\right)=q_{0}$. As $v_{j} \in W_{\eta_{j}^{\prime}, r_{1}}^{s}\left(\hat{y}_{j}\right), \bar{q}_{j} \in K$, and $t_{j}^{\prime} \rightarrow \infty$ and as $f_{\eta_{j}^{\prime}}^{n}=f_{\eta_{j}}^{n}$ for $n \geq 0$, we have $d\left(v_{j}^{\prime}, y_{j}^{\prime}\right) \rightarrow 0$; hence $d\left(x_{0}, y_{0}\right) \geq \frac{1}{C_{2} C_{3} M_{0}^{6}} \delta$. Since $q_{1} \in K \subset[0,1) \times \Lambda^{\prime \prime \prime}$, the measure $\omega_{\left(\varsigma_{1}, \xi_{1}\right)}$ has an atom at $q_{1}$. By the invariance of $\omega$, it follows that $\omega_{\left(s_{0}, \xi_{0}\right)}$ has an atom at $q_{0}$. On the other hand, $x_{0} \neq y_{0}$ yet $d\left(x_{0}, y_{0}\right) \leq C_{2} C_{3} M_{0}^{6} \delta<\varepsilon_{1}$. As $p_{0} \in K \subset[0,1) \times \Lambda^{\prime \prime \prime}$, this contradicts the choice of $\varepsilon_{1}$.
Remark 12.3. In the above proof, we have that $q_{0} \in W_{r_{1}}^{u}\left(p_{0}\right)$. Thus one can modify the above proof to conclude that the skew product $F:(X, \mu) \rightarrow(X, \mu)$ has positive fiber-wise entropy. In this way, one can show that for any hyperbolic, $\hat{\nu}$-stationary measure $\hat{\mu}$ such that
(1) $E_{\omega}^{s}(x)$ is not non-random, and
(2) $\hat{\mu}$ is not $\hat{\nu}$-a.s. invariant
the $\hat{\mu}$ entropy $h_{\hat{\mu}}\left(\mathcal{X}^{+}(M, \hat{\nu})\right)$ is positive. Under the positive entropy hypothesis, the authors showed in an earlier version of this paper that $\hat{\mu}$ must then be SRB. However, one still needs to perform the more detailed analysis in Section 10 to rule out the existence of a $\hat{\nu}$-a.s. invariant, hyperbolic measure $\hat{\mu}$ with zero entropy and such that $E_{\omega}^{s}(x)$ is not non-random to derive the full result in Theorem 3.1.

## 13. Proofs of remaining theorems

### 13.1. Proof of Theorem 3.4.

Proof. Let $\hat{\mu}$ be as in Theorem 3.4, and assume $\hat{\mu}$ is not finitely supported and that the stable distribution $E_{\omega}^{s}(x)$ is non-random. It follows from Theorem 3.1 that $\hat{\mu}$ is SRB. Let $F: \Sigma \times M \rightarrow \Sigma \times M$ be the canonical skew product constructed in Section 4.1, and let $\mu$ be the $F$-invariant measure defined by Proposition 4.2. Then the conditional measures of $\mu$ along almost every unstable manifold $W^{u}(x, \xi)$ for the skew product $F$ are absolutely continuous. Define the ergodic basin $B \subset \Sigma \times M$ of $\mu$ to be the set of $(\xi, x) \in \Sigma \times M$ such that

$$
\lim _{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi\left(f_{\xi}^{n}(x)\right)=\int \phi d \hat{\mu}
$$

for all $\phi: M \rightarrow \mathbb{R}$ continuous. By the pointwise ergodic theorem and the separability of $C^{0}(M)$, we have $\mu(B)=1$. Furthermore, for points $(\xi, x) \in B$ whose fiber-wise stable manifold $W^{s}(\xi, x)$ is defined we have

$$
W^{s}(\xi, x) \subset B
$$

We have the following "transverse" absolute continuity property. Given a typical $\xi \in \Sigma$ and certain continuous families of fiber-wise local stable manifolds $\mathcal{S}:=$ $\left\{W_{\xi, r}^{s}(x)\right\}_{x \in Q}$, consider two manifolds $T_{1}$ and $T_{2}$ everywhere uniformly transverse to the collection $\mathcal{S}$. Define the holonomy map from $T_{1}$ to $T_{2}$ by "sliding along" elements of $\mathcal{S}$. Such holonomy maps were shown by Pesin to be absolutely continuous in the deterministic volume-preserving setting [Pes. For fiber-wise stable manifolds associated to skew products satisfying (IC), such holonomy maps are also known to be absolutely continuous. See [LY3, (4.2)] or LQ, III.5] for further details and references to proofs.

The above absolute continuity property implies that if $\hat{\mu}$ is $\operatorname{SRB}$ (whence $\mu$ is fiber-wise SRB) and if $A \subset \Sigma \times M$ is any set with $\mu(A)>0$, then for a positive measure subset of $\xi$,

$$
\bigcup_{(x) \in A \cap M_{\xi}} W_{\xi}^{s}(x) \subset M_{\xi}
$$

has positive Lebesgue measure in $M_{\xi}$. It follows that for the ergodic basin $B$,

$$
\left(\hat{\nu}^{\mathbb{Z}} \times m\right)(B)>0 .
$$

We note that if $\eta \in \Sigma_{\text {loc }}^{-}(\xi)$, then (under the natural identification of subsets of $M_{\eta}$ and $M_{\xi}$ )

$$
B \cap M_{\eta}=B \cap M_{\xi}
$$

since $f_{\xi}^{n}=f_{\eta}^{n}$ for $n \geq 0$. Define $\hat{B}$ to be the ergodic basin of $\nu^{\mathbb{N}} \times \hat{\mu}$ for the skew product $\hat{F}: \Sigma_{+} \times M$; that is, $(\omega, x) \in \hat{B}$ if

$$
\lim _{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi\left(f_{\omega}^{n}(x)\right)=\int \phi d \hat{\mu}
$$

for all $\phi: M \rightarrow \mathbb{R}$ continuous. We have that $\hat{B}$ is the image of $B$ under the natural projection $\Sigma \times M \rightarrow \Sigma_{+} \times M$ whence $\left(\hat{\nu}^{\mathbb{N}} \times m\right)(\hat{B})>0$.

Define a measure

$$
\hat{m}=\frac{1}{\left(\hat{\nu}^{\mathbb{N}} \times m\right)(\hat{B})}\left(\hat{\nu}^{\mathbb{N}} \times m\right) \Gamma_{\hat{B}}
$$

on $\Sigma_{+} \times M$. Since both the set $\hat{B}$ and the measure $\hat{\nu}^{\mathbb{N}} \times m$ are $\hat{F}$-invariant (recall that $m$ is $\hat{\nu}$-a.s. invariant), the measure $\hat{m}$ is $\hat{F}$-invariant. Furthermore, for $\hat{m}$-a.e. $(\omega, x)$ and any continuous $\phi: M \rightarrow \mathbb{R}$, the Birkhoff sums satisfy

$$
\lim _{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi\left(f_{\omega}^{n}(x)\right)=\int \phi d \hat{\mu}
$$

which implies that $\hat{m}$ is ergodic for $F$ and, in particular, is an ergodic component of $\hat{\nu}^{\mathbb{Z}} \times m$. This implies (see e.g. [Kif, Proposition I.2.1]) that $\hat{m}$ is of the form $\hat{m}=\hat{\nu}^{\mathbb{Z}} \times m_{0}$ for $m_{0}$ an ergodic component of $m$ for $\mathcal{X}^{+}(M, \nu)$.

Then, for any continuous function $\phi: M \rightarrow \mathbb{R}, \hat{\nu}^{\mathbb{N}}$-a.e. $\omega \in \Sigma_{+}$, and $m_{0}$-a.e. $x \in M$, we have

$$
\lim _{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi\left(f_{\omega}^{n}(x)\right)=\int \phi d \hat{\mu}
$$

Furthermore, since $\hat{\nu} \times m_{0}$ is invariant and ergodic for $\hat{F}$, for $\hat{\nu}^{\mathbb{N}}$-a.e. $\omega \in \Sigma_{+}$and $m_{0}$-a.e. $x \in M$ we also have that

$$
\lim _{n \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \phi\left(f_{\omega}^{n}(x)\right)=\int \phi d m_{0}
$$

In particular, $\int \phi d \hat{\mu}=\int \phi d m_{0}$ for all continuous $\phi: M \rightarrow \mathbb{R}$, whence $\hat{\mu}=m_{0}$.
13.2. Proof of Theorem 5.1, Let $M$ be a compact surface, and let $\mu$ be a nonatomic Borel probability on $M$. Let $f \in \operatorname{Diff}_{\mu}^{2}(M)$ as in Theorem 5.1] In particular, $f$ is ergodic, hyperbolic, and, as $\mu$ has no atoms, $f$ has one positive and one negative exponent which we denote by $\lambda_{f}^{s}<0<\lambda_{f}^{u}$.
13.2.1. Preliminary constructions and observation. Let $K \subset \operatorname{Diff}_{\mu}^{2}(M)$ be a fixed compact subset with $f \in K$. Moreover, assume that $K$ is symmetric in that if $g \in K$, then $g^{-1} \in K$. For this section set

$$
\Sigma:=\Sigma_{K}=K^{\mathbb{Z}}
$$

Let $\sigma: \Sigma \rightarrow \Sigma$ be the left shift, $F: \Sigma \times M \rightarrow \Sigma \times M$ the canonical invertible skew products, and $D F: \Sigma \times T M \rightarrow \Sigma \times T M$ the fiber-wise derivative. With $X=\Sigma \times M$, we observe that $F$ and $D F$ are continuous transformations of $X$ and $T X$. In what follows, we will study the fiber-wise exponents of the cocycle $D F$ as the measures on $\Sigma$ change. We rely on tools developed in the study of continuity properties of Lyapunov exponents appearing in many sources including [BBB, BGMV, BNV, Via].

Write $\mathcal{M}(K)$ for the space of all Borel probability measures on $K$. Given $\nu \in \mathcal{M}(K)$, equip $\Sigma$ with the shift-invariant measure $\nu^{\mathbb{Z}}$. For any $\nu \in \mathcal{M}(K)$, we have that $\mu$ is $\nu$-stationary. Moreover, as $\mu$ is preserved by every element of $K$, the measure $\nu^{\mathbb{Z}} \times \mu$ is $F$-invariant and coincides with the measure given by Proposition 4.2 We will say that $\mu$ is ergodic for $\nu$ if it is ergodic as a $\nu$-stationary measure.

We make some preliminary observations.
Claim 13.1. Let $\nu \in \mathcal{M}(K)$ with $\nu(f)>0$. Then $\mu$ is ergodic for $\nu$.
Proof. Suppose $\mu=\mu_{1}+\mu_{2}$ where $\mu_{i}$ are non-trivial, $\nu$-stationary, mutually singular measures. Then

$$
\mu_{1}=\int_{g \neq f} g_{*} \mu_{1} d \nu(g)+\nu(f) f_{*} \mu_{1}
$$

By the $f$-ergodicity of $\mu, f_{*} \mu_{1}$ is not mutually singular with respect to $\mu_{2}$. This contradicts that $\mu_{1}$ is mutually singular with respect to $\mu_{2}$.

For $\nu \in \mathcal{M}(K)$, we recall the definition of Lyapunov exponents guaranteed by Proposition2.1 for the stationary measure $\mu$. We recall that the exponent is $\nu^{\mathbb{N}}$-a.s. independent of choice of word. We write

$$
\lambda_{\nu}^{1}(x) \leq \lambda_{\nu}^{2}(x)
$$

for the Lyapunov exponents of $\mu$ for words defined by $\nu$ at the point $x$ with the convention that if $\mu$ has only one exponent at $x$ we declare $\lambda_{\nu}^{1}(x)=\lambda_{\nu}^{2}(x)$. Given $\nu$ and $x$ with $\lambda_{\nu}^{1}(x) \neq \lambda_{\nu}^{2}(x)$ and $\xi \in \Sigma=K^{\mathbb{Z}}$ write

$$
\begin{equation*}
T_{x} M=E_{\xi}^{1}(x) \oplus E_{\xi}^{2}(x) \tag{13.1}
\end{equation*}
$$

for the associated Lyapunov splitting. If $\lambda_{\nu}^{1}(x)=\lambda_{\nu}^{2}(x)$, then we write $E_{\xi}^{1}(x)=$ $E_{\xi}^{2}(x)=T_{x} M$.

Consider an involution on $\mathcal{M}(K)$ defined as follows. For $g \in K$ define $\theta(g):=$ $g^{-1}$. For $\nu \in \mathcal{M}(K), \theta_{*} \nu$ is the measure

$$
\begin{equation*}
\theta_{*} \nu(A):=\nu(\theta(A)) \tag{13.2}
\end{equation*}
$$

for $A \subset K$. We have that $\theta_{*}: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$ is involutive.
Lemma 13.2. For $\nu \in \mathcal{M}(K)$ and $\mu$-a.e. $x$ we have

$$
\lambda_{\nu}^{1}(x)=-\lambda_{\theta(\nu)}^{2}(x) .
$$

Proof. On $\Sigma:=K^{\mathbb{Z}}$, define the involution $\Psi: \Sigma \rightarrow \Sigma$ given by

$$
\Psi:\left(\ldots, g_{-2}, g_{-1} \cdot g_{0}, g_{1}, \ldots\right) \mapsto\left(\ldots, g_{1}^{-1}, g_{0}^{-1} \cdot g_{-1}^{-1}, g_{2}^{-1}, \ldots\right)
$$

We have $\Psi_{*}\left(\nu^{\mathbb{Z}}\right)=\left(\theta_{*} \nu\right)^{\mathbb{Z}}$.
Consider a $\mu$-generic $x$ and $\nu^{\mathbb{Z}}$-generic $\xi$. Then

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \log \left\|D f_{\Psi(\xi)}^{n} \upharpoonright_{E_{\xi}^{1}(x)}\right\|=\lim _{n \rightarrow \infty} \frac{1}{n} \log \left\|D f_{\xi}^{-n} \upharpoonright_{E_{\xi}^{1}(x)}\right\|=-\lambda_{\xi}^{1}(x) .
$$

Similarly

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \log \left\|D f_{\Psi(\xi)}^{n} \upharpoonright_{E_{\xi}^{2}(x)}\right\|=-\lambda_{\xi}^{2}(x) .
$$

Since $\Psi$ takes $\nu^{\mathbb{Z}}$-generic words to $\left(\theta_{*} \nu\right)^{\mathbb{Z}}$-generic words, this completes the proof.

Consider $\nu \in \mathcal{M}(K)$. We remark that if the fiber-wise exponents were both positive or both negative, the measure $\mu$ would necessarily by atomic. Hence for $\nu \in \mathcal{M}(K)$ we have $\lambda_{\nu}^{1}(x) \leq 0 \leq \lambda_{\nu}^{2}(x)$.
13.2.2. Invariant measures for the projectivized cocycle. With $X:=\Sigma \times M$ and $T X=\Sigma \times T M$, let $\mathbb{P} T X$ denote the projectivized tangent bundle

$$
\mathbb{P} T X:=\Sigma \times \mathbb{P} T M
$$

Given $(\xi, x, v)$ in $T X$ with $v \neq 0$, write $(\xi, x,[v])$ for the class in $\mathbb{P} T X$. We write $\mathbb{P} D F: \mathbb{P} T X \rightarrow \mathbb{P} T X$ to denote the action induced by $D F$ on $\mathbb{P} T X$.

For a fixed $\nu \in \mathcal{M}(K)$ let $\eta$ be a $\mathbb{P} D F$-invariant, Borel probability measure on $\mathbb{P} T X$ which projects to $\nu^{\mathbb{Z}} \times \mu$ under the natural projection $\mathbb{P} T X \rightarrow X$. Given such an $\eta$ write $\left\{\eta_{\xi}\right\}$ for the family of conditional measures induced by the projection $\mathbb{P} T X \rightarrow \Sigma$.

Given $\zeta, \xi \in \Sigma$ we have a natural identification of $\{\xi\} \times \mathbb{P} T M$ and $\{\zeta\} \times \mathbb{P} T M$, and we view $\xi \mapsto \eta_{\xi}$ as a measurable map from $X$ to the space of measures on PTM.

Recall we have two natural partitions of $\Sigma$ : the partitions into local stable and unstable sets $\left\{\Sigma_{\text {loc }}^{-}\right\}$and $\left\{\Sigma_{\text {loc }}^{+}\right\}$. The conditional measures on $\Sigma$ induced by either partition is naturally identified with $\nu^{\mathbb{N}}$. Recall (cf. Section 4.3 and ignore the modification in Section 9.1) that we write $\hat{\mathcal{F}}$ for the $\sigma$-algebra of local unstable sets on $\Sigma$. We will say $\eta$ is a $u$-measure if $\xi \mapsto \eta_{\xi}$ is $\hat{\mathcal{F}}$-measurable. Alternatively $\eta$ is a $u$-measure if for $\nu^{\mathbb{Z}}$-a.e. $\xi \in \Sigma$ and $\nu^{\mathbb{N}}$-a.e. $\zeta \in \Sigma_{\text {loc }}^{+}(\xi)$, we have

$$
\eta_{\xi}=\eta_{\zeta}
$$

We similarly define s-measures and define $\eta$ to be an su-measure if it is simultaneously an $s$ - and $u$-measure. We remark that $u$-measures correspond to $\nu$-stationary measures on $\mathbb{P} T M$ projecting to $\mu$. (See Via, Chapter 5] for more details.)

If $\eta$ is an $s u$-measure, then the level sets of the map $\zeta \mapsto \eta_{\zeta}$ are essentially saturated by local stable and local unstable sets. Since the measure $\nu^{\mathbb{Z}}$ has product structure, if $\eta$ is an su-measure, the assignment $\zeta \mapsto \eta_{\zeta}$ is $\nu^{\mathbb{Z}}$-a.s. constant. In particular, if $\nu$ is an $s u$-measure, there is a measure $\eta_{0}$ on $\mathbb{P} T M$ projecting to $\mu$ on $M$ with $\eta=\nu^{\mathbb{Z}} \times \eta_{0}$. If $\eta$ is assumed $\mathbb{P} D F$-invariant it follows that $\eta_{0}$ is $\nu$-a.s. invariant.

Claim 13.3. Let $\nu_{j} \in \mathcal{M}(K)$ converge to $\nu$ in the weak-* topology. Let $\eta_{j}$ be a sequence of $\mathbb{P} D F$-invariant $u$-measures, projecting to $\left(\nu_{j}\right)^{\mathbb{Z}} \times \mu$. Then the set of weak-* accumulation points of $\left\{\eta_{j}\right\}$ is non-empty and consists of $\mathbb{P} D F$-invariant $u$-measures projecting to $\nu^{\mathbb{Z}} \times \mu$.

The above claim follows for instance from [Via, Proposition 5.18].
We note that there always exist $\mathbb{P} D F$-invariant $s$ - and $u$-measures. However, the existence of $\mathbb{P} D F$-invariant su-measures is unexpected, absent the existence of a $\nu$-a.s. invariant subbundle $V \subset T M$. However, there is a dynamical situation where every $\mathbb{P} D F$-invariant measure is an su-measure.

Proposition 13.4 (Led2, AV]). Suppose $\nu \in \mathcal{M}(K)$ is such that $\lambda_{\nu}^{1}(x)=0=$ $\lambda_{\nu}^{2}(x)$ for $\mu$-a.e. $x$. Then any $\mathbb{P} D F$-invariant measure $\eta$ for the projectivized cocycle $\mathbb{P D F}: \mathbb{P} T X \rightarrow \mathbb{P} T X$ projecting to $\nu^{\mathbb{Z}} \times \mu$ is an su-measure.

In what follows, we will primarily focus on measures $\nu$ such that $\mu$ is ergodic and has two distinct Lyapunov exponents $\lambda_{\nu}^{1}<\lambda_{\nu}^{2}$ for $D F$. In this case we have two canonical measures $\eta_{\nu}^{1}$ and $\eta_{\nu}^{2}$ given by

$$
\begin{equation*}
d \eta_{\nu}^{j}(\xi, x,[v]):=d \delta_{E_{\xi}^{j}(x)}([v]) d \mu(x) d \nu^{\mathbb{Z}}(\xi) \tag{13.3}
\end{equation*}
$$

where $E_{\xi}^{j}(x)$ is the associated subspace of the Lyapunov splitting (13.1). By the $D F$-invariance of the distributions $E_{\xi}^{j}(x)$, we have that the measures $\eta_{\nu}^{j}$ are $\mathbb{P} D F$ invariant. Furthermore, it follows from Proposition 4.5 that $\eta_{\nu}^{2}$ is a $u$-measure and $\eta_{\nu}^{1}$ is an $s$-measure.

In the above setting, the measures defined by (13.3) are the only ergodic $\mathbb{P} D F$ invariant measures on $\mathbb{P} T X$ projecting to $\nu^{\mathbb{Z}} \times \mu$. Indeed, see the following claim.

Claim 13.5. Let $\nu \in \mathcal{M}(K)$ be such that $\mu$ is ergodic for $\nu$ and has two distinct Lyapunov exponents. Then any $\mathbb{P} D F$-invariant probability measure $\eta$ projecting to $\nu^{\mathbb{Z}} \times \mu$ is of the form

$$
\eta=a \eta_{\nu}^{1}+(1-a) \eta_{\nu}^{2}
$$

for some $a \in[0,1]$.
Proof. For $(\xi, x, v) \in T X \backslash E^{1}(\xi, x)$ write $v=v^{1}+v^{2}$ with $v^{j} \in E_{\xi}^{j}(x)$ and define $\psi(\xi, x, v) \in[0, \infty)$ by

$$
\psi(\xi, x, v)=\frac{\left\|v^{1}\right\|}{\left\|v^{2}\right\|}
$$

As $\psi(\xi, x, t v)=\psi(\xi, x, v), \psi$ descends to a function

$$
\psi: \mathbb{P} T X \backslash E^{1}(\xi, x) \rightarrow[0, \infty)
$$

For $(\xi, x, v) \in T X \backslash E^{1}(\xi, x)$, we have that

$$
D F^{n}(\xi, x, v)=\left(\sigma^{n}(\xi), f_{\xi}^{n}(x), D_{x} f_{\xi}^{n} v^{1}+D_{x} f_{\xi}^{n} v^{2}\right),
$$

and hence for any sufficiently small $\varepsilon>0$ and $\nu^{\mathbb{Z}} \times \mu$-a.e. $(\xi, x)$ there is a $c$ with

$$
\psi\left(\mathbb{P} D F^{n}(\xi, x,[v])\right) \leq c \exp \left(n\left(\lambda_{\nu}^{1}-\lambda_{\nu}^{2}+2 \varepsilon\right)\right) \psi(\xi, x,[v])
$$

In particular, for almost every $(\xi, x,[v]) \in \mathbb{P} T X \backslash E^{1}(\xi, x)$ we have

$$
\psi\left(\mathbb{P} D F^{n}(\xi, x,[v])\right) \rightarrow 0
$$

as $n \rightarrow \infty$. By Poincaré recurrence, we conclude that $\eta\left(\psi^{-1}(0, \infty)\right)=0$.

In the context of Theorem 5.1 we have the following characterization of sumeasures.

Lemma 13.6. Let $\nu \in \mathcal{M}(K)$ be such that $\nu(f)>0$. Then there exists a $\mathbb{P} D F$ invariant, su-measure projecting to $\nu^{\mathbb{Z}} \times \mu$ if and only if one of the subbundles $\left\{E_{f}^{u}, E_{f}^{s}\right\}$ or their union $E_{f}^{u} \cup E_{f}^{s}$ is $\nu$-a.s. invariant.
Proof. The only if case is clear. Indeed, if $E_{f}^{u}$ is $\nu$-a.s. invariant, then $\eta$ defined by $d \eta=d \delta_{E_{f}^{u}} d \mu d \nu^{\mathbb{Z}}$ is an $s u$-measure. If the union $E_{f}^{u} \cup E_{f}^{s}$ is $\nu$-a.s. invariant, we may take

$$
d \eta=\frac{1}{2} d \delta_{E_{f}^{u}} d \mu d \nu^{\mathbb{Z}}+\frac{1}{2} d \delta_{E_{f}^{s}} d \mu d \nu^{\mathbb{Z}}
$$

To prove the converse, suppose $\eta$ is a $\mathbb{P} D F$-invariant, su-measure on $\mathbb{P} T X$. As remarked above, there is a measure $\eta_{0}$ on $\mathbb{P} T M$, projecting to $\mu$, such that $\eta=$ $\nu^{\mathbb{Z}} \times \eta_{0}$. Furthermore, such $\eta_{0}$ is $D g$-invariant for $\nu$-a.e. $g$. Since $\nu(f)>0$, we have $D f_{*}\left(\eta_{0}\right)=\eta_{0}$. However, by the hyperbolicity of $f$ and arguments analogous to the proof of Claim 13.5, the only such measures are supported on $E_{f}^{u} \cup E_{f}^{s}$.
13.2.3. Characterization of discontinuity of exponents. Let $\mathcal{M}_{\operatorname{erg}}(K) \subset \mathcal{M}(K)$ be the set of $\nu$ such that $\mu$ is ergodic for $\nu$. Then for $\nu \in \mathcal{M}_{\text {erg }}(K)$ the Lyapunov exponents $\lambda_{\nu}^{1} \leq \lambda_{\nu}^{2}$ are independent of $x$. We study the continuity properties of the maps

$$
\lambda_{(\cdot)}^{j}: \mathcal{M}_{\text {erg }} \rightarrow \mathbb{R}
$$

as $\nu$ varies in $\mathcal{M}_{\text {erg }}(K)$ with the weak-* topology. The arguments here are well known. (See, for example, BNV,Via and references therein.)

Proposition 13.7. Let $\nu \in \mathcal{M}_{\operatorname{erg}}(K)$ be a point of discontinuity for one of $\lambda_{(\cdot)}^{1}$, $\lambda_{(\cdot)}^{2}$. Then
(1) $\lambda_{\nu}^{1}<\lambda_{\nu}^{2}$, and
(2) there exists a $\mathbb{P} D F$-invariant su-measure $\eta$ projecting to $\nu^{\mathbb{Z}} \times \mu$.

Proof. We first consider the case where $\lambda_{\nu}^{1}=\lambda_{\nu}^{2}$. Recall then that $\lambda_{\nu}^{1}=\lambda_{\nu}^{2}=0$. Suppose $\lambda_{(\cdot)}^{1}$ is discontinuous at $\nu \in M_{\text {erg }}(K)$. Then there is some $\varepsilon>0$ and a sequence $\nu_{j} \rightarrow \nu$ in $\mathcal{M}_{\operatorname{erg}}(K)$ with $\lambda_{\nu_{j}}^{1}<-\varepsilon<0$ for every $j$. For such $j$, we have two distinct exponents $\lambda_{\nu_{j}}^{1}<0 \leq \lambda_{\nu_{j}}^{2}$. By the pointwise ergodic theorem we have

$$
\lambda_{\nu_{j}}^{1}=\int \log \left\|D_{x} f_{\xi} \upharpoonright_{E_{\xi}^{1}(x)}\right\| d \mu(x) d \nu^{\mathbb{Z}}(\xi)=\int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{\nu_{j}}^{1}(\xi, x,[v]),
$$

where $\eta_{\nu_{j}}^{1}$ are as defined in (13.3). Let $\eta_{0}$ be an accumulation point of $\left\{\eta_{\nu_{j}}^{1}\right\}$. Passing to subsequences assume $\eta_{\nu_{j}}^{1} \rightarrow \eta_{0}$. Since each $\eta_{\nu_{j}}$ is $\mathbb{P} D F$-invariant, it follows that $\eta_{0}$ is $\mathbb{P} D F$-invariant.

Note that $(\xi, x, E) \mapsto\left\|D_{x} f_{\xi} \upharpoonright_{E}\right\|$ is a continuous function on $\mathbb{P} T X$. By weak-* convergence we have
(13.4) $-\varepsilon \geq \lim _{j \rightarrow \infty} \int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{\nu_{j}}^{1}(\xi, x,[v])=\int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{0}(\xi, x,[v])$.

From the pointwise ergodic theorem, (13.4) implies that for $\left(\nu^{\mathbb{Z}} \times \mu\right)$-a.e. $(\xi, x) \in$ $\Sigma \times M$ there is a $v \in T_{x} M$ with

$$
\lim _{n \rightarrow \infty} \frac{1}{n}\left\|D f_{\xi}^{n}(v)\right\|<-\varepsilon
$$

contradicting that $\lambda_{\nu}^{1}=0$. This shows that if $\lambda_{\nu}^{1}=0$, then $\lambda_{(\cdot)}^{1}$ is continuous at $\nu$. Similarly, $\lambda_{(\cdot)}^{2}$ is continuous at $\nu$ if $\lambda_{\nu}^{2}=0$.

We now assume that $\lambda_{\nu}^{1}<\lambda_{\nu}^{2}$. Suppose again that $\lambda_{(\cdot)}^{1}$ is discontinuous at $\nu$. Then there is a convergent sequence $\nu_{j} \rightarrow \nu$ in $\mathcal{M}_{\text {erg }}(K)$ with

$$
\lim _{j \rightarrow \infty} \lambda_{\nu_{j}}^{1} \neq \lambda_{\nu}^{1}
$$

We may then select a sequence of $\mathbb{P} D F$-invariant $s$-measures $\eta_{j}$ projecting to $\nu_{j}^{\mathbb{Z}} \times \mu$ with

$$
\lambda_{\nu_{j}}^{1}:=\int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{j}(\xi, x,[v])
$$

Indeed, if $\lambda_{\nu_{j}}^{1}<\lambda_{\nu_{j}}^{2}$, we may take the canonical $s$-measures $\eta_{j}=\eta_{j}^{1}$. Otherwise we have $\lambda_{\nu_{j}}^{1}=\lambda_{\nu_{j}}^{2}=0$, and hence, by Proposition 13.4 we may take $\eta_{j}$ to be any $\mathbb{P} D F$-invariant measure with projection $\nu_{j}^{\mathbb{Z}} \times \mu$.

Let $\eta_{0}$ be any accumulation point of $\left\{\eta_{j}\right\}$. Again, $\eta_{0}$ is $\mathbb{P} D F$-invariant, and by Lemma 13.5 we have

$$
\eta_{0}=\alpha \eta_{\nu}^{1}+\beta \eta_{\nu}^{2}, \quad \alpha+\beta=1 .
$$

Moreover, by weak-* convergence we have

$$
\begin{aligned}
\alpha \lambda_{\nu}^{1}+\beta \lambda_{\nu}^{2} & =\int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d\left(\alpha \eta_{\nu}^{1}+\beta \eta_{\nu}^{2}\right)(\xi, x,[v]) \\
& =\int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{0}(\xi, x,[v]) \\
& =\lim _{j \rightarrow \infty} \int \log \left\|D_{x} f_{\xi} \upharpoonright_{[v]}\right\| d \eta_{j}(\xi, x,[v]) \\
& =\lim _{j \rightarrow \infty} \lambda_{\nu_{j}}^{1} \neq \lambda_{\nu}^{1}
\end{aligned}
$$

It follows that $\alpha \neq 1$, whence $\beta \neq 0$. By Claim 13.3, $\eta_{0}$ is an $s$-measure. On the other hand, we have that $\eta_{\nu}^{1}$ is an $s$-measure and $\eta_{\nu}^{2}$ is an $u$-measure, whence

$$
\eta_{\nu}^{2}=\frac{1}{\beta}\left(\eta_{0}-\alpha \eta_{\nu}^{1}\right)
$$

is an $s u$-measure.
13.2.4. Proof of Theorem 5.1: irreducible case. We prove the conclusion of Theorem 5.1](a), Let $f$ be as in Theorem 5.1. Under the hypotheses of Theorem 5.1)(a) we may find $g_{1}, g_{2} \in \Gamma$ with $D g_{1} E_{f}^{u} \not \subset E_{f}^{s} \cup E_{f}^{u}$ and $D g_{2} E_{f}^{s} \not \subset E_{f}^{s} \cup E_{f}^{u}$. Indeed, without loss of generality we may assume there is $g_{2} \in \Gamma$ with $D g_{2} E_{f}^{s}(x) \not \subset$ $\left\{E_{f}^{u}\left(g_{2}(x)\right), E_{f}^{s}\left(g_{2}(x)\right)\right\}$ for all $x \in A$ with $\mu(A)>0$. Let $g \in \Gamma$ be such that $D g E_{f}^{u}(x) \neq E_{f}^{u}(x)$ for all $x \in B$ with $\mu(B)>0$. If $D g E_{f}^{u}(x)=E_{f}^{s}(g(x))$ for almost every $x \in B$, then there is some $k$ with $\mu\left(f^{k}(g(B)) \cap A\right)>0$. Then take $g_{1}=g_{2} \circ f^{k} \circ g$.

Let $K=\left\{f, f^{-1}, g_{1}, g_{1}^{-1}, g_{2}, g_{2}^{-1}\right\}$. Then $\mathcal{M}(K)$ is the simplex $\Delta$ given by the convex hull of

$$
\left\{\delta_{f}, \delta_{f-1}, \delta_{g_{1}}, \delta_{g_{1}^{-1}}, \delta_{g_{2}}, \delta_{g_{2}^{-1}}\right\} .
$$

We write int $(\Delta)$ for the interior of the simplex $\Delta$.

Proof of Theorem 5.1](a). Note that for $\nu \in \operatorname{int}(\Delta)$ we have $\nu(f)>0$, whence $\mu$ is ergodic for $\nu$. It follows from the choice of $g_{i}$, Proposition 13.7, and Lemma 13.6 that every $\nu \in \operatorname{int}(\Delta)$ is a continuity point of the functions $\nu \mapsto \lambda_{\nu}^{1}, \nu \mapsto \lambda_{\nu}^{2}$. Indeed, were $\nu$ a discontinuity point, there would exist a $\mathbb{P} D F$-invariant $s u$-measure projecting to $\nu^{\mathbb{Z}} \times \mu$ which by Lemma 13.6 would imply a union of the two distributions $E_{f}^{s} \cup E_{f}^{u}$ is $D g_{1}$ and $D g_{2}$ invariant. Moreover, for $\nu \in \operatorname{int}(\Delta)$, at least one $\lambda_{\nu}^{1}, \lambda_{\nu}^{2}$ is non-zero. Indeed, by Proposition [13.4, if $\lambda_{\nu}^{1}=\lambda_{\nu}^{2}=0$, then there exists a $\mathbb{P} D F$-invariant su-measure over $\nu^{\mathbb{Z}} \times \mu$ which again, by Lemma 13.6, contradicts the choice of $g_{i}$.

Let $P, N \subset \Delta$ be the sets

$$
P=\left\{\nu \in \Delta \mid \lambda_{\nu}^{2}>0\right\}, \quad N=\left\{\nu \in \Delta \mid \lambda_{\nu}^{1}<0\right\} .
$$

By the continuity of $\lambda^{j}$ the sets $P$ and $N$ are open in int( $\Delta$ ). Furthermore, the simplex $\Delta$ is invariant under the involution (13.2) whence $P$ is non-empty if and only if $N$ is non-empty. Since there are no $\nu \in \operatorname{int}(\Delta)$ with all exponents of $\mu$ of the same sign or all zero, it follows that $\{P, N\}$ is an open cover of $\operatorname{int}(\Delta)$. In particular, there exists a $\nu_{0} \in \operatorname{int}(\Delta)$ such that $\lambda_{\nu}^{1}<0<\lambda_{\nu}^{2}$.

The conclusion then follows from Theorem4.10 for $\nu_{0}$. Indeed, we have that $\mu$ is an ergodic, hyperbolic, $\nu_{0}$-stationary measure that is not finitely supported. Recall that sub- $\sigma$-algebras $\mathcal{F}$ and $\mathcal{G}$ are on $\Sigma \times M$. If $(\xi, x) \mapsto E_{\xi}^{1}(x)$ were $\mathcal{F}$-measurable, then since is it $\mathcal{G}$-measurable, we have $E_{\xi}^{1}(x)=V(x)$ for some $\nu_{0}$-a.s. invariant $\mu$-measurable line field $V \subset T M$. As $\nu_{0}(f)>0$, by the hyperbolicity of $f$, we can conclude that $V(x)$ coincides with either $E_{f}^{u}(x)$ or $E_{f}^{s}(x)$ for almost every $x$. By the ergodicity of $f$ and $f$-invariance of $V, E_{f}^{u}$, and $E_{f}^{s}$, it follows that $V(x)=E_{f}^{s}(x)$ a.s. or $V(x)=E_{f}^{u}(x)$ a.s. The hypotheses on $g_{i}$ ensure no such $V(x)$ exists, and thus the measure $\nu_{0}^{\mathbb{Z}} \times \mu$ is fiber-wise-SRB for the skew product $F$.

Repeating the above argument, and using the fact that $\mu$ is $\nu_{0}$-a.s. invariant, we conclude that $\nu^{\mathbb{Z}} \times \mu$ is fiber-wise-SRB for the skew product $F^{-1}$. It follows from the transverse absolute continuity property of stable and unstable manifolds discussed in the proof of Theorem [3.4 that $\mu$ is absolutely continuous.
13.2.5. Proof of Theorem [5.1: reducible case. We prove Theorem 5.1](b)] Theorem 5.1](c) is proved similarly. Note that in this case, the continuity of exponents follows immediately from the hypotheses.

Let $f$ be as in Theorem 5.1](b) and take $g \in \Gamma$ with $D g E_{f}^{s}(x) \neq E_{f}^{s}(g(x))$ for a positive measure set of $x$. Let $K=\left\{f, f^{-1}, g, g^{-1}\right\}$. For $t \in[0,1]$ write

$$
\nu_{t}:=t \delta_{f}+(1-t) \delta_{g} .
$$

Note that for $t>0$ we have $\nu_{t} \in \mathcal{M}_{\text {erg }}(K)$.
Write $V(x)=E_{f}^{u}(x)$. By hypotheses, the line field $V$ is preserved by $f$ and $g$. Define

$$
\begin{equation*}
\chi(t):=\int t \log \left\|D_{x} f \upharpoonright_{V_{x}}\right\|+(1-t) \log \left\|D_{x} g \upharpoonright_{V_{x}}\right\| d \mu(x) . \tag{13.5}
\end{equation*}
$$

It follows that $\chi(t)$ is a Lyapunov exponent for the $\nu_{t}$-stationary measure $\mu$. Fixing a Riemannian structure on $M$, define the average Jacobian $J\left(\nu_{t}\right)=\int t \log \left|\operatorname{det} D_{x} f\right|+$ $(1-t) \log \left|\operatorname{det} D_{x} g\right| d \mu(x)$. Then from (2.4)

$$
\begin{equation*}
\tilde{\chi}(t):=J\left(\nu_{t}\right)-\chi(t) \tag{13.6}
\end{equation*}
$$

is also a Lyapunov exponent. This establishes the following.

Claim 13.8. For $t \in(0,1]$ the Lyapunov exponents $\lambda_{\nu_{t}}^{j}$ are continuous.
We continue the proof of the theorem.
Proof of Theorem 5.1](b). Let $t \rightarrow 1$. By the hyperbolicity of $f$, from (13.5) and (13.6), for $t$ sufficiently close to $1, \mu$ has one positive and one negative exponent. Moreover, for $t$ sufficiently close to 1 , it follows that the stable bundle for the random dynamics $E_{\omega}^{s}(x)$ does not coincide with $E_{f}^{u}(x)$ on a set of positive measure. Thus, were $E_{\omega}^{s}(x)$ non-random, as $\nu_{t}(f)>0$ by the ergodicity of $f$, the line bundle $E_{\omega}^{s}(x)$ would have to coincide with $E_{f}^{s}$. As $g$ does not preserve $E_{f}^{s}$, we conclude that $E_{\omega}^{s}(x)$ is not non-random.

As $\mu$ is not finitely supported, by Theorem 3.1 it follows that $\mu$ is an $\operatorname{SRB} \nu_{t^{-}}$ stationary measure for all sufficiently large $t<1$. We show $\mu$ is SRB for $f$. Let $\delta^{u}$ denote the unstable dimension of $\mu$ with respect to the single diffeomorphism $f: M \rightarrow M$. We show below that $\delta^{u}=1$ which implies $\mu$ is SRB for $f$. This follows from the following entropy trick.

Let $D=\operatorname{dim}(\mu)$. Recall the fiber-wise entropy and dimension formulas for skew products given by Proposition 6.10 Similar formulas hold for the individual diffeomorphism $f$. Suppose for the sake of contradiction that $\delta^{u}<1$. Then given any $\varepsilon>0$, for all sufficiently large $0<t<1$,

$$
\delta^{u} \lambda_{\nu_{1}}^{2}=\left(D-\delta^{u}\right)\left(-\lambda_{\nu_{1}}^{1}\right)>(D-1)\left(-\lambda_{\nu_{t}}^{1}\right)=\lambda_{\nu_{t}}^{2} \geq \lambda_{\nu_{1}}^{2}-\varepsilon .
$$

As $\varepsilon \rightarrow 0$ as $t \rightarrow 1$ this yields a contradiction. We thus have $\delta^{u}=1$.
13.3. Proof of Proposition 5.5] and Theorem [5.6. Recall the joint cone condition and relevant notation from Section 5.3.

Proof of Proposition 5.5. If $A$ and $B$ do not commute, it follows that $E_{A}^{s} \neq E_{B}^{s}$ and $E_{A}^{u} \neq E_{B}^{u}$. Then for $n>0$ large enough, we have that $A^{-n} C^{s}$ and $B^{-n} C^{s}$ are disjoint. We take $f$ and $g$ sufficiently close to $L_{A}$ and $L_{B}$ so that for some $\kappa>1$ and any $x \in M$
(1) $D_{f(x)} f^{-1} C^{s} \subset C^{s}$ and $D_{g(x)} g^{-1} C^{s} \subset C^{s}$;
(2) if $v \in C^{s}$, then $\left\|D_{g(x)} g^{-1} v\right\|>\kappa\|v\|$ and $\left\|D_{f(x)} f^{-1} v\right\|>\kappa\|v\|$;
(3) $D_{f^{n}(x)} f^{-n} C^{s}$ and $D_{g^{n}(x)} g^{-n} C^{s}$ are disjoint in $T_{x} \mathbb{T}^{2}$.

We further assume analogous properties to the above hold relative to the unstable cones.

Let $\Sigma_{+}=\{f, g\}^{\mathbb{N}}$. Given $\omega=\left(f_{0}, f_{1}, f_{2}, \ldots\right) \in \Sigma_{+}$define

$$
E_{\omega}^{s}(x):=\bigcap_{i=0}^{M} D_{\left(f_{M} \circ \cdots \circ f_{0}\right)(x)}\left(f_{M} \circ \cdots \circ f_{0}\right)^{-1}\left(C^{s}\right)
$$

The set $E_{\omega}^{s}(x)$ is invariant under scaling; moreover, the cone conditions ensure $E_{\omega}^{s}(x)$ is non-empty for every $\omega$ and every $x$. Note that if $v \in E_{\omega}^{s}(x)$, then for any $j \geq 0,\left\|D\left(f_{j} \circ \cdots \circ f_{0}\right) v\right\| \in C^{s}$; hence we have

$$
\left\|D\left(f_{j} \circ \cdots \circ f_{0}\right) v\right\| \leq \kappa^{-j}\|v\| .
$$

Similarly, if $u \in C^{u}$, then $\left\|D\left(f_{j} \circ \cdots \circ f_{0}\right) u\right\| \in C^{u}$ for any $j$, and hence we have $\left\|D\left(f_{j} \circ \cdots \circ f_{0}\right) v\right\| \geq \kappa^{j}\|v\|$. It follows that every $\nu$-stationary measure is hyperbolic with one exponent of each sign. We claim that $E_{\omega}^{s}(x)$ is a one-dimensional subspace.

Indeed, if otherwise there are non-zero $v, u \in E_{\omega}^{s}(x)$ with $v=u+w$ for $w \in C^{u}$, then for $M$ sufficiently large we obtain a contradiction as

$$
\begin{aligned}
\kappa^{M}\|w\| \leq\left\|D\left(f_{M} \circ \cdots \circ f_{0}\right) w\right\| & \leq\left\|D\left(f_{M} \circ \cdots \circ f_{0}\right) u\right\|+\left\|D\left(f_{M} \circ \cdots \circ f_{0}\right) v\right\| \\
& \leq \kappa^{-M}(\|v\|+\|u\|) .
\end{aligned}
$$

In particular, for any $\nu$-stationary measure, $E_{\omega}^{s}(x)$ coincides with the stable Lyapunov subspace for the word $\omega$ at $x$.

Recall that the cones $D_{f^{n}(x)} f^{-n} C^{s}$ and $D_{g^{n}(x)} g^{-n} C^{s}$ are disjoint. As the set of words $\omega=\left(f_{0}, f_{1}, f_{2}, \ldots\right) \in \Sigma$ with $f_{i}=f$ for $0 \leq i \leq n$ and the set of words $\omega=\left(f_{0}, f_{1}, f_{2}, \ldots\right) \in \Sigma$ with $f_{i}=g$ for all $0 \leq i \leq n$ have positive $\nu^{\mathbb{N}}$-measure, it follows that the distribution $E_{\omega}^{s}(x)$ is not non-random for every $\nu$-stationary measure.

It then follows from Theorem 3.1 that any ergodic, $\nu$-stationary measure $\mu$ on $\mathbb{T}^{2}$ is either SRB or finitely supported. Moreover, fixing $f$, by choosing a generic perturbation $g$, for any periodic point $p$ for $f$ we may further assume that $p$ is not a periodic point for $g$. Then, as $f$ and $g$ have no common finite invariant subsets, there are no finitely supported $\nu$-stationary measures.

Proof of Theorem 5.6. Recall in the statement of Theorem[5.6] we set $\nu_{0}=\sum p_{k} \delta_{L_{A_{k}}}$. We take $\tilde{\nu}_{0}=\sum p_{k} \delta_{A_{k}}$ on $\operatorname{SL}(2, \mathbb{Z})$. Consider $\mu$ any $\nu_{0}$-stationary measure. The Lyapunov exponents of $\mu$ coincide with the Lyapunov exponents of the random product of matrices given by $\tilde{\nu}_{0}$. In particular, the Lyapunov exponents of $\mu$ are constant a.s. and independent of the choice of $\mu$. As $\Gamma \subset \operatorname{SL}(2, \mathbb{Z})$ is infinite and does not have $\mathbb{Z}$ as a finite-index subgroup, it follows that $\Gamma$ is not contained in a compact subgroup and that any line $L \in \mathbb{R P}^{1}$ has infinite $\Gamma$-orbit. By a theorem of Furstenberg ([Fur, Theorem 8.6]; see also [Via, Theorem 6.11]) it follows that the random product of matrices given by $\tilde{\nu}_{0}$ has one positive and one negative Lyapunov exponent. The same is then true for any $\nu_{0}$-stationary measure on $\mathbb{T}^{2}$. Moreover, as $\Gamma$ is not virtually $\mathbb{Z}$, one can find hyperbolic elements $B_{1}, B_{2} \in \Gamma$ that satisfy a joint cone property (defined in Section 5.3) and such that $B_{1}$ and $B_{2}$ do not commute. Write

$$
\begin{equation*}
B_{1}=A_{i_{1}} A_{i_{2}} \cdots A_{i_{\ell}}, \quad B_{2}=A_{j_{1}} A_{j_{2}} \cdots A_{j_{p}} \tag{13.7}
\end{equation*}
$$

in terms of the generators.
For each $1 \leq k \leq n$ take a neighborhood $L_{A_{k}} \in U_{k} \subset \operatorname{Diff}{ }^{2}\left(\mathbb{T}^{2}\right)$ sufficiently small so that
(1) $\left|g_{k}^{ \pm 1}\right|_{C^{2}} \leq C$ for all $g_{k} \in U_{k}$ and some $C>0$, and
(2) writing

$$
f_{1}=g_{i_{1}} \circ \cdots \circ g_{i_{\ell}}, \quad f_{2}=g_{j_{1}} \circ \cdots \circ g_{j_{p}}
$$

as in (13.7) for any choice of $g_{i_{\ell}} \in U_{i_{\ell}}$ and $g_{j_{m}} \in U_{j_{m}}, f_{1}$ and $f_{2}$ are sufficiently close to $B_{1}$ and $B_{2}$ so that Proposition 5.5 holds.
In particular, such $f_{1}$ and $f_{2}$ satisfy a joint cone condition, are Anosov diffeomorphisms of $\mathbb{T}^{2}$, and $E_{f_{1}}^{s}(x) \neq E_{f_{2}}^{s}(x)$ and $E_{f_{1}}^{u}(x) \neq E_{f_{2}}^{u}(x)$ for any $x \in \mathbb{T}^{2}$.

Take $U \subset \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right)$ in the theorem to be the set $U=\left\{g \in \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right):|g|_{C^{2}}<\right.$ $C\}$. Let $\nu$ be a probability measure on $U$. We moreover assume $\nu$ is sufficiently close to $\nu_{0}$ so that $\nu\left(U_{k}\right)>0$ for each $1 \leq k \leq n$.

We introduce some notation. Given $f \in \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right)$, consider $\mathbb{P} D f$ acting on the projectivized tangent bundle $\mathbb{P} T \mathbb{T}^{2}$. We naturally identify $\nu$ with a measure on
$\left\{\mathbb{P} D f: f \in \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right)\right\}$. Consider a $\nu$-stationary probability measure $\eta$ on $\mathbb{P} T \mathbb{T}^{2}$. Note that the projection of $\eta$ onto $\mathbb{T}^{2}$ is also a $\nu$-stationary measure.

Given $f \in \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right)$, write

$$
\Phi(f, x, E)=\log \left(\left\|D_{x} f \upharpoonright_{E}\right\|\right)
$$

Note that $\Phi: \operatorname{Diff}^{2}\left(\mathbb{T}^{2}\right) \times \mathbb{P} T \mathbb{T}^{2} \rightarrow \mathbb{R}$ is continuous and uniformly bounded on $U$.
Lemma 13.9. For all $\nu$ sufficiently close to $\nu_{0}$, every ergodic, $\nu$-stationary measure on $\mathbb{T}^{2}$ has a positive Lyapunov exponent.

Proof. Suppose $\nu_{k} \rightarrow \nu_{0}$ on $U$ in the weak-* topology and that for each $k$, there is an ergodic, $\nu_{k}$-stationary measure $\mu_{k}$ with only non-positive exponents. For each $k$ we may select a $\nu_{k}$-stationary probability measure $\eta_{k}$ on $\mathbb{P} T \mathbb{T}^{2}$ projecting to $\mu_{k}$ such that

$$
\iint \Phi(f, x, E) d \eta_{k}(x, E) d \nu_{k}(f) \leq 0
$$

Indeed, the existence and construction of $\eta_{k}$ is essentially the same as in the proof of Proposition 13.7 and (13.3).

As $\mathbb{P} T \mathbb{T}^{2}$ is compact, let $\eta_{0}$ be an accumulation point of $\left\{\eta_{k}\right\}$. Then (see, for example, Via, Proposition 5.9]) $\eta_{0}$ is $\nu_{0}$-stationary. Moreover, $\eta_{0}$ projects to a $\nu_{0}$-stationary measure $\mu_{0}$ on $\mathbb{T}^{2}$ and by weak-* convergence (and boundedness of $\Phi(f, x, E)$ on $U)$

$$
\iint \Phi(f, x, E) d \eta_{0}(x, E) d \nu_{0}(f) \leq 0
$$

Recall we define $\tilde{\nu}_{0}=\sum p_{k} \delta_{A_{k}}$ to be a measure on $\Gamma \subset \operatorname{SL}(2, \mathbb{Z})$. Note that $T \mathbb{T}^{2}$ is parallelizable so $\mathbb{P} T \mathbb{T}^{2}=\mathbb{T}^{2} \times \mathbb{R} \mathbb{P}^{1}$. Then define a factor measure $\tilde{\eta}_{0}$ on $\mathbb{R} \mathbb{P}^{1}$ by

$$
\tilde{\eta}_{0}(D):=\eta_{0}\left(\mathbb{T}^{2} \times D\right)
$$

We have that $\tilde{\eta}_{0}$ is a $\tilde{\nu}_{0}$-stationary measure for the natural action of $\operatorname{SL}(2, \mathbb{R})$ on $\mathbb{R P}^{1}$. Moreover, with $\tilde{\Phi}(A, E)=\log \left\|A \upharpoonright_{E}\right\|$ we have

$$
\iint \tilde{\Phi}(A, E) d \tilde{\eta}(E) d \tilde{\nu}_{0}(A) \leq 0
$$

On the other hand, by a theorem of Furstenberg ([Fur, Theorem 8.5], Via, Theorem $6.8]$ ) this is impossible under our hypotheses on $\Gamma$.

Take $\nu$ sufficiently close to $\nu_{0}$ so that every ergodic, $\nu$-stationary measure on $\mathbb{T}^{2}$ has a positive Lyapunov exponent. Consider $\mu$ an ergodic, $\nu$-stationary measure on $\mathbb{T}^{2}$. Suppose that all exponents of $\mu$ were non-negative. By the invariance principle in AV , it would follow that $\mu$ is invariant for $\nu$-a.e. $f \in \operatorname{Diff}^{2}\left(\mathbb{T}_{2}\right)$. In particular, the sets of $f_{1}$ and $f_{2}$ constructed above for which $\mu$ is simultaneously $f_{1}$ and $f_{2}$-invariant have positive measure. As $f_{1}$ does not preserve $E_{f_{2}}^{s}, E_{f_{2}}^{u}$, or their union, Theorem 5.1 implies that either $\mu$ is atomic or is absolutely continuous. If $\mu$ were absolutely continuous, then, as $\nu$-a.e. $f$ preserves an absolutely continuous measure, it follows from (2.4) that $\mu$ is necessarily hyperbolic. Hence, for all $\nu$ satisfying Lemma 13.9, every ergodic, $\nu$-stationary measure either is atomic or is hyperbolic with one exponent of each sign.

In the case that $\mu$ is hyperbolic with one exponent of each sign, we claim that the stable line fields for $\mu$ are not non-random.

Definition 13.10. $\nu$ is strongly expanding if, for any $\nu$-stationary measure $\eta$ on $\mathbb{P} T \mathbb{T}^{2}$,

$$
\iint \Phi(f, x, E) d \eta(x, E) d \nu(f)>0 .
$$

Let $\mu$ be an ergodic, hyperbolic, $\nu$-stationary measure with one exponent of each sign. Suppose the stable line bundle is non-random. That is, $E_{\omega}^{s}(x)=V(x)$ for some measurable line field $V(x)$ on $T \mathbb{T}^{2}$. Let $\eta$ be the measure on $\mathbb{P} T \mathbb{T}^{2}$ defined as follows: for measurable $\psi: \mathbb{P} T \mathbb{T}^{2} \rightarrow \mathbb{R}$ set

$$
\int \psi(x, E) d \eta(x, E)=\int \psi(x, V(x)) d \mu(x)
$$

It follows from the invariance of $E_{\omega}^{s}(x)$ that $\eta$ is a $\nu$-stationary measure. Moreover, from the pointwise ergodic theorem we have

$$
\iint \Phi d \eta d \nu<0
$$

Thus, see the following claim.
Claim 13.11. If $\nu$ is strongly expanding, then the stable line bundle for any $h y$ perbolic, $\nu$-stationary measure $\mu$ is not non-random.

As in the previous lemma we have the following.
Lemma 13.12. Every $\nu$ sufficiently close to $\nu_{0}$ is strongly expanding.
From the above, it follows that for all $\nu$ sufficiently close to $\nu_{0}$, any ergodic $\nu$ stationary measure $\mu$ which is not atomic is hyperbolic with one exponent of each sign and, moreover, the stable line field for $\mu$ is not non-random. From Theorem3.1] it follows that if $\mu$ is non-atomic, then $\mu$ is SRB for $\nu$.

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[^1]:    ${ }^{1}$ Writing the cocycle as $f_{\xi}^{n}$ is standard in the literature but is somewhat ambiguous. We write $\left(f_{\xi}\right)^{-1}$ to indicate the diffeomorphism that is the inverse of $f_{\xi}: M \rightarrow M$. The symbol $f_{\xi}^{-1}$ indicates $\left(f_{\theta^{-1}(\xi)}\right)^{-1}$.

[^2]:    ${ }^{2}$ Recall that given a sub- $\sigma$-algebra $\mathcal{A}$ of a Lebesgue probability space $(\Omega, \mathcal{B}, \mu)$, there is a unique (up to a.s. equivalence) measurable partition $\alpha$, called the partition into atoms. If $\left\{\mu_{\omega}^{\alpha}\right\}$ denotes a family of conditional measures induced by the partition $\alpha$, then $\mathbb{E}(f \mid \mathcal{A})(\omega)=\int f d \mu_{\omega}^{\alpha}$.

